Boundedness for Multilinear Littlewood-Paley Operators on Hardy and Herz-Hardy Spaces

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(Presented by Oscar Blasco)

AMS Subject Class. (2000): 42B20, 42B25

Received December 26, 2003

1. Introduction

Let \( T \) be a Calderon-Zygmund operator, a classical result of Coifman, Rochberg and Weiss (see [7]) states that the commutator \([b, T] = T(bf) - bTf\) (where \( b \in BMO(R^n) \)) is bounded on \( L^p(R^n) \) for \( 1 < p < \infty \); Chanillo (see [2]) proves a similar result when \( T \) is replaced by the fractional integral operator. However, it was observed that \([b, T]\) is not bounded, in general, from \( H^p(R^n) \) to \( L^p(R^n) \) for \( p \leq 1 \). But, the boundedness holds if \( b \) belongs to Lipschitz spaces \( Lip_\beta(R^n) \) (see [3],[15]). This shows the difference of \( b \in BMO(R^n) \) and \( b \in Lip_\beta(R^n) \). The purpose of this paper is to prove the boundedness properties for some multilinear operators generated by Littlewood-Paley operators and Lipschitz functions on Hardy and Herz-Hardy spaces.

2. Preliminaries and results

In this paper, we will consider a class of multilinear operators related to Littlewood-Paley operators, whose definitions are following.

Let \( m \) be a positive integer and \( A \) be a function on \( R^n \). We denote

\[
R_{m+1}(A; x, y) = A(x) - \sum_{|\alpha| \leq m} \frac{1}{\alpha!} D^\alpha A(y)(x - y)^\alpha,
\]

Supported by the NNSF (Grant: 10271071).
\[ Q_{m+1}(A; x, y) = R_m(A; x, y) - \sum_{|\alpha|=m} \frac{1}{\alpha!} D^\alpha A(x - y)^\alpha \]

and \( \Gamma(x) = \{(y, t) \in R^{n+1}_+ : |x - y| < t\} \) as well as the characteristic function of \( \Gamma(x) \) by \( \chi_{\Gamma(x)} \).

Fix \( \varepsilon > 0 \) and \( \mu > 1 \). Let \( \psi \) be a fixed function which satisfies the following properties:

1. \( \int_{R^n} \psi(x)dx = 0 \),
2. \( |\psi(x)| \leq C(1 + |x|)^{-(n+1)} \),
3. \( |\psi(x + y) - \psi(x)| \leq C|y|^{\varepsilon}(1 + |x|)^{-(n+1+\varepsilon)} \) when \( 2|y| < |x| \);

The multilinear Littlewood-Paley operators are defined by

\[ g_A^\psi(f)(x) = \left( \int_0^\infty |F_t^{\psi,A}(f)(x)|^2 \frac{dt}{t} \right)^{1/2}, \]

\[ S_A^\psi(f)(x) = \left[ \int \int_{\Gamma(x)} |F_t^{\psi,A}(f)(x, y)|^2 \frac{dydt}{t^{n+1}} \right]^{1/2} \]

and

\[ g_\mu^A(f)(x) = \left[ \int \int_{R^{n+1}_+} \left( \frac{t}{t + |x - y|} \right)^{nm} |F_t^{\psi,A}(f)(x, y)|^2 \frac{dydt}{t^{n+1}} \right]^{1/2}, \]

where

\[ F_t^{\psi,A}(f)(x) = \int_{R^n} \frac{R_{m+1}(A; x, y)}{|x - y|^m} \psi_t(x - y)f(y)dy, \]

\[ F_t^{\psi,A}(f)(x, y) = \int_{R^n} \frac{R_{m+1}(A; x, z)}{|x - z|^m} f(z)\psi_t(y - z)dz, \]

and \( \psi_t(x) = t^{-n}\psi(x/t) \) for \( t > 0 \). The variants of \( g_A^\psi, S_A^\psi \) and \( g_\mu^A \) are defined by

\[ \tilde{g}_\psi^A(f)(x) = \left( \int_0^\infty |\tilde{F}_t^{\psi,A}(f)(x)|^2 \frac{dt}{t} \right)^{1/2}, \]

\[ \tilde{S}_\psi^A(f)(x) = \left[ \int \int_{\Gamma(x)} |\tilde{F}_t^{\psi,A}(f)(x, y)|^2 \frac{dydt}{t^{n+1}} \right]^{1/2}, \]

and

\[ \tilde{g}_\mu^A(f)(x) = \left[ \int \int_{R^{n+1}_+} \left( \frac{t}{t + |x - y|} \right)^{nm} |\tilde{F}_t^{\psi,A}(f)(x, y)|^2 \frac{dydt}{t^{n+1}} \right]^{1/2}, \]
where
\[
\tilde{F}_t^{\psi,A}(f)(x) = \int_{\mathbb{R}^n} \frac{Q_{m+1}(A;x,y)}{|x-y|^m} \psi_t(x-y) f(y) dy
\]
and
\[
\tilde{F}_t^{\psi,A}(f)(x,y) = \int_{\mathbb{R}^n} \frac{Q_{m+1}(A;x,z)}{|x-z|^m} \psi_t(y-z) f(z) dz.
\]
We denote that \(F_t^{\psi}(f)(y) = f * \psi_t(y)\). We also define that
\[
g_\mu(f)(x) = \left( \int_0^\infty |F_t^{\psi}(f)(x)|^2 \frac{dt}{t} \right)^{1/2},
\]
\[
S_\psi(f)(x) = \left( \int \int_{\Gamma(x)} |F_t^{\psi}(f)(y)|^2 \frac{dydt}{t^{n+1}} \right)^{1/2},
\]
and
\[
g_\mu(f)(x) = \left( \int \int_{\mathbb{R}^n+1} \left( \frac{t}{t+|x-y|} \right)^{n\mu} \left| F_t^{\psi}(f)(y) \right|^2 \frac{dydt}{t^{n+1}} \right)^{1/2},
\]
which are the Littlewood-Paley operators (see [19]). For \(S_\psi^A, \tilde{S}_\psi^A\) and \(g_\mu^A, \tilde{g}_\mu^A\), we have the following pointwise estimates (see [19, p.317]):
\[
S_\psi^A(f)(x) \leq C g_\mu^A(f)(x) \quad \text{and} \quad \tilde{S}_\psi^A(f)(x) \leq C \tilde{g}_\mu^A(f)(x).
\]
Let \(\psi = \varphi * \chi_B\), where \(B\) is a ball of \(\mathbb{R}^n\). It is easy to see that
\[
F_t^{\psi,A}(f)(x) = \frac{1}{t^n} \int_{|x-y| \leq t} F_t^{\varphi,A}(f)(x,y) dy,
\]
thus
\[
g_\mu^A(f)(x) \leq C S_\varphi^A(f)(x) \quad \text{and} \quad \tilde{g}_\mu^A(f)(x) \leq C \tilde{S}_\varphi^A(f)(x).
\]
Notice that if \(\varphi\) satisfies the properties (1),(2) and (3), then \(\psi\) also satisfies similar estimates.

Note that when \(m = 0\), \(g_\psi^A, S_\psi^A\) and \(g_\mu^A\) are just the commutator of Littlewood-Paley operators (see [1],[12],[13]), while when \(m > 0\), they are non-trivial generalizations of the commutators. It is well known that multilinear operators are of great interest in harmonic analysis and have been widely studied by many authors (see [3-6],[8],[9]). In [3] and [20], authors obtain the boundedness of multilinear singular integral operators generated by singular integrals and Lipschitz functions on \(L^p(p > 1)\) and some Hardy spaces. The
main purpose of this paper is to discuss the boundedness properties of the multilinear Littlewood-Paley operators on Hardy and Herz-Hardy spaces. Let us first introduce some definitions (see [10],[16],[17],[18]). Throughout this paper, $M(f)$ will denote the Hardy-Littlewood maximal function of $f$, $Q$ will denote a cube of $\mathbb{R}^n$ with side parallel to the axes. Denote the Hardy spaces by $H^p(\mathbb{R}^n)$. It is well known that $H^p(\mathbb{R}^n)(0 < p \leq 1)$ has the atomic decomposition characterization (see [19]). For $\beta > 0$, the Lipschitz space $\text{Lip}_\beta(\mathbb{R}^n)$ is the space of functions $f$ such that

$$||f||_{\text{Lip}_\beta} = \sup_{x, h \in \mathbb{R}^n \setminus 0} |f(x + h) - f(x)|/|h|^\beta < \infty.$$ 

Let $B_k = \{x \in \mathbb{R}^n : |x| \leq 2^k\}$ and $C_k = B_k \setminus B_{k-1}$ for $k \in \mathbb{Z}$. Denote by $\chi_k$ the characteristic function of $C_k$ and $\tilde{\chi}_k$ the characteristic function of $C_k$ for $k \geq 1$ and $\tilde{\chi}_0$ the characteristic function of $B_0$.

**Definition 1.** Let $0 < p, q < \infty, \alpha \in \mathbb{R}$.

(1) The homogeneous Herz space is defined by

$$K^{\alpha,p}_q(\mathbb{R}^n) = \{f \in L^q_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) : ||f||_{K^{\alpha,p}_q(\mathbb{R}^n)} < \infty\},$$

where

$$||f||_{K^{\alpha,p}_q} = \left[ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} ||f \chi_k||_{L^q}^p \right]^{1/p}.$$

(2) The nonhomogeneous Herz space is defined by

$$K^{\alpha,p}_q(\mathbb{R}^n) = \{f \in L^q_{\text{loc}}(\mathbb{R}^n) : ||f||_{K^{\alpha,p}_q(\mathbb{R}^n)} < \infty\},$$

where

$$||f||_{K^{\alpha,p}_q} = \left[ \sum_{k=1}^{\infty} 2^{k\alpha p} ||f \chi_k||_{L^q}^p + ||f \chi_{B_0}||_{L^q}^p \right]^{1/p}.$$

**Definition 2.** Let $\alpha \in \mathbb{R}, 0 < p, q < \infty$.

(1) The homogeneous Herz type Hardy space is defined by

$$H^{\alpha,p}_q(\mathbb{R}^n) = \{f \in S'(\mathbb{R}^n) : G(f) \in K^{\alpha,p}_q(\mathbb{R}^n)\},$$

and

$$||f||_{H^{\alpha,p}_q} = ||G(f)||_{K^{\alpha,p}_q};$$
(2) The nonhomogeneous Herz type Hardy space is defined by
\[ HK_q^{\alpha,p}(\mathbb{R}^n) = \{ f \in S'(\mathbb{R}^n) : G(f) \in K_q^{\alpha,p}(\mathbb{R}^n) \}, \]
and
\[ \|f\|_{HK_q^{\alpha,p}} = \|G(f)\|_{K_q^{\alpha,p}}; \]
where \( G(f) \) is the grand maximal function of \( f \).

The Herz type Hardy spaces have the atomic decomposition characterization.

**Definition 3.** Let \( \alpha \in \mathbb{R}, 1 < q < \infty \). A function \( a(x) \) on \( \mathbb{R}^n \) is called a central \((\alpha, q)\)-atom (or a central \((\alpha, q)\)-atom of restrict type), if
1) \( \text{Supp } a \subset B(0, r) \) for some \( r > 0 \) (or for some \( r \geq 1 \)),
2) \( \|a\|_{L^q} \leq |B(0, r)|^{-\alpha/n} \),
3) \( \int a(x)x^\gamma dx = 0 \) for \( |\gamma| \leq [\alpha - n(1 - 1/q)] \).

**Lemma 1.** (See [17]) Let \( 0 < p < \infty, 1 < q < \infty \) and \( \alpha \geq n(1 - 1/q) \). A temperate distribution \( f \) belongs to \( H\dot{K}_q^{\alpha,p}(\mathbb{R}^n) \) (or \( HK_q^{\alpha,p}(\mathbb{R}^n) \)) if and only if there exist central \((\alpha, q)\)-atoms (or central \((\alpha, q)\)-atoms of restrict type) \( a_j \) supported on \( B_j = B(0, 2^j) \) and constants \( \lambda_j, \sum_j |\lambda_j|^p < \infty \) such that \( f = \sum_{j=-\infty}^{\infty} \lambda_j a_j \) (or \( f = \sum_{j=0}^{\infty} \lambda_j a_j \)) in the \( S'(\mathbb{R}^n) \) sense, and
\[ \|f\|_{H\dot{K}_q^{\alpha,p}} \approx \left( \sum_j |\lambda_j|^p \right)^{1/p} . \]

We will prove the following theorems in Section 4.

**Theorem 1.** Let \( 0 < \beta \leq 1, \max(n/(n + \beta), n/(n + \varepsilon)) < p \leq 1 \) and \( 1/p - 1/q = \beta/n \). If \( D^\alpha A \in Lip_\beta(\mathbb{R}^n) \) for \( |\alpha| = m \). Then \( g^A_\mu \) is bounded from \( H^p(\mathbb{R}^n) \) to \( L^q(\mathbb{R}^n) \).

**Theorem 2.** Let \( 0 < \beta < \min(1, \varepsilon) \). If \( D^\alpha A \in Lip_\beta(\mathbb{R}^n) \) for \( |\alpha| = m \). Then \( \tilde{g}^A_\mu \) is bounded from \( H^{n/(n+\beta)}(\mathbb{R}^n) \) to \( L^1(\mathbb{R}^n) \).

**Theorem 3.** Let \( 0 < \beta < \min(1, \varepsilon) \). If \( D^\alpha A \in Lip_\beta(\mathbb{R}^n) \) for \( |\alpha| = m \). Then \( g^A_\mu \) is bounded from \( H^{n/(n+\beta)}(\mathbb{R}^n) \) to weak \( L^1(\mathbb{R}^n) \).
**Theorem 4.** Let $0 < \beta \leq 1$, $0 < p < \infty$, $1 < q_1, q_2 < \infty$, $1/q_1 - 1/q_2 = \beta/n$ and $n(1 - 1/q_1) \leq \alpha < \min(n(1 - 1/q_1) + \beta, n(1 - 1/q_1) + \varepsilon)$. If $D^{\alpha}A \in Lip_{\beta}(R^n)$ for $|\alpha| = m$. Then $g^A_\mu$ is bounded from $H^{\alpha,p}_{q_1}(R^n)$ to $\dot{K}^{\alpha,p}_{q_2}(R^n)$.

**Remark 1.** By the pointwise estimates of $g^A_\psi$, $S^A_\psi$ and $g^A_{\mu}$ (or $\tilde{g}^A_\psi$, $\tilde{S}^A_\psi$ and $\tilde{g}^A_{\mu}$), Theorem 1, 2, 3 and 4 also hold for $g^A_\psi$ and $S^A_\psi$ (or $\tilde{g}^A_\psi$ and $\tilde{S}^A_\psi$).

**Remark 2.** Theorem 4 also hold for the nonhomogeneous Herz type Hardy space.

### 3. Some lemmas

We begin with some preliminary lemmas.

**Lemma 2.** (See [6]) Let $A$ be a function on $R^n$ and $D^{\alpha}A \in L^q(R^n)$ for $|\alpha| = m$ and some $q > n$. Then

$$|R_m(A; x, y)| \leq C|x - y|^m \sum_{|\alpha| = m} \left( \frac{1}{\dot{Q}(x, y)} \int_{\dot{Q}(x, y)} |D^{\alpha}A(z)|^q dz \right)^{1/q},$$

where $\dot{Q}(x, y)$ is the cube centered at $x$ and having side length $5\sqrt{n}|x - y|$.

**Lemma 3.** (See [3, p.418, Theorem 2.3]) Let $T^A$ be the multilinear operators defined by

$$T^A(f)(x) = \int_{R^n} \frac{R_{m+1}(A; x, y)}{|x - y|^{n+m}} f(y) dy.$$ 

If $0 < \beta < 1$, $1 < p < n/\beta$, $1/q = 1/p - \beta/n$ and $D^{\alpha}A \in Lip_{\beta}(R^n)$ for $|\alpha| = m$. Then $T^A$ is bounded from $L^p(R^n)$ to $L^q(R^n)$, that is

$$||T^A(f)||_{L^q} \leq C||f||_{L^p}.$$

**Lemma 4.** Let $0 < \beta \leq 1$, $1 < p < n/\beta$, $1/q = 1/p - \beta/n$ and $D^{\alpha}A \in Lip_{\beta}(R^n)$ for $|\alpha| = m$. Then $g^A_\psi$, $S^A_\psi$ and $g^A_{\mu}$ are all bounded from $L^p(R^n)$ to $L^q(R^n)$.

**Proof.** By the pointwise estimates of $g^A_\psi$, $S^A_\psi$ and $g^A_{\mu}$, we only need to give the proof of $g^A_\mu$. Note that

$$\int_0^\infty \frac{tdt}{(t + |x - z|)^{2n+2}} = C|x - z|^{-2n}$$
and
\[
 t^{-n} \int_{\mathbb{R}^n} \left( \frac{t}{t + |x - y|} \right)^{n\mu} \frac{dy}{(t + |y - z|)^{2n+2}} \leq CM \left( \frac{1}{(t + |x - z|)^{2n+2}} \right) \leq C \left( \frac{1}{(t + |x - z|)^{2n+2}} \right),
\]
by using Minkowski’s inequality and the condition of \( \psi \), we obtain
\[
g^A_{\mu}(f)(x) \leq C \int_{\mathbb{R}^n} \frac{|f(z)||R_{m+1}(A; x, z)|}{|x - z|^m} \cdot \left[ \int_0^\infty \left( t^{-n} \int_{\mathbb{R}^n} \left( \frac{t}{t + |x - y|} \right)^{n\mu} \frac{dy}{(t + |y - z|)^{2n+2}} \right)^{1/2} dt \right]^{1/2} dz \leq C \int_{\mathbb{R}^n} \frac{|f(z)||R_{m+1}(A; x, z)|}{|x - z|^m+n} dz.
\]
Thus, the lemma follows from Lemma 3.

4. Proofs of theorems

Proof of Theorem 1. It suffices to show that there exists a constant \( C > 0 \) such that for every \( H^p \)-atom \( a \),
\[
 ||g^A_{\mu}(a)||_{L^q} \leq C.
\]
Let \( a \) be a \( H^p \)-atom, that is that \( a \) supported on a cube \( Q = Q(x_0, r) \), \( ||a||_{L^\infty} \leq |Q|^{-1/p} \) and \( \int a(x)x^\gamma dx = 0 \) for \( |\gamma| \leq [n(1/p - 1)] \). We write
\[
 \int_{\mathbb{R}^n} [g^A_{\mu}(a)(x)]^q dx = \left( \int_{2Q} + \int_{(2Q)^c} \right) [g^A_{\mu}(a)(x)]^q dx = I_1 + I_2.
\]
For \( I_1 \), taking \( 1 < p_1 < n/\beta \) and \( q_1 \) such that \( 1/p_1 - 1/q_1 = \beta/n \), by Hölder’s inequality and the \( (L^{p_1}, L^{q_1}) \)-boundedness of \( g^A_{\mu} \) (see Lemma 4), we get
\[
 I_1 \leq C ||g^A_{\mu}(a)||_{L^{q_1}}^q |2Q|^{1-q/q_1} \leq C ||a||_{L^{p_1}}^q |Q|^{1-q/q_1} \leq C.
\]
To obtain the estimate of \( I_2 \), we need to estimate \( g^A_{\mu}(a)(x) \) for \( x \in (2Q)^c \). Let \( \bar{Q} = 5\sqrt{n}Q \) and \( \bar{A}(x) = A(x) - \sum_{|\alpha| = m} \frac{1}{\alpha!} (D^\alpha A)_{\bar{Q}} x^\alpha \). Then \( R_m(A; x, y) = \)
\( R_m(\tilde{A}; x, y) \) and \( D^\alpha \tilde{A}(y) = D^\alpha A(y) - (D^\alpha A)_Q \). We write, by the vanishing moment of \( a \),

\[
F_t^{\psi, A}(a)(x, y) = \int_{R^n} \left[ \frac{\psi_t(y - z)R_m(\tilde{A}; x, z)}{|x - z|^m} - \frac{\psi_t(y - x_0)R_m(\tilde{A}; x, x_0)}{|x - x_0|^m} \right] dz
\]

\[
- \sum_{|\alpha|=m} \frac{1}{\alpha!} \int_{R^n} \frac{\psi_t(y - z)(x - z)^\alpha D^\alpha \tilde{A}(z)}{|x - z|^m} a(z) dz.
\]

By Lemma 2 and the following inequality, for \( b \in Lip_\beta(R^n) \),

\[
|b(x) - b_Q| \leq \frac{1}{|Q|} \int_{Q} ||b||_{Lip_\beta} |x - y|^\beta dy \leq ||b||_{Lip_\beta} (|x - x_0| + r)^\beta,
\]

we get

\[
|R_m(\tilde{A}; x, y)| \leq \sum_{|\alpha|=m} ||D^\alpha A||_{Lip_\beta} (|x - y| + r)^{m+\beta}.
\]

On the other hand, by the formula (see [6]):

\[
R_m(\tilde{A}; x, y) - R_m(\tilde{A}; x, x_0) = \sum_{|\eta|<m} \frac{1}{\eta!} R_{m-|\eta|}(D^\eta \tilde{A}; x_0, y)(x - x_0)^\eta
\]

and note that \( |x-y| \sim |x-x_0| \) for \( y \in Q \) and \( x \in R^n \setminus Q \), we obtain, similar to the proof of Lemma 4,

\[
g^A_\mu(a)(x) \leq C \sum_{|\alpha|=m} ||D^\alpha A||_{Lip_\beta} \int \left[ \frac{|y - x_0|}{|x - x_0|^{n+1-\beta}} + \frac{|y - x_0|^\beta}{|x - x_0|^{n+\varepsilon - \beta}} \right]
\]

\[
+ \sum_{|\eta|<m} \frac{|y - x_0|^{m+\beta - |\eta|}}{|x - x_0|^{n+m-|\eta|}} + \frac{|y - x_0|^\beta}{|x - x_0|^n} |a(y)| dy
\]

\[
\leq C \sum_{|\alpha|=m} ||D^\alpha A||_{Lip_\beta} \left[ \frac{|Q|^{\beta/n+1 - 1/p}}{|x - x_0|^n} + \frac{|Q|^\beta/n+1 - 1/p}{|x - x_0|^{n+\varepsilon - \beta}} \right].
\]
Thus,
\[
I_2 \leq \sum_{k=1}^{\infty} \int_{2^{k+1}Q \setminus 2^k Q} |T^A(a)(x)|^q dx
\]
\[
\leq C \left( \sum_{|\alpha|=m} ||D^\alpha A||_{\text{Lip}_\beta} \right)^q \sum_{k=1}^{\infty} \left[ 2^{kqn(1/p-(n+\beta)/n)} + 2^{kqn(1/p-(n+\epsilon)/n)} \right]
\]
\[
\leq C \left( \sum_{|\alpha|=m} ||D^\alpha A||_{\text{Lip}_\beta} \right)^q,
\]
which together with the estimate for \(I\) yields the desired result. This finishes the proof of Theorem 1.

From Theorem 1, we get

**Corollary.** Let \(0 < \beta \leq 1\). If \(D^\alpha A \in \text{Lip}_\beta(R^n)\) for \(|\alpha| = m\). Then \(g^A, S^A_\psi\) and \(g^A_\mu\) are all bounded from \(L^{n/(n+\beta)}(R^n)\) to \(\text{BMO}(R^n)\).

**Proof of Theorem 2.** It suffices to show that there exists a constant \(C > 0\) such that for every \(H^{n/(n+\beta)}\)-atom \(a\) supported on \(Q = Q(x_0, r)\), we have
\[
||\tilde{g}^A_\mu(a)||_{L^1} \leq C.
\]
We write
\[
\int_{R^n} \tilde{g}^A_\mu(a)(x) dx = \left[ \int_{2Q} + \int_{(2Q)^c} \right] \tilde{g}^A_\mu(a)(x) dx := J_1 + J_2.
\]
For \(J_1\), by the following equality
\[
Q_{m+1}(A; x, z) = R_{m+1}(A; x, z) - \sum_{|\alpha|=m} \frac{1}{\alpha!} (x-z)^\alpha (D^\alpha A(x) - D^\alpha A(z)),
\]
we have, similar to the proof of Lemma 4,
\[
\tilde{g}^A_\mu(a)(x) \leq g^A_\mu(a)(x) + C \sum_{|\alpha|=m} \int_{R^n} \frac{|D^\alpha A(x) - D^\alpha A(y)|}{|x-y|^n} |a(y)| dy,
\]
thus, \(\tilde{g}^A_\mu\) is \((L^p, L^q)\)-bounded by Lemma 4 and [1],[2], where \(n/\beta > p > 1\) and 
\[
1/q = 1/p - \beta/n.\]
We see that
\[
J_1 \leq C ||\tilde{g}^A_\mu(a)||_{L^q}|2Q|^{1-1/q} \leq C ||a||_{L^p}|Q|^{1-1/q} \leq C|Q|^{1+1/p-1/q-(n+\beta)/n} \leq C.
\]
To obtain the estimate of \( J_2 \), we denote that \( \tilde{A}(x) = A(x) - \sum_{|\alpha|=m} \frac{1}{\alpha!} (D^\alpha A)_{2B} x^\alpha \). Then \( Q_m(A; x, y) = Q_m(\tilde{A}; x, y) \). We write, by the vanishing moment of \( a \) and \( Q_{m+1}(A; x, y) = R_m(A; x, y) - \sum_{|\alpha|=m} \frac{1}{\alpha!} (x - y)^\alpha D^\alpha A(x) \), for \( x \in (2Q)^c \),

\[
\tilde{F}_t^\psi,A(a)(x, y) = \int_{\mathbb{R}^n} \frac{\psi_t(y - z)R_m(\tilde{A}; x, z)}{|x - z|^m} a(z)dz - \sum_{|\alpha|=m} \frac{1}{\alpha!} \int_{\mathbb{R}^n} \frac{\psi_t(y - z)D^\alpha \tilde{A}(z)(x - z)^\alpha}{|x - z|^m} a(z)dz \\
= \int_{\mathbb{R}^n} \left[ \psi_t(y - z)R_m(\tilde{A}; x, z) - \frac{\psi_t(x - x_0)R_m(\tilde{A}; x, x_0)}{|x - x_0|^m} \right] a(z)dz \\
- \sum_{|\alpha|=m} \frac{1}{\alpha!} \int_{\mathbb{R}^n} \left[ \frac{\psi_t(y - z)(x - z)^\alpha}{|x - z|^m} - \frac{\psi_t(x - x_0)(x - x_0)^\alpha}{|x - x_0|^m} \right] D^\alpha \tilde{A}(z)a(z)dz,
\]

thus, similar to the proof of Theorem 1, we obtain, for \( x \in (2Q)^c \)

\[
|\tilde{g}_t^A(a)(x)| \leq C|Q|^{-\beta/n} \sum_{|\alpha|=m} ||D^\alpha A||_{Lip_\beta} \left( \frac{|Q|^{1/n}}{|x - x_0|^{n+1-\beta}} + \frac{|Q|^{\varepsilon/n}}{|x - x_0|^{n+\varepsilon-\beta}} \right) \\
+ C|Q|^{-\beta/n} \sum_{|\alpha|=m} ||D^\alpha \tilde{A}(x)|| \left( \frac{|Q|^{1/n}}{|x - x_0|^{n+1}} + \frac{|Q|^{\varepsilon/n}}{|x - x_0|^{n+\varepsilon}} \right),
\]

so that,

\[
J_2 \leq C \sum_{|\alpha|=m} ||D^\alpha A||_{Lip_\beta} \sum_{k=1}^\infty [2^{k(\beta-1)} + 2^{k(\beta-\varepsilon)}] \leq C,
\]

which together with the estimate for \( J_1 \) yields the desired result. This finishes the proof of Theorem 2. \( \blacksquare \)

**Proof of Theorem 3.** By the following equality

\[
R_{m+1}(A; x, z) = Q_{m+1}(A; x, z) + \sum_{|\alpha|=m} \frac{1}{\alpha!} (x - z)^\alpha (D^\alpha A(x) - D^\alpha A(z))
\]

and similar to the proof of Lemma 4, we get

\[
g_\mu^A(f)(x) \leq \tilde{g}_\mu^A(f)(x) + C \sum_{|\alpha|=m} \int_{\mathbb{R}^n} \frac{|D^\alpha A(x) - D^\alpha A(z)|}{|x - z|^n} |f(z)|dz,
\]
from Theorem 1, 2 and [15], we obtain

\[ |\{ x \in \mathbb{R}^n : g_\mu^A(f)(x) > \lambda \} | \]
\[ \leq |\{ x \in \mathbb{R}^n : \tilde{g}_\mu^A(f)(x) > \lambda/2 \} | \]
\[ + \left| \left\{ x \in \mathbb{R}^n : \sum_{|\alpha|=m} \int_{\mathbb{R}^n} \frac{|D^\alpha A(x) - D^\alpha A(z)|}{|x-z|^n} |f(z)|dz > C\lambda \right\} \right| \]
\[ \leq C\lambda^{-1} \| f \|_{H^{n/(n+\beta)}}. \]

This completes the proof of Theorem 3. \[ \square \]

**Proof of Theorem 4.** Let \( f \in H\dot{K}_1^{\alpha,p}(\mathbb{R}^n) \) and \( f(x) = \sum_{j=-\infty}^{\infty} \lambda_j a_j(x) \) be the atomic decomposition for \( f \) as in Lemma 1. We write

\[ \|g_\mu^A(f)\|_{\dot{K}_2^{\alpha,p}}^p \leq C \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left( \sum_{j=-\infty}^{\infty} \lambda_j \|g_\mu^A(a_j)\chi_k\|_{L^q_2} \right)^p \]
\[ + C \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left( \sum_{j=k-2}^{\infty} \lambda_j \|g_\mu^A(a_j)\chi_k\|_{L^q_2} \right)^p \]
\[ = L_1 + L_2. \]

For \( L_2 \), by the \((L^q, L^{q_2})\) boundedness of \( g_\mu^A \) (see Lemma 4), we get

\[ L_2 \leq C \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left( \sum_{j=k-2}^{\infty} \lambda_j \|a_j\|_{L^{q_1}} \right)^p \]
\[ \leq \left\{ \begin{array}{l}
    C \sum_{j=-\infty}^{\infty} |\lambda_j|^p \left( \sum_{k=-\infty}^{j+2} 2^{(k-j)\alpha p/2} \right)^{p/p'}, 0 < p \leq 1 \\
    C \sum_{j=-\infty}^{\infty} |\lambda_j|^p \left( \sum_{k=-\infty}^{j+2} 2^{(k-j)\alpha p/2} \right)^{(j+2)/(j+2)} \left( \sum_{k=-\infty}^{j+2} 2^{(k-j)\alpha p/2} \right)^{p/p'}, p > 1 \\
\end{array} \right. \]
\[ \leq C \sum_{j=-\infty}^{\infty} |\lambda_j|^p \leq C \| f \|^p_{H\dot{K}_1^{\alpha,p}}. \]

For \( L_1 \), similar to the proof of Theorem 1, we have, for \( x \in C_k, j \leq k-3, \)

\[ g_\mu^A(a_j)(x) \leq C \left( \frac{|B_j|^{\beta/n}}{|x|^n} + \frac{|B_j|^{\varepsilon/n}}{|x|^{n+\varepsilon-\beta}} \right) \int |a_j(y)|dy \]
\[ \leq C (2^{j(\beta+n(1-1/n)-\alpha)}|x|^{-n} + 2^{j(\varepsilon+n(1-1/n)-\alpha)}|x|^\beta-n-\varepsilon), \]
thus
\[ ||g^A_{\alpha}(a_j)\xi_k||_{L^p} \leq C 2^{-k\alpha} (2^{(j-k)(\beta+n(1/q_1)-\alpha)} + 2^{(j-k)(\epsilon+n(1/q_1)-\alpha)}) \]

and
\[ L_1 \leq C \sum_{k=-\infty}^{\infty} \left( \sum_{j=-\infty}^{k-3} \left| \lambda_j \right| (2^{(j-k)(\beta+n(1/q_1)-\alpha)} + 2^{(j-k)(1/2+n(1/q_1)-\alpha)}) \right)^p \]
\[ \leq \left\{ \begin{array}{ll}
C \sum_{j=-\infty}^{\infty} \left| \lambda_j \right|^p \sum_{k=j+3}^{\infty} \left( 2^{(j-k)(\beta+n(1/q_1)-\alpha)} + 2^{(j-k)(\gamma+n(1/q_1)-\alpha)} \right)^p, & 0 < p \leq 1 \\
C \sum_{j=-\infty}^{\infty} \left| \lambda_j \right|^p \left[ \sum_{k=j+3}^{\infty} \left( 2^{(j-k)p(\beta+n(1/q_1)-\alpha)/2} + 2^{(j-k)(\gamma+n(1/q_1)-\alpha)/2} \right) \right], & p > 1
\end{array} \right.
\]
\[ \leq C \sum_{j=-\infty}^{\infty} \left| \lambda_j \right|^p \leq C ||f||_{HK^{\alpha,p}}^p. \]

This completes the proof of Theorem 4.

ACKNOWLEDGEMENTS

The author would like to express his deep gratitude to the referee for his valuable comments and suggestions.

REFERENCES

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