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The Positive Supercyclicity Theorem

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We present some recent results related with supercyclic operators, also some of its consequences. We will finalize with new related questions.

A bounded linear operator T on a separable Banach space $\mathcal{B}$ is said to be *supercyclic* if there exists a vector $x \in \mathcal{B}$ such that the set

$$\{\lambda T^n x \,:\, n \in \mathbb{N},\, \lambda \in \mathbb{C}\}$$

is dense in $\mathcal{B}$. Of course, supercyclic operators cannot exist in non separable spaces.

Supercyclicity is an intermediate property among the hypercyclicity and the cyclicity (an operator $T \in \mathcal{L}(\mathcal{B})$ is called *hypercyclic* if the orbit $\{T^n x\}$ of some vector x is dense in $\mathcal{B}$, and is it called *cyclic* if the linear span of the orbit $\{T^n x\}$ of some vector x is dense in $\mathcal{B}$). These notions are in the heart of operator theory and they are intimately related to the invariant subspace problem. The right reference on hypercyclic operators is a recent survey of Große-Erdman (see [7]).

Supercyclicity was introduced in the sixties by Hilden and Wallen (see [9]). In 1991 this property is studied extensively, thanks to the interesting work of Godefroy and Shapiro (see [5]). In the trivial case $\mathcal{B} = \mathbb{C}$ each non-zero linear operator is supercyclic, however if $2 \leq \dim \mathcal{B} < \infty$ then there are not supercyclic operators. The first example of supercyclic operator in infinite dimensional Banach spaces (moreover hypercyclic) was discovered by Rolewicz in 1969 (see [13]). This expository paper deals with positive supercyclicity

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(this concept was introduced in [2]). An operator T is said to be *positive supercyclic* if the set

$$\{rT^n x : r \in \mathbb{R}_+, n \in \mathbb{N}\}$$

is dense for some x.

In [2] arises the following question

Question. Which supercyclic operators are positive supercyclic?

The structure of the set of supercyclic operators is quite exotic (in some sense, this structure is inherited by the trivial case in dimension 1).

Following Herrero [8] the class of supercyclic operators is divided into two classes:

(1) Supercyclic operators T for which the point spectrum of its adjoint is empty: $\sigma_p(T^\star) = \emptyset$.
(2) Supercyclic operators T for which the point spectrum of $T^\star$ is a non-zero complex number α. In that case T is decomposed in the following direct sum $T = R \oplus \alpha 1_{\mathbb{C}}$ (in [6] is established that an operator with that decomposition is supercyclic if and only if $(1/\alpha)R$ is hypercyclic).

In the trivial case ($\mathcal{B} = \mathbb{C}$) if $\alpha = r\mathrm{e}^{2\pi\theta\mathrm{i}}$ with $\mathrm{e}^{2\pi\theta\mathrm{i}}$ a root of the unity, then T is clearly not positive supercyclic. Moreover, it was pointed in [2] that if $T = R \oplus \alpha 1_{\mathbb{C}}$ with $\alpha = r\mathrm{e}^{2\pi\theta\mathrm{i}}$ and $\mathrm{e}^{2\pi\theta\mathrm{i}}$ a root of the unity, then T is not positive supercyclic.

On the other hand if $\alpha = r\mathrm{e}^{2\pi\theta\mathrm{i}}$ with θ irrational, observe that T is trivially positive supercyclic.

Our result on positive supercyclicity was originated solving the following question, which was formulated by several authors at the same time:

Question. Let T be a hypercyclic operator in $\mathcal{L}(\mathcal{B})$ and suppose that $\alpha \in \partial\mathbb{D}$. Is αT hypercyclic?

If $\alpha \neq 1$ there exists a lot of examples for which αT is not hypercyclic, for instance each hypercyclic compact perturbation of the identity (see [4]). However the above question has a positive answer (see [10]). Moreover the proof can be extended for semigroups of operators, we can obtain the following general result:

Theorem. (cf. [10]) *Let $\mathcal{M}$ be a semigroup of operators for which there exists a vector x such that the set $\{\lambda S x : S \in \mathcal{M}, \lambda \in \partial\mathbb{D}\}$ is dense. Suppose* *that there exists a bounded linear operator T such that $\sigma_p(T^\star) = \emptyset$ and $ST = TS$ for all $S \in \mathcal{M}$, then the set $\{Sx : S \in \mathcal{M}\}$ is dense.*

As a consequence we obtain the Positive Supercyclicity Theorem:

Theorem. (cf. [10]) *Let T be a supercyclic operator satisfying $\sigma_p(T^\star) = \emptyset$, then T is positive supercyclic.*

Let us observe that the above theorem simplify obviously the concept of supercyclicity. In fact, when the Positive Supercyclicity Theorem was discovered, several consequences were obtained. We show some of these consequences.

Let $(\mathcal{B}, \leq)$ a Banach lattice, and T a positive bounded linear operator on $\mathcal{B}$. The invariant subspace problem is open even in this situation, that is for positive operators on a general Banach lattice (cf. [1]). The invariant subspace problem can be easily reformulated in terms of cyclic vectors. Namely: *An operator T has a non-trivial invariant closed subspace if and only if there exists a non-cyclic vector (non trivial) for T.*

The following partial result can be obtained directly form the Positive Supercyclicity Theorem:

Corollary. *If T is a positive operator on a Banach lattice $(\mathcal{B}, \leq)$ then each non-zero vector in the positive conus of $\mathcal{B}$ is a non-supercyclic vector for T.*

As a second consequence, let us recall the well-known result of Brown, Chevreau and Pearcy [3], each Hilbert space contraction whose spectrum contains the unit circle has a nontrivial closed invariant subspace. Equivalently, there is a nonzero vector which is not cyclic.

And it is a well known open question to reduce the hypothesis on the spectrum.

Question. Let T be a Hilbert space contraction whose spectral radius is 1 (that is, contains at most a point of the unit circle). Is there a non cyclic (nontrivial) vector for T?

In this direction, Müller proved recently (see [12]) that each Hilbert space contraction (or more generally, a power bounded operator) with spectrum containing at least one point from the unit circle, has a nonzero vector which is not supercyclic. The main ingredient in the proof is the Positive Supercyclicity Theorem.

Theorem. (cf. [12]) *Let T be a power bounded operator on a Hilbert space H. Suppose $r(T) = 1$. Then there exists a non-zero vector x in H which is not supercyclic for T.*

Finally, solving a question of Salas, we use the Positive Supercyclicity Theorem to prove that the classical Volterra operator (see [11])

$$\int_0^x f(t)\,\mathrm{d}t$$

defined on $L^p[0,1]$, $1 \leq p < \infty$, is not supercyclic.

As we can see, more things are known about supercyclic operators with the Positive Supercyclicity Theorem. It remains a case in which we do not know if there exists positive supercyclicity. We conjecture the following (cf. [10]):

Conjecture. Let us suppose that T is a supercyclic operator with $\sigma_p(T^\star) = \{\alpha\}$ with $\alpha = r\mathrm{e}^{2\pi\theta\mathrm{i}}$ and θ irrational. Then T is positive supercyclic.

Another interesting problem is the following.

Problem. Let T be a power bounded operator on a Hilbert space H. Suppose $r(T) = 1$. Does it exist a non-trivial real subspace for T?

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