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1, 2, 4, 8, ... What Comes Next?

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1. Number theory begat Hurwitz algebras

Mathematicians were always interested, when a sum of n squares of integers times another sum of n squares of integers is a sum of n squares of integers again? For $n = 1$ the corresponding identity:

$$x_1^2 y_1^2 = (x_1 y_1)^2$$

immediately follows from the commutativity of integers. For $n = 2$ the corresponding identity

$$(x_1^2 + x_2^2)(y_1^2 + y_2^2) = (x_1y_1 + x_2y_2)^2 + (x_1y_2 - x_2y_1)^2$$

is not so trivial and has been used by many mathematicians. Diophantos used it sometime between 2nd to 4th century but maybe even he was not the first. The corresponding identity for four squares

$$\begin{aligned} &(x_1^2 + x_2^2 + x_3^2 + x_4^2)(y_1^2 + y_2^2 + y_3^2 + y_4^2) = \\ = \ &(x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4)^2 + (x_1y_2 - x_2y_1 - x_3y_4 + x_4y_3)^2 + \\ + \ &(x_1y_3 + x_2y_4 - x_3y_1 - x_4y_2)^2 + (x_1y_4 - x_2y_3 + x_3y_2 - x_4y_1)^2 \end{aligned}$$

was first announced by Euler in his letter to Goldbach in 1748[1]. In 1818 C.F. Degen[2] found the first example of a similar identity for the sum of eight squares

[1] Thanks to Garry Tee for these two historical comments.
[2] Mem Acad.Sci. St. Petersbourg **8**, (1818), 207–219.

which then went unnoticed. It was followed by numerous publications where a number of false identities for the sum of sixteen squares were suggested.

In 1843 Hamilton noted that the existence of an identity

$$(x_1^2 + \cdots + x_n^2)(y_1^2 + \cdots + y_n^2) = (z_1^2 + \cdots + z_n^2), \tag{1}$$

where the z_i's are bilinear functions of x_i's and y_i's, is equivalent to the existence of a division algebra with dimension n over the field $\mathbb{R}$ of real numbers. Indeed, in this case we can define in $\mathbb{R}^n$ the following multiplication: if $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$, then set $x \cdot y = z$, where $z = (z_1, \ldots, z_n)$ and the z_i are functions of x's and y's determined by the identity (1). Since for such an algebra (1) can be written as

$$|x||y| = |x \cdot y| \tag{2}$$

then it is clear that $x \cdot y = 0$ implies $x = 0$ or $y = 0$. In general, this algebra may not be associative. The identities for the sums of 1, 2 and 4 squares follow immediately from the identity (2) for real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$, and quaternions $\mathbb{H}$.

In 1845 Cayley constructed an eight-dimensional division algebra of octonions $\mathbb{O}$ giving the identity for the sum of eight squares. As later appeared, Graves constructed the same algebra one year earlier, in December 1843. He called octonions octaves. This algebra is not associative but every two elements in $\mathbb{O}$ generate an associative (usually quaternion) subalgebra. Algebras with this property are called alternative because in such algebras associator

$$(x, y, z) = (xy)z - x(yz)$$

is an alternating function of its arguments.

To familiarize ourselves with quaternions and octonions we might think that

$$\mathbb{H} = \{(\alpha, u) \mid \alpha \in \mathbb{R}, u \in \mathbb{R}^3\}$$

with componentwise addition, and the multiplication

$$(\alpha, u)(\beta, v) = (\alpha\beta - (u, v)\ ,\ \alpha v + \beta u + u \times v).$$

Also, we can view octonions as 2×2 matrices

$$\mathbb{O} = \left\{ \begin{pmatrix} \alpha & u \\ v & \beta \end{pmatrix} \mid \alpha, \beta \in \mathbb{R},\ u, v \in \mathbb{R}^3 \right\}$$

with componentwise addition, and the multiplication

$$\begin{pmatrix} \alpha_1 & u_1 \\ v_1 & \beta_1 \end{pmatrix} \begin{pmatrix} \alpha_2 & u_2 \\ v_2 & \beta_2 \end{pmatrix} = \begin{pmatrix} \alpha_1\alpha_2 + (u_1, v_2) & \alpha_1 u_2 + \beta_2 u_1 - v_1 \times v_2 \\ \alpha_2 v_1 + \beta_1 v_2 + u_1 \times u_2 & \beta_1\beta_2 + (v_1, u_2) \end{pmatrix}.$$

In attempts to obtain division algebras of larger dimensions Dickson generalized the process that leads us from $\mathbb{R}$ to $\mathbb{C}$, from $\mathbb{C}$ to $\mathbb{H}$, and from $\mathbb{H}$ to $\mathbb{O}$ in the following way. Let A be an algebra with 1, of dimension n over $\mathbb{R}$, with an involution $a \mapsto \bar{a}$ such that for every $a \in A$ elements both $a + \bar{a}$ and $a\bar{a}$ are scalars

$$a + \bar{a} = t(a)1, \qquad a\bar{a} = n(a)1,$$

$t(a), n(a) \in \mathbb{R}$. Then one can construct an algebra B with involution, of dimension $2n$ over $\mathbb{R}$, by defining it as the set of all ordered pairs $b = (a_1, a_2)$ with componentwise addition, multiplication defined by

$$(x_1, x_2)(y_1, y_2) = (x_1 y_1 - y_2 \bar{x}_2 \ , \ \bar{x}_1 y_2 + y_1 x_2), \tag{3}$$

and involution

$$\overline{(x_1, x_2)} = (\bar{x}_1, -x_2). \tag{4}$$

This doubling process, which is known now as Cayley-Dickson process [5], reproduces the sequence of division algebras

$$\mathbb{R} \to \mathbb{C} \to \mathbb{H} \to \mathbb{O}$$

but it leads nowhere further. The next algebra does not have a division. We see how dramatic this picture is. We have to pay at each step. For the sake of constructing a division algebra of larger dimension we sacrifice one good thing after another. At first step we had to introduce nonidentical involution, then we lose commutativity, then associativity. Finally at the dimension 8 we have nothing valuable to sacrifice with.

Therefore we cannot get more identities for sums of squares going along these lines, and we cannot get them in any other way because in 1898 Hurwitz proved that such identities exist only for $n = 1, 2, 4, 8$. This means also that absolute valued division algebras, i.e. division algebras with the property (2) exist only in dimensions $n = 1, 2, 4, 8$. Sometimes, altogether they are called Hurwitz algebras.

2. Applications to differential geometry and topology

The existence of these four algebras leads to the fact that mathematics in these dimensions looks different than in the others. For instance, for $n = 1, 2, 4, 8$, due to (2), a continuous multiplication can be induced from the corresponding Hurwitz algebra to the $(n-1)$-dimensional sphere S^{n-1} which consists of vectors of length 1. This multiplication converts it into a topological group for $n = 2, 4$ and a topological loop for $n = 8$. Due to this, spheres S_0, S_1, S_3 and S_7 are parallelizable. Parallelizability of a manifold means that the tangent space at each point is isomorphic to any other tangent space by an isomorphism induced by a parallel transport along a curve, and that this isomorphism is independent of the choice of curve joining the two points.

To see the parallelizability of S_0, S_1, S_3 and S_7, take the standard basis $\{e_1, \ldots, e_n\}$ of $\mathbb{R}^n$ $(n = 1, 2, 4, 8)$, where $e_1 = 1$ is the unit element of the Hurwitz algebra, and arbitrary $x \in S^{n-1}$. Consider vectors $w_1(x) = e_1 \cdot x = x, w_2(x) = e_2 \cdot x, \ldots, w_n(x) = e_n \cdot x$. Since multiplication by x does not change norms, it does not change angles. Therefore the vectors $w_1(x), \ldots, w_n(x)$ are orthogonal, and the vectors $w_2(x), \ldots, w_n(x)$ are tangential to the sphere S^{n-1} at point x. The mappings $x \mapsto w_i(x)$ are clearly diffeomorphisms.

In fact, we do not need a Hurwitz algebra to parallelize S^{n-1}. A small additional effort allows us to prove that existence of an arbitrary division algebra of dimension n over the reals implies parallelizability of S^{n-1}. But even such division algebras in other dimensions do not exist. It follows from the famous theorem of Bott and Milnor that spheres are parallelizable only in dimensions $n = 1, 2, 4, 8$.

An m-dimensional vector bundle over S^n is a real vector space E together with a continuous projection mapping $p\colon E \to S^n$, such that, for every $x \in S^n$, the fibre $E_x = p^{-1}(x)$ is an m-dimensional vector space, and for every point in S^n there exists neighborhood U and m cross-sections $v_1, \ldots, v_n\colon U \to E$ (that is continuous mappings with $p \circ v_i = Id$) such that for every $x \in U$, the m vectors $v_1(x), \ldots, v_m(x)$ form a basis of E_x. Vector bundle is one of the most fundamental concept of differential topology and differential geometry. A central role among vector bundles over S^n play the four so-called *Hopf bundles* ρ_1, ρ_2, ρ_4, ρ_8, associated with the division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$, respectively. They are constructed as follows. For $n = 1, 2, 4, 8$, we divide S^n into its upper and low hemispheres and we assume that both contain the equator S^{n-1}, which points are temporarily duplicated:

$$S^n = H^+ \cup H^-, \qquad H^+ \cap H^- = S^{n-1}.$$

Then, due to existence of a division algebra of dimension n, we can identify it with $\mathbb{R}^n$. We form the trivial vector bundles $H^+ \times \mathbb{R}^n$ and $H^+ \times \mathbb{R}^n$ on the upper and lower hemispheres and identify each point $(x, v) \in S^{n-1} \times \mathbb{R}^n$ with $(x, x \cdot v)$, where the dot denotes the multiplication in the division algebra. For $n = 1$ we obtain the famous Mobius strip.

The Hopf bundles, especially ρ_8, play a pivotal role in proving Bott's isomorphism and Bott's periodicity theorem, which essentially says, that no new phenomena can be found beyond $n = 8$.

3. Applications to geometry

A few words about geometry. The most striking feature of the Hilbert's *Grundlangen der Geometrie* of 1899 was the discovery of the relation between geometric incidence theorems (lock theorems) and axioms for algebraic structures. To formulate these relations we choose projective geometry instead of Euclidean, as did Hilbert.

Adding Desargues's theorem as a "lock incidence theorem" to the "trivial" incidence axioms, one gets a class of geometries which can be described algebraically as that of projective geometries over an associative division algebra. Adding Pappus-Pascal's theorem algebraically means postulating commutativity. In 1932-34 Moufang discovered that for projective planes, the little Desargues's theorem is equivalent to being a projective geometry over an alternative division algebra. The projective plane over octonions satisfies the little Desargues's theorem but not the full Desargues's theorem. This effect disappears in higher dimensions where the little Desargues's theorem implies Desargues's theorem [3].

Let A be now an arbitrary Hurwitz algebra, one of the four. Let us consider the set of Hermitian 3×3 matrices

$$H_3(A) = \left\{ \begin{pmatrix} \alpha & a & b \\ \bar{a} & \beta & c \\ \bar{b} & \bar{c} & \gamma \end{pmatrix}, \quad \alpha, \beta, \gamma \in \mathbb{R};\ a, b, c \in A \right\}$$

with the ordinary addition and the symmetrized multiplication

$$X \odot Y = \frac{1}{2}(XY + YX).$$

Thus we obtained a sequence of simple Jordan matrix algebras

$$H_3(\mathbb{R}) \to H_3(\mathbb{C}) \to H_3(\mathbb{H}) \to H_3(\mathbb{O})$$

whose dimensions are 6, 9, 15, 27. The last Jordan algebra is especially interesting. It is exceptional in the sense that each of the other three belongs to an infinite series of simple algebras but this is not the case with $H_3(\mathbb{O})$ [5].

A natural way to construct a Lie algebra is to take the set of derivations $\mathrm{Der}(U)$ of some algebra U. Indeed, the commutator $[D_1, D_2] = D_1D_2 - D_2D_1$ of two derivations is again a derivation, and $\mathrm{Der}(U)$ is a Lie algera with respect to the usual addition of linear transformations and the commutator as the multiplication.

Let A be one of the four Hurwitz algebras, and J be one of the four Jordan matrix algebras constructed from the Hurwitz algebras. Define

$$A_0 = \{a \in A \mid t(a) = 0\}, \quad J_0 = \{X \in J \mid \mathrm{tr}(X) = 0\},$$

which are the sets of elements with zero trace. These sets A_0 and J_0 will be closed under the following operations:

$$a * b = ab - \frac{1}{2}t(ab), \quad X * Y = XY - \frac{1}{3}\mathrm{tr}(XY).$$

We form the set

$$L = \mathrm{Der}(A) \oplus (A_0 \otimes J_0) \oplus \mathrm{Der}(J)$$

with the following multiplication [,]. This multiplication agrees with the ordinary commutators in $\mathrm{Der}(A)$ and $\mathrm{Der}(J)$ and satisfies $[\mathrm{Der}(A), \mathrm{Der}(J)] = 0$. Also

$$[a \otimes X, D] = aD \otimes X, \qquad [a \otimes X, E] = a \otimes XE$$

for all $D \in \mathrm{Der}(A)$, $E \in \mathrm{Der}(J)$ and $a \in A_0$, $X \in J_0$. Moreover,

$$[a \otimes X, b \otimes Y] = \frac{1}{12}\mathrm{tr}(XY)D_{a,b} + (a * b) \otimes (X * Y) + \frac{1}{2}t(ab)D_{X,Y},$$

where $a, b \in A_0$, $X, Y \in J_0$ and

$$D_{a,b} = R_{[a,b]} - L_{[a,b]} - 3[L_x, R_y], \quad D_{X,Y} = [R_X, R_Y]$$

are inner derivations of A and J, written with the help of operators of right and left multiplications $R_x \colon a \mapsto ax$, $L_x \colon a \mapsto xa$. The algebra L so constructed is a Lie algebra. This construction is known as Tits's construction [4].

If in this construction we take $A = \mathbb{O}$ and $J = \mathbb{R}$, $H_3(\mathbb{R})$, $H_3(\mathbb{C})$, $H_3(\mathbb{H})$, $H_3(\mathbb{O})$, we obtain all five exceptional simple Lie algebras:

$$\mathbf{G}_2, \ \mathbf{F}_4, \ \mathbf{E}_6, \ \mathbf{E}_7, \ \mathbf{E}_8,$$

of dimensions 14, 52, 78, 133, 248. These Lie algebras and corresponding Lie groups lead to exceptional geometries [1].

REFERENCES

[1] FREUDENTHAL, H., Lie Groups in foundation of geometry, *Adv. in Math.,* **1** (1965), 145–190.

[2] HIRZEBRUCH, F., Division algebras and topology, in "Numbers", Graduate Texts in Mathematics, Springer-Verlag, 281–302.

[3] PICKERT, G., "Projective Ebenen," Springer-Verlag, Berlin-Heidelberg-New York, 1975.

[4] SCHAFER, R.D., "An Introduction to Nonassociative Algebras," Academic Press, New York-London, 1966.

[5] ZHEVLAKOV, K.A., SLINKO, A.M., SHESTAKOV, I.P., SHIRSHOV, A.I., "Rings that are Nearly Associative," Academic Press, New York-London, 1982.