An Example of a Banach Space $V$ such that all the Degree $d \geq 2$ Hypersurfaces of $P(V)$ are Singular

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1. THE EXAMPLE

For any complex Banach space let $P(V)$ denote the projective space of all one-dimensional linear subspaces of $V$. For any integer $d \geq 1$ let $P^d(V)$ be the set of all continuous degree $d$ complex valued homogeneous polynomials on $V$. By definition a degree $d$ hypersurface of $P(V)$ is the zero-locus of some $f \in P^d(V)$, $f \neq 0$. Hence a degree one hypersurface is just a closed hyperplane. Here we want to show the existence of a Banach space $V$ (not separable) such that for every integer $d \geq 2$ there is no smooth degree $d$ hypersurface of $P(V)$. We do not have any example of a separable Banach space with the same property. Smooth hypersurfaces are important, because one hopes to use analytic tools on them ([3]). This example shows that the situation for Banach projective spaces is dramatically different from the situation for finite-dimensional projective spaces, in which Bertini’s theorem states that a general choice of a finite number of polynomial equations define a nonsingular set. The same example gives the non-existence of non-linear smooth complete intersection with finite codimension. This example is not new. We just extracted it from the literature ([2] and [1, Prop. 8]) and proved that it is interesting also from this new point of view.

EXAMPLE. Fix an uncountable discrete set $A$ and let $C_0(A)$ be the Banach space of all complex valued functions on $A$ which vanish at infinity, with the supremum norm. For any open subset $U$ of $C_0(A)$ every holomorphic function

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on $U$ depends only from a countable number of variables ([2] or [1, Prop. 8]). Hence every continuous homogeneous degree $d$ polynomial $f$ on $C_0(A)$, $f \neq 0$, depends only from countably many variables. Hence the degree $d$ hypersurface $\{f = 0\} \subset P(C_0(A))$ is a cone with as vertex $W$ a projective space $P(B)$ with uncountable algebraic dimension. If $d \geq 2$ every point of $W$ is a singular point of the hypersurface $\{f = 0\}$. Since $B$ has a supplement isomorphic to $C_0(A')$ with $A'$ countable, it is easy to see that for every integer $s \geq 1$ and any continuous homogeneous polynomials $f_i$ on $V$, $1 \leq i \leq s$, $f_i \neq 0$, $\deg(f_i) \geq 2$ for all $i$, the closed analytic subset $\{f_1 = \cdots = f_s = 0\}$ of $P(C_0(A))$ is a cone with infinite-dimensional vertex and in particular it is not smooth.

REFERENCES

