On Convexity, Smoothness and Renormings in the Study of Faces of the Unit Ball of a Banach Space

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It is well known (see [6]) for a subset $C$ of a bounded closed convex subset $H$ of a $T_2$ locally convex topological vector space $X$ that

(i) $C$ is a face of $H$ if it is closed convex and for every $x, y \in H$ and every $\alpha \in (0, 1)$ such that $\alpha x + (1 - \alpha) y \in C$, then $x, y \in C$;

(ii) $C$ is an exposed face of $H$ if there exists $f$ in $X^*$ such that $C = \{x \in H : f(x) = \sup \{f(H)\}\}$;

(iii) $C$ is a strongly exposed face of $H$ if there exists $f$ in $X^*$ verifying that $C = \{x \in H : f(x) = \sup \{f(H)\}\}$ and for every open subset $U$ of $H$ with $C \subseteq U$, there exists $\delta > 0$ such that $\text{slc}(H, f, \delta) \subseteq U$ (where $\text{slc}(H, f, \delta) = \{h \in H : f(h) \geq \sup \{f(H)\} - \delta\}$ is the slice of $H$ determined by $f$ and $\delta$).

If $c$ is an element of $H$, then

(i) $c$ is an extreme point of $H$ if $\{c\}$ is a face of $H$ (see [1]);

(ii) $c$ is an exposed point of $H$ if $\{c\}$ is an exposed face of $H$ (see [6]);

(iii) $c$ is a strongly exposed point of $H$ if $\{c\}$ is a strongly exposed face of $H$ (see [6]).

It is well known for a point $x$ of the unit sphere of a Banach space $X$ that

(i) $x$ is a rotund point of $B_X$ if every $y$ in $S_X$, such that $\|(x + y)/2\| = 1$, verifies that $x = y$ (see [8]);

(ii) $x$ is a locally uniformly rotund point of $B_X$ if every sequence $(y_n)_{n \in \mathbb{N}}$ in $S_X$, such that $\|(x + y_n)/2\|_{n \in \mathbb{N}}$ converges to 1, verifies that $(y_n)_{n \in \mathbb{N}}$ converges to $x$ (see [4]).
It is said that a Banach space is \((locally \ uniformly)\ rotund\) if every point of its unit sphere is a (locally uniformly) rotund point of its unit ball.

It is clear that every locally uniformly rotund point is a strongly exposed point (for the vector topology given by the norm). Nevertheless, there exist rotund points which are not strongly exposed points. A Banach space is said to be \(strongly\ exposed\) if every point of its unit sphere is a strongly exposed point of its unit ball.

It is well known (see [9, Chapter 5.3]) for a point \(x\) of the unit sphere of a Banach space \(X\) that

(i) \(x\) is a smooth point of \(B_X\) if every sequence \((f_n)_{n\in\mathbb{N}}\) in \(S_{X^*}\), such that \((f_n(x))_{n\in\mathbb{N}}\) converges to 1, verifies that \((f_n)_{n\in\mathbb{N}}\) is \(\omega^*\)-convergent;

(ii) \(x\) is a strongly smooth point of \(B_X\) if every sequence \((f_n)_{n\in\mathbb{N}}\) in \(S_{X^*}\), such that \((f_n(x))_{n\in\mathbb{N}}\) converges to 1, verifies that \((f_n)_{n\in\mathbb{N}}\) is convergent.

It can be checked (see [7]) that

(i) \(x\) is a smooth point of \(B_X\) if and only if the norm of \(X\) is Gâteaux differentiable at \(x\);

(ii) \(x\) is a strongly smooth point of \(B_X\) if and only if the norm of \(X\) is Fréchet differentiable at \(x\).

It is said that a Banach space is \((strongly)\ smooth\) if every point of its unit sphere is a (strongly) smooth point of its unit ball.

It is well known for a Banach space \(X\) that

(i) \(X\) has the Efimov-Stechkin property if for every sequence \((x_n)_{n\in\mathbb{N}}\) in \(S_X\) and for every \(f\) in \(S_{X^*}\) such that \((f(x_n))_{n\in\mathbb{N}}\) converges to 1, then \((x_n)_{n\in\mathbb{N}}\) has a convergent subsequence (see [9, pp. 478–479] and [10]);

(ii) \(X\) is \(almost-rotund\) if all closed convex subsets of \(S_X\) are compact (see [3]).

We are very interested in (strongly exposed) faces, which allows us to characterize Efimov-Stechkin property and rotundity.

**Theorem 1.** Let \(X\) be a Banach space. The following assertions are equivalent:

(i) \(X\) has the Efimov-Stechkin property.

(ii) \(X\) is reflexive, almost-rotund and every exposed face of \(B_X\) is a strongly exposed face of \(B_X\).
Theorem 2. Let \( X \) be a Banach space. The following assertions are equivalent:

(i) \( X \) is rotund.
(ii) If \( C \) is a closed convex subset of \( S_X \) such that \( B_X \setminus C \) is convex, then \( C \) is a face of \( B_X \).

On the other hand, smoothness techniques can be used to characterize rotundity in a local way. Following this line, we extend a result of Bandyopadhyay and Lin (see [5]).

Theorem 3. Let \( X \) be a Banach space and let \( x \in S_X \). The following assertions are equivalent:

(i) \( x \) is a rotund point of \( B_X \).
(ii) For every \( y \in S_X \setminus \{x\} \),
\[
\lim_{t \to 0^+} \left( \frac{\|x + ty\| - \|x\|}{t} \right) < 1.
\]

Theorem 4. Let \( X \) be a Banach space and let \( x \in S_X \). If \( x \) is a strongly exposed point of \( B_X \) and a strongly smooth point of \( B_X \), then it is a locally uniformly rotund point of \( B_X \).

Corollary 5. Let \( X \) be a Banach space. Then, \( X \) is locally uniformly rotund if it is strongly exposed and its norm is Fréchet differentiable in \( S_X \).

Finally, exposed faces can be characterized using some renorming techniques. In this way, we can prove the following theorems.

Theorem 6. Let \( X \) be a Banach space. Let \( C \) be a nonempty subset of \( S_X \). The following statements are equivalent:

(i) \( C \) is an exposed face of \( B_X \).
(ii) There exists an equivalent norm \( \| \|_0 \) on \( X \) such that \( B_X \subseteq B_{X_0} \subseteq \sqrt{2}B_X \), \( S_{X_0} \cap S_X = C \cup -C \), and \( C \) is a maximal face of \( B_{X_0} \), where \( X_0 \) denotes the space \( X \) with the norm \( \| \|_0 \).

Corollary 7. Let \( X \) be a Banach space and let \( x \in S_X \). The following statements are equivalent:

(i) \( x \) is an exposed point of \( B_X \).
(ii) There exists an equivalent norm \( \| \cdot \|_0 \) on \( X \) such that \( B_X \subseteq B_{X_0} \subseteq \sqrt{2} B_X \), \( S_{X_0} \cap S_X = \{ x, -x \} \), and \( x \) is a rotund point of \( B_{X_0} \), where \( X_0 \) denotes the space \( X \) with the norm \( \| \cdot \|_0 \).

**Theorem 8.** Let \( X \) be a Banach space and let \( x \in S_X \). The following statements are equivalent:

(i) \( x \) is a strongly exposed point of \( B_X \).

(ii) There exists an equivalent norm \( \| \cdot \|_0 \) on \( X \) such that \( B_X \subseteq B_{X_0} \subseteq \sqrt{2} B_X \), \( S_{X_0} \cap S_X = \{ x, -x \} \), and \( x \) is a locally uniformly rotund point of \( B_{X_0} \), where \( X_0 \) denotes the space \( X \) with the norm \( \| \cdot \|_0 \).

Part of these results will appear in [2].

**References**


