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On the Convergence and Unconditionally Convergence of Series of Operators on Banach Spaces

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We study Kalton's theorem [10] on the unconditional convergence of series of compact operators and we use some matrix techniques to obtain sufficient conditions, weaker than the previous one, on the convergence and unconditional convergence of series of compact operators.

Let X and Y be two Banach spaces. We denote by $CL(X,Y)$ (resp. $K(X,Y)$) the Banach spaces of bounded (resp. compact) linear maps from X to Y. Kalton [10] proved that if $\sum_i T_i$ is subseries convergent in $K(X,Y)$ with respect to the weak operator topology (WOT) and X^* contains no subspace isomorphic to l_∞, then $\sum_i T_i$ is subseries convergent with respect to the norm topology. The two main ideas of the proof are the following: (a) if X^* does not have a copy of l_∞, then a series $\sum_i x_i^*$ in X^* is unconditionally convergent (uco) with respect to the $*$-w topology if and only if $\sum_i x_i^*$ is uco with repect to the norm; (b) if $T \in K(X,Y)$ and WOT-$\lim_i T_i^* = T^*$ in $K(X^*,Y^*)$, then w-$\lim_i T_i = T$.

We are interested in matrix techniques, and so we refer the reader to [5], where Antosik and Swartz proved Kalton's theorem by means of a matrix argument. It is obtained a k-matrix (i.e., a matrix which verifies the hypothesis of Basic Matrix theorem [5]) which allow us to conclude that $\lim_i \|T_i\| = 0$.

In 1997 Qingying Bu and Congxin Wu [6] proved a similar result to Kalton's one. Their argument is based on interesting characterizations of relatively compact subsets of BMC(X) (i.e., the Banach space of the bounded multiplier convergent series in X). The converse result, which is also proved, characterizes the property "X^* does not have a copy of c_0" in terms of the unconditionally convergence of series of operators.

We first introduce the property S_1, which is based on the Seever property [14] and on the concept of FQ-σ family [12]. This concept was introduced by Samaratunga and Sember and it has been studied by other authors like Swartz [16] or Wu Junde and Lu Shijie [8] obtaining interesting matrix results. Let $\mathcal{F}$ be a natural family (i.e., a subfamily of $P(\mathbb{N})$ which contains the family $\phi_0(\mathbb{N})$ of the finite subsets of $\mathbb{N}$). We will say that $\mathcal{F}$ has the property S_1 if for any pair $[(A_i),(B_i)]$ of mutually disjoint sequences of disjoint subsets of $\phi_0(\mathbb{N})$ there exist $B \in \mathcal{F}$ and an infinite set $M \subset \mathbb{N}$ such that $A_i \subset B$ and $B_i \subset B^c$.

It can be proved the following result.

Theorem 1. *Let $(x_{ij})_{i,j}$ be a matrix in a normed space X such that $(x_{ij})_i$ is a Cauchy sequence for $j \in \mathbb{N}$ and, for $i \in \mathbb{N}$, the series $\sum_j x_{ij}$ is subseries convergent. Then the following conditions are equivalent:*

(1) *There exists a natural family $\mathcal{F}$ with the property S_1 such that $\left(\sum_{j\in B} x_{ij}\right)_i$ is Cauchy, for $B \in \mathcal{F}$.*

(2) *The sequence $\left(\sum_{j\in A_n} x_{ij}\right)_i$ is Cauchy uniformly on $n \in \mathbb{N}$, for every disjoint sequence $(A_n)_n$ of $\phi_0(\mathbb{N})$.*

As a corollary we obtain the following result which generalizes the Orlicz-Pettis theorem: each $\mathcal{F}$-weak convergent series $\sum_i x_i$ in a Banach space X (i.e., $\sum_{i\in A} x_i$ is convergent for each $A \in \mathcal{F}$), where $\mathcal{F}$ denotes a natural family with the property S_1, is uco.

These results allow us to characterize the unconditional convergence of a series in $K(X,Y)$, where X and Y are Banach spaces.

Let $\mathcal{F}$ be a natural family with the property S_1 and let $\sum_i T_i$ be a series in $K(X,Y)$ such that: (a) $\sum_i T_i^*$ is $\mathcal{F}$-convergent in $K(Y^*,X^*)$ with respect to the WOT; (b) $\sum_i T_i$ is not a uco series. Since the rank of a compact operator is separable, we can suppose that Y is separable. This assumption and the Banach-Alaoglu theorem allow us to obtain a matrix $(g_iT_j)_{i,j} = (T_j^* g_i)_{i,j}$ as the one in Theorem 1. This matrix verifies property (1) in this result but it lacks property (2), which is impossible.

This argument shows that *a series $\sum_i T_i$ is uco in $K(X,Y)$ if and only if $\sum_i T_i^*$ is $\mathcal{F}$-convergent in $K(Y^*,X^*)$ with respect to the WOT, where $\mathcal{F}$ denotes a natural family with the property S_1*.

As a corollary we obtain the following result.

Theorem 2. *Let X, Y be two Banach spaces and let $\mathcal{F}$ be a natural family with the property S_1. If X^* does not have a copy of l_∞, then a series $\sum_i T_i$ in $K(X,Y)$ is uco if and only if $\sum_i T_i$ is $\mathcal{F}$-convergent in $K(X,Y)$ with respect to the WOT.*

These results remain if the natural family with the property S_1 is replaced by a natural Boolean algebra with the Vitali-Hahn-Saks property (VHS) [13]. Let us observe that our generalization of the Orlicz-Pettis theorem remains valid if we consider a natural Boolean algebra (VHS) instead a natural family with the property S_1.

From what it has been said, property S_1 generalizes the notion of FQ-σ family. Our interest is now centred on the concept of IQ-σ family [12]. This concept is also studied in [15] and [11], obtaining summation theorems where the unconditional convergence of series is replaced by the convergence of series.

In [3] we introduce property P_1 which generalizes the notion of IQ-σ family. Analysis similar to that in the case of property S_1 allow us to obtain the following two theorems in relation to the convergence of series in $K(X,Y)$

Theorem 3. *Let X, Y be two Banach spaces and let $\mathcal{F}$ be a natural family with property P_1. Let $\sum_i T_i$ be a series in $K(X,Y)$.*

(a) *If $\sum_i T_i^*$ is $\mathcal{F}$-convergent in $K(Y^*, X^*)$ with respect to the strong operator topology (SOT), then $\sum_i T_i$ is a convergent series in $K(X,Y)$ (with respect to the norm topology).*

(b) *If $\sum_i T_i$ is $\mathcal{F}$-convergent in $K(X,Y)$ with respect to the SOT and $\sum_{i\in B} T_i^*(g)$ is convergent in X^* for each $g \in Y^*$ and $B \in \mathcal{F}$, then $\sum_i T_i$ is a convergent series.*

(c) *If X^* does not have a copy of l_∞ and $\mathcal{F}$ also verifies that any real and $\mathcal{F}$-convergent series is uco, then each series $\sum_i T_i$ is $\mathcal{F}$-convergent in $K(X,Y)$ with respect to the SOT is convergent.*

Part of these results will appear in [2].

References

[1] Aizpuru, A., Pérez-Fernández, F.J., Spaces of S-bounded multiplier convergent series, *Acta Math. Hungar.* **87** (1-2) (2000), 135 – 146.

[2] Aizpuru, A., Gutiérrez-Dávila, A., Sala, A., A Riemann type theorem for series of operators on Banach spaces, *Bull. Austral. Math. Soc.* **68** (2003), 13 – 20.

[3] Aizpuru, A., Gutiérrez-Dávila, A., Interchange of series and applications, *Bull. Belg. Math. Soc.* (to appear).

[4] Aizpuru, A., Gutiérrez-Dávila, A., Unconditionally Cauchy series and uniform convergence on matrices (preprint, 2001).

[5] Antosik, P., Swartz, C. "Matrix Methods in Analysis", Lectures Notes in Math. 1113, Springer-Verlag, Berlin-New York, 1985.

[6] Qingying Bu, Congxin Wu, Unconditionally convergent series of operators on Banach spaces, *J. Math. Anal. Appl.* **207** (1997), 291 – 299.

[7] Diestel, J., Jarchow, H., Tonge, A., "Absolutely Summing Operators", Cambridge Studies in Advanced Math. 43, Cambridge, 1995.

[8] Wu Junde, Lu Shijie, A summation theorem and its applcations, *J. Math. Anal. Appl.* **257** (2001), 77 – 87.

[9] Kaftal, V., Weiss, G., A Riemann type theorem for unconditional convergence of operators, *Proc. Amer. Math. Soc.* **98** (3) (1986), 431 – 435.

[10] Kalton, N.J., Spaces of compact operators, *Math. Ann.* **208** (1974), 267 – 278.

[11] Ronglu Li, Minkang Shin, On convergence of double series, *J. Math. Anal. Appl.* **240** (1999), 481 – 490.

[12] Samaratunga, R.T., Sember, J., Summability and substructures of $2^{\mathbb{N}}$, *Southeast Asia Bull. Math.* **12** (1988), 11 – 21.

[13] Schachermayer, W., On some classical measure theoretics theorems for non-sigma-complete Boolean algebras, *Dissertationes Math.* (*Rozprawy Mat.*) **214** (1982).

[14] Seever, G.L., Measure on F-spaces, *Trans. Amer. Math. Soc.* **133** (1968), 267 – 280.

[15] Swartz, C., Antosik's interchange theorem, in "Funtion Spaces", Lecture Notes in Pure and Appl. Math. 172, Dekker, New York, 1995, 361 – 369.

[16] Swartz, C., A general Hahn-Schur theorem, *SEA Bull. Math.* **20** (1996), 57 – 58.