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Some Generalizations of Kadison's Theorem: A Survey

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1. Introduction

Let H be a complex Hilbert space and let $\mathcal{K}(H)$ and $\mathcal{L}(H)$ denote respectively the space of compact and bounded linear operators on H. A well known result of Kadison [16] (see also Chapter 6 of [9]) describes surjective isometries of these spaces as $T \to UTV$ or $T \to UT^{tr}V$, where U and V are unitaries and ^{tr} denotes the Banach space adjoint of an operator via the identification of H with H^*. For general Banach spaces X and Y, in this article we will consider various interpretations of the above result in order to completely describe surjective isometries of $\mathcal{K}(X,Y)$ and $\mathcal{L}(X,Y)$. Clearly for surjective isometries $V \in \mathcal{L}(X)$ and $U \in \mathcal{L}(Y)$, $T \to UTV$ is a surjective isometry of $\mathcal{L}(X,Y)$ leaving compacts invariant. Such isometries we shall call as standard isometries. Note that if $U : X \to Y^*$ and $V : X^* \to Y$ are surjective isometries then $T \to VT^*U$ is an isometry of $\mathcal{K}(X,Y)$ and also of $\mathcal{L}(X,Y)$. We assume throughout this note that X is not isometric to Y^* or X^* is not isometric to Y. This hypothesis is missing from the statement of the theorems in [18] and [33] .

In general isometries of $\mathcal{K}(X,Y)$ or $\mathcal{L}(X,Y)$ need not be of the above form (see examples below). In this article we will focus only on the following three variations of the Kadison's theorem. We recall that $\mathcal{L}(H)$ is the bidual of $\mathcal{K}(H)$ and under the canonical embedding the above isometry on $\mathcal{L}(H)$ is the bi-transpose of the isometry on $\mathcal{K}(H)$. And thus any onto isometry of $\mathcal{L}(H)$ is the bi-transpose of an isometry of $\mathcal{K}(H)$.

1. When are the isometries of $\mathcal{K}(X,Y)$ and $\mathcal{L}(X,Y)$ describable in the standard form?

2. When the isometries are not of the above form is there a 'matching' description of isometries of $\mathcal{K}(X,Y)$ and $\mathcal{L}(X,Y)$? In such a case does the isometry leave the space of compact operators invariant?
3. Suppose the bidual of $\mathcal{K}(X,Y)$ is canonically identified as a space of operators, when are the isometries of the latter bi-transpose of isometries of $\mathcal{K}(X,Y)$?

In the first part of the paper we consider these questions for $Y = C(K)$ for some compact set K. We note that even though in general there is no exact analogue of Kadison's theorem, under some additional hypothesis on X and K, the space of compact operators does form an invariant subspace for the group of isometries on $\mathcal{L}(X, C(K))$. By considering an M-ideal condition we obtain some exact analogues of the Kadison's theorem. Our results extend a recent description of surjective isometries of $\mathcal{L}(X,Y)$ for certain pairs of Banach spaces given in [18]. Versions of Kadison's theorem for other operator ideals can be found in [36]. Some of the results of this paper, taken in conjugation with the results from Section 4 of [37] can also be applied to other operator ideals.

In the third section of the paper as an application of our results we show that for a metrizable compact set K and for certain uniformly smooth spaces X the range of a 'local surjective isometry' on $\mathcal{L}(X, C(K))$ contains all compact operators. For a Banach space X which is an M-ideal in its bidual (we always consider X as canonically embedded in its bidual), we show that algebraic reflexivity of the group of isometries of X implies the algebraic reflexivity of the group of isometries of the bidual X^{**}. As a consequence we get a new geometric proof of the algebraic reflexivity of $\mathcal{G}(\mathcal{L}(\ell^2))$, first proved in [24]. These sections cover some parts of [33] and [34].

In the last section of the paper we initiate the study of nice surjections. Let $\Phi : E \to E$ be a linear map such that for all $e^* \in \partial_e E_1^*$, $e^* \circ \Phi \in \partial_e E_1^*$. This hypothesis implies, $\|\Phi\| = 1$. We say that Φ^* preserves the extreme points. Such operators are called nice operators. See [22] for a description of nice operators on C^*-algebras and some function spaces. Their description indicates the similarities with the Theorem of Kadison. Our investigation is also motivated by the recent work [13] where the authors studied nice isomorphisms on certain function spaces. For a compact convex set K, let $A(K)$ denote the space of affine continuous functions equipped with the supremum norm. We give a complete description of nice surjections of $A(K)$ when K is a Choquet simplex. We give some partial answers to the algebraic reflexivity of this class of operators.

For a Banach space X by X_1 we denote the closed unit ball, by $S(X)$ the unit sphere and by $\mathcal{G}(X)$ the group of isometries. $\partial_e K$ denotes the set of extreme points. We assume that K is canonically embedded in $C(K)^*(A(K)^*)$ when these spaces have the weak*-topology. We refer to the monographs [1] and [2] for results on convexity theory and Choquet simplexes. Let Γ denote the unit circle. We assume that all Banach spaces under consideration are infinite dimensional.

2. Main results

We first consider the case when $Y = C(K)$ for a compact set K. Here the surjective isometries of Y are given by the classical Banach-Stone theorem in terms of homeomorphisms of K and extreme points (unitaries) of $C(K)_1$. We assume that K is homeomorhically identified with the set of Dirac measures in $C(K)_1^*$ equipped with the weak*-topology. It is well known that the space $\mathcal{K}(X, C(K))$ via the map $T \to T^*|K$ is an onto isometry of this space with $C(K, X^*)$, the space of X^*-valued continuous maps on K, equipped with the supremum norm. Thus the description of surjective isometries of $\mathcal{K}(X, C(K))$ is given by the study of vector-valued Banach-Stone theorems [4]. Note that for any $\phi : K \to \mathcal{G}(X^*)$ that is continuous when $\mathcal{G}(X^*)$ is equipped with the strong operator topology, $f \to \phi \circ f$ is a surjective isometry of $C(K, X^*)$ (where $(\phi \circ f)(k) = \phi(k)(f(k))$. Now we are ready to give an example where the isometries of $\mathcal{K}(X, C(K))$ are not of the standard form.

Example 1. Let X be any Banach space such that there is an isometry V of X^* that is not weak*-continuous. For example when $X = C(K')$ for an infinite set K', then for any measurable function g on K such that $|g| \equiv 1$ that is not continuous, $\mu \to g\mu$ is an onto isometry of $C(K)^*$ that is not weak*-continuous. Now $f \to V \circ f$ is a surjective isometry of $C(K, X^*)$ that is not in the standard form. If it were, then by the Banach-Stone theorem, there exists a homeomorphism ψ of K, a $U \in \mathcal{G}(X)$ and a unitary g in $C(K)$such that, $V(f(k)) = g(k)U^*(f(k))$ for all $k \in K$ and $f \in C(K, X^*)$. Thus for any $x^* \in X^*$, by taking the constant function x^*, we see that $V(x^*) = g(k)U^*(x^*)$ for all k. Hence V is weak*-continuous. A contradiction.

However even when all elements of $\mathcal{G}(X^*)$ are weak*-continuous (for example when X is reflexive or a predual of a von Neumann algebra) by taking a non-constant function $\rho : K \to \mathcal{G}(X^*)$ that is continuous w.r.t the strong operator topology we can again produce isometries that are not of standard form.

Let X be a Banach space such that the centralizer $Z(X^*)$ of X^* (see [4] for the definition) is trivial. It follows from Theorem 8.10 in [4] that any surjective isometry Ψ of $C(K,X^*)$ is given by $\Psi(f)(k) = \rho(k)(f(\psi(k)))$ for $f \in C(K,X^*)$ and $k \in K$. Here ψ is a homeomorphism of K and ρ is as above.

We now recall that $\mathcal{L}(X,C(K))$ can be identified with $W^*C(K,X^*)$ the space of X^*-valued functions on K that are continuous when X^* has the weak*-topology, equipped with the supremum norm via the same transformation $T \to T^*|K$. Thus motivated by our second question one can ask if the above description of isometries of $C(K,X^*)$ will also yield a complete description of isometries of $W^*C(K,X^*)$? Clearly in general composition with ρ need not give raise to weak* continuous functions. Thus it is more reasonable to consider situations where all isometries of $W^*C(K,X^*)$ leave $C(K,X^*)$ invariant. A positive solution was given by [7], Theorem 4. We recall that a Banach space has the Namioka-Phelps property if weak* and norm topologies coincide on $S(X^*)$.

THEOREM 2. *Let K be a compact first countable space and suppose X^* has the Namioka-Phelps property then any surjective isometry of $W^*C(K,X^*)$ has a form identical to that of a surjective isometry of $C(K,X^*)$ and hence leaves $C(K,X^*)$ invariant.*

Proof. Let Φ be a surjective isometry. It was proved in [30] that for spaces with the Namioka-Phelps property $Z(X^*)$ is trivial. Thus it follows from Theorem 4 of [7] that there exists a homeomorphism ϕ of K and a $\rho : K \to \mathcal{G}(X^*)$ that is continuous when $\mathcal{G}(X^*)$ has the strong operator topology, such that $\Phi(f)(k) = \rho(k)(f(\phi(k))$ for $k \in K$ and $f \in W^*C(K,X^*)$. Thus $\Phi(C(K,X^*)) \subset C(K,X^*)$. ∎

Remark 3. It is worth recalling that $\mathcal{K}(\ell^2)$ has the Namioka-Phelps property [21] and any surjective isometry of the dual is weak*-continuous.

The following proposition shows that the condition 'X is not isometric to Y^* or Y is not isometric to X^*' that we had imposed is automatically satisfied in the case of $Y = C(K)$. Since we are dealing with only infinite dimensional spaces, the failing of the condition implies that $C(K)$ is isometric to its bidual. That this fails for an infinite K is probably a folk lore result, but as we are unable to give an exact reference we present below its proof which is based on M-structure theory. It can also be deduced from a more general result on von Neumann algebras as noted in the remark below.

PROPOSITION 4. *For an infinite set K, $C(K)$ is not isometric to its bidual.*

Proof. Suppose $C(K)$ is isometric to its bidual. By a well known theorem of Dixmier and Grothendieck (see [20] Theorems 10, 11 in Chapter 3) this assumption implies that K is a hyperstonean space and $C(K)^*$ is the unique predual of $C(K)$. It is well known that isolated points of K correspond precisely to M-summands of dimension one. Let S denote the set of isolated points of K which is non-empty since $C(K)$ is a dual space. Then $K = \beta(S) \cup K'$ where both the sets are clopen and disjoint. Now as $C(K)^*$ is the dual of $C(K)$, any point of K corresponds to a L-ideal of dimension one in $C(K)^*$ and thus gives rise to a M-summand of dimension one in $C(K)$ which in turn gives an isolated point of K. Hence we get in particular $|\beta(S)| = |S|$ which is a contradiction since by Theorem 9.2 in [10] for an infinite discrete set S, $|\beta(S)| = 2^{2^{|S|}}$. ∎

Remark 5. It is perhaps a folk lore result in von Neumann algebra theory that an infinite dimensional von Neumann algebra is not isometric to its bidual. For the sake of completeness we note here a proof using a recent result from [35]. Let $V = W^*$ be a von Neumann algebra. If V^{**} is isometric to V, we get that V has preduals of all order, this contradicts the result from [35].

We next consider the third question mentioned above. We begin by considering the case of $\mathcal{K}(X, C(K)) = C(K, X^*)$. It is well known that $C(K, X^*) = C(K) \otimes_\epsilon X^*$ (see [8] Chapter VIII for matters relating to tensor products, we let ϵ, π stand for the injective and projective tensor products respectively). Let X be a reflexive space. Then $(C(K) \otimes_\epsilon X^*)^* = C(K)^* \otimes_\pi X$. Thus $\mathcal{K}(X, C(K))^{**} = \mathcal{L}(X, C(K)^{**})$. Thus the bidual of $\mathcal{K}(X, C(K))$ consists of $C(K)^{**}$-valued operators. We recall that $C(K)^{**} = C(K')$ for some compact hyperstonean space K'. Since we are considering infinite dimensional spaces, K' is not first countable thus the isometries are not covered by the description given in the above theorem. When K is not a dispersed space it can be seen that K' is not -purely atomic in the sense defined in [6]. The best available description of the isometries of $W^*C(K', X^*)$ is given by the following theorem from [6] adapted to our set up.

THEOREM 6. *Let X be a separable reflexive Banach space and K' a not purely atomic hyperstonean space. Let Φ be a surjective isometry of $W^*C(K', X)$. There exists a homeomorphism ϕ of K' and a dense open set $O \subset K'$ and a $\rho : O \to \mathcal{G}(X)$ that is continuous w.r.t strong operator topology such that $\Phi(f)(k) = \rho(k)(f(\phi(k)))$ for all $k \in O$.*

We next consider situations that yield a positive answer to the 3rd question raised here.

We recall from [12] Chapter I that a closed subspace $M \subset X$ is an M-ideal if there is a projection $P : X^* \to X^*$ such that $ker(P) = M^\perp$ and $\|x^*\| = \|P(x^*)\| + \|x^* - P(x^*)\|$ for all $x^* \in X^*$. It is well known that $\mathcal{K}(\ell^2)$ is an M-ideal in $\mathcal{L}(\ell^2)$ and more generally $\mathcal{K}(\ell^p, \ell^q)$ is an M-ideal in $\mathcal{L}(\ell^p, \ell^q)$ for $1 < p \leq q < \infty$ (Example VI.4.1 in [12]). Also for $1 < q < p < \infty$, $\mathcal{L}(\ell^p, \ell^q) = \mathcal{K}(\ell^p, \ell^q)$. We refer to Chapter VI of [12], [37], [17] and [19] for several examples of pairs of Banach space X, Y for which $\mathcal{K}(X, Y)$ is an M-ideal in $\mathcal{L}(X, Y)$ from among classical function spaces. The basic idea we would like to use in this set up is Proposition III.2.2 from [12] which states that if X is an M-ideal in its bidual then any surjective isometry of X^{**} is the bi-transpose of an isometry of X. Thus what we are considering is a very natural generalization of Kadison's theorem.

Remark 7. It follows from Corollary 2.5 of [23] that in the infinite dimensional case $\mathcal{K}(X, C(K))$ is not an M-ideal in $\mathcal{L}(X, C(K))$. Thus this set up is different from the ones we have considered earlier.

Remark 8. When X is a non-reflexive M-embedded space, it is not isometric to its bidual because X being a proper M-ideal it is not isometric to any dual space by Corollary II.3.6 in [12].

First we shall describe conditions when $\mathcal{K}(X, Y)$ is an M-ideal in its bidual and the bidual can be realized as $\mathcal{L}(X, Y^{**})$.

It is known that being an M ideal in the bidual (we always consider the canonical embedding of a space in its bidual) is a hereditary property and if a dual space is an M-ideal in its bidual then it is reflexive (see [12] Chapter III). Thus since X^* and Y embed in $\mathcal{K}(X, Y)$ we get that if $\mathcal{K}(X, Y)$ is a M-ideal in its bidual then X is reflexive and Y is an M-ideal in its bidual. We also note from Chapter III of [12] that Y^* has the Radon-Nikodym property.

Proposition 9. *Let $\mathcal{K}(X, Y)$ be an M-ideal in its bidual. Suppose X^* or Y has the metric approximation property. Then $\mathcal{L}(X, Y^{**})$ is the bidual of $\mathcal{K}(X, Y)$ and $\mathcal{G}(\mathcal{L}(X, Y^{**}))$ leaves $\mathcal{K}(X, Y)$ invariant. In particular when Y is reflexive $\mathcal{L}(X, Y)$ is the bidual of $\mathcal{K}(X, Y)$.*

Proof. As noted above the hypothesis implies that X is reflexive and Y is an M-ideal in its bidual. Our assumption of metric approximation property ensures that $\mathcal{K}(X, Y) = X^* \otimes_\epsilon Y$. Thus $\mathcal{K}(X, Y)^{**} = (X^* \otimes_\epsilon Y)^{**} = (X \otimes_\pi$

$Y^*)^* = \mathcal{L}(X, Y^{**})$. Now the M-ideal condition along with proposition III.2.2 in [12] implies that $\mathcal{K}(X, Y)$ is an invariant subspace for $\mathcal{G}(\mathcal{L}(X, Y^{**}))$. ∎

Remark 10. Since under the above hypothesis $\mathcal{K}(X, Y) \subset \mathcal{L}(X, Y) \subset \mathcal{L}(X, Y^{**})$ we have in particular $\mathcal{K}(X, Y)$ is an M-ideal in $\mathcal{L}(X, Y)$.

Remark 11. To see a concrete application, let β_0 be the space of analytic functions f on the unit disk such that $\|f\|_\beta = \sup(1 - |z|^2)|f(z)| < \infty$ and $\lim_{|z|\to 1}(1 - |z|^2)f(z) = 0$, the so called little Bloch space. The Bloch space β consists of those analytic functions without the limit condition. See Section 4.4 of [9] for a description of isometries of the little Bloch space. Here every isometry is surjective. It is known that β is the bidual of β_0. It follows from Corollary 4.9 in [17] that for any Banach space X, $\mathcal{K}(X, \beta_0)$ is an M-ideal in $\mathcal{L}(X, \beta)$. Thus when X is reflexive we have that any surjective isometry of $\mathcal{L}(X, \beta)$ leaves the compacts invariant.

Our next results extends Theorem 1.2 of [18], see also Theorem 1 of [14] and [33].

THEOREM 12. *Suppose X^* or Y has the metric approximation property and $\mathcal{K}(X, Y)$ is an M-ideal in its bidual $\mathcal{L}(X, Y^{**})$. Assume further that X and Y^* are strictly convex. Then the isometries of $\mathcal{K}(X, Y)$ and $\mathcal{L}(X, Y^{**})$ have the standard form.*

Proof. We first note that we only need to show that the isometries of $\mathcal{K}(X, Y)$ are of the standard form. From the M-ideal condition we have that any surjective isometry of $\mathcal{L}(X, Y^{**})$ is the bi-transpose of an isometry of $\mathcal{K}(X, Y)$. Since the bidual identification we have set up is exactly same as in the case of the Hilbert space, we conclude that the isometries of $\mathcal{L}(X, Y^{**})$ are also standard.

By our assumption we have $\mathcal{K}(X, Y) = X^* \otimes_\epsilon Y$. Thus we only need to indicate the modifications needed in the proof of Theorem 1.1 in [18]. Note that since Y is an M-ideal in its bidual, any surjective isometry of Y^{**} is the bi-transpose of an isometry of Y and in particular any isometry of Y^* is weak*-continuous. Thus in the proof of Step III in Theorem 1.1 the operator T_2 defined on Y^* is weak* continuous. Also Y^* has the Radon-Nikodym property. Hence the conclusion follows as in the proof of Theorem 1.1 taking into account the correct form of remarks made after the proof of Theorem 1.1 in [18]. ∎

Remark 13. Note that the M-ideal condition was mainly needed to show that the isometries of $\mathcal{L}(X, Y^{**})$ are of the standard form. Thus if one assumes that X is reflexive, Y is an M-ideal in its bidual, both X and Y^* are strictly convex with one of them having the metric approximation property then the surjective isometries of $\mathcal{K}(X, Y)$ are of the standard form. In this context it may be worth noting that any space Y that is an M-ideal in its bidual can be renormed so that Y^* is strictly convex and in the new norm Y is still an M-ideal in its bidual (see Theorem III,4.6 e) in [12]).

Remark 14. Part of the motivation for getting the isometries in the standard form is the possibility of using the concrete description of isometries of the component spaces that may be available. As an illustration we note that it follows from Proposition 6.6 in [17] that for the Schatten class C_p for $2 \leq p < \infty$, $\mathcal{K}(\ell^p, C_p)$ is an M-ideal in $\mathcal{L}(\ell^p, C_p)$. See [9] for a description of the isometries of the component spaces.

3. An application

Let $\Phi : X \to X$ be a linear map. Φ is said to be a local surjective isometry if for every $x \in X$ there exists a $\Psi_x \in \mathcal{G}(X)$ such that $\Phi(x) = \Psi_x(x)$. An interesting question is for what Banach spaces X, such a Φ is always surjective. This property is also known as algebraic reflexivity of $\mathcal{G}(X)$. We refer to [27] Chapter 3 for a very comprehensive account of this problem and its variations. Here we consider only the complex scalar field. It was shown in [26] that for a first countable compact set K, $\mathcal{G}(C(K))$ is algebraically reflexive. These questions were considered for the space $C(K, X)$ in [15]. Even when the algebraic reflexivity can not be decided it may be possible to prove surjectivity of Φ under some additional hypothesis on Φ. Taking this approach in [31] it was proved that in the case of affine continuous functions on a metrizable Choquet simplex , if in addition Φ^* preserves extreme points of the dual unit ball then it is surjective. In the following proposition even though we can not show that Φ is onto we can at least show that the range is a large space. For $1 < p \neq 2 < \infty$, ℓ^p satisfies the hypothesis of the following proposition [5].

Proposition 15. *Let K be a first countable compact Hausdorff space and let X be a unifromly smooth Banach space such that $\mathcal{G}(X^*)$ is algebraically reflexive. Let $\Phi : \mathcal{L}(X, C(K)) \to \mathcal{L}(X, C(K))$ be a local surjective isometry. Then range(Φ) contains all compact operators.*

Proof. Since X^* is uniformly convex, it has the Namioka-Phelps property thus it follows from Theorem 2 that the restriction of any surjective isometry of $\mathcal{L}(X, C(K))$ is a surjective isometry of $\mathcal{K}(X, C(K))$. Therefore by our hypothesis Φ is a local surjective isometry on $\mathcal{K}(X, C(K))$. Since $\mathcal{G}(X^*)$ is algebraically reflexive it follows from Theorem 7 in [15] that $\mathcal{G}(\mathcal{K}(X, C(K))$ is algebraically reflexive. Therefore Φ is surjective on $\mathcal{K}(X, C(K))$. ∎

In the following theorem we once again use the identification of $\mathcal{K}(X, C(K))$ with $C(K, X^*)$ and $\mathcal{L}(X, C(K))$ with $W^*C(K, X^*)$.

THEOREM 16. *Let K be a metric space and X a uniformly smooth space such that $\mathcal{G}(X^*)$ is algebraically reflexive. Let Φ be a local surjective isometry of $W^*C(K, X^*)$. For any $f \in W^*C(K, X^*)$ there exists a sequence $\{f_n\}_{n \geq 1} \subset C(K, X^*)$ such that $\Phi(f_n)(k) \to f(k)$ for all $k \in K$.*

Proof. Let $\Phi : W^*C(K, X^*) \to W^*C(K, X^*)$ be a local surjective isometry. As before by Theorem 2 we have that $\Phi|C(K, X^*)$ is a local surjective isometry. From Theorem 7 in [15] we have that $\Phi|C(K, X^*)$ is surjective and again by Theorem 2, there exists a homeomorphism ϕ and a weight function ρ such that $\Phi(f)(k) = \rho(k)(f(\phi(k))$ for all $k \in K$ and for $f \in C(K, X^*)$.

Now let $f \in W^*C(K, X^*)$. Since K is a metric space and X^* is reflexive, it follows from the results in [3] (see also [32]) that there exists a sequence $\{g_n\}_{n \geq 1} \subset C(K, X^*)$ such that $g_n(k) \to f(k)$ for every $k \in K$. Let $f_n(k) = \rho^{-1}(k)(g_n(\phi^{-1}(k))$. Then $\{f_n\}_{n \geq 1} \subset C(K, X^*)$. We know that $\Phi(f_n)(k) = \rho(k)(f_n(\phi(k)))$ for all n and k. Thus $\Phi(f_n)(k) = \rho(k)(f_n(\phi(k))) = g_n(k) \to f(k)$. ∎

The methods used in [15] for proving the algebraic reflexivity of $C(K, X)$ rely among other things on the availability of a complete description of extreme points of the dual unit ball of $C(K, X)$. No such description is available of the extreme points of the dual unit ball of $W^*C(K, X^*)$. See page 267 of [12] for an example of an extreme point of the dual unit ball of $\mathcal{K}(c_0, C(K))$ that is not extreme in the dual unit ball of $\mathcal{L}(c_0, C(K))$. See also Section 4 for more information on the additional condition assumed on the Φ below.

THEOREM 17. *Let K be a first countable space and let X be as in the above theorem. Suppose in addition Φ^* preserves extreme points of the dual unit ball (i.e, a nice operator) and X is also uniformly convex. Then Φ is a $C(K)$-module map in the sense that there is an onto homeomorphism ϕ of K*

such that $\Phi(gf)(k) = g(\phi(k))\Phi(f)(k)$ *for* $g \in C(K)$, $f \in W^*C(K, X^*)$ *and* $k \in K$.

Proof. As in the previous theorem we get the structure of $\Phi|C(K, X^*)$.

Let $f \in W^*C(K, X^*)$, $g \in C(K)$ and $k \in k$. We will verify the module identity at a unit vector x_0. Let $\delta(k) \otimes x_0$ denote the functional defined $(\delta(k) \otimes x_0)(F) = F(k)(x_0)$. Since x_0 is an extreme point it is well known that $\delta(k) \otimes x_0$ is an extreme point of the unit ball of $C(K, X^*)^*$. It follows from Theorem 0.2 in [29] that as X is uniformly reflexive, x_0 is also a denting point and hence $\delta(k) \otimes x_0$ is an extreme points of the unit ball of $(W^*C(K, X^*))^*$.

Note by the structure of $\Phi|C(K, X^*)$, $\Phi^*(\delta(k) \otimes x_0) = \delta(\phi(k)) \otimes \rho(k)(x_0)$. Now by our hypothesis $\Phi^*(\delta(k) \otimes x_0)$ is an extreme point of the unit ball of $W^*C(K, X^*)$. Note that since $\{\delta(k) \otimes x : k \in K, \ \|x\| = 1\}$ is a norming set for $W^*C(K, X^*)$, the unit ball of $W^*C(K, X^*)^*$ is the weak* closed convex hull of $\{\delta(k) \otimes x : k \in K, \ \|x\| = 1\}$. Since $\Phi^*(\delta(k) \otimes x_0)$ is an extreme point by Milman's converse of the Krein-Milman theorem, we get a net of unit vectors $\{x_\alpha\}$ and a net $k_\alpha \subset K$ such that $\delta(k_\alpha) \otimes x_\alpha \to \Phi(\delta(k) \otimes x_0)$ in the weak* topology of $W^*C(K, X^*)$. We assume w.l.o.g that $k_\alpha \to k'$.

Note that if $h \in C(K)$ and $F \in C(K, X^*)$ then $h(\phi(k))F(k)(\rho(k)(x_0) = \Phi^*(\delta(k) \otimes x_0)(hF) = \lim h(k_\alpha)(\delta(k_\alpha) \otimes x_0)(F) = h(k')\Phi^*(\delta(k) \otimes x_0)(F) = h(k')F(k)(\rho(k)(x_0))$. Therefore we have $\phi(k) = k'$. Finally $\Phi^*(\delta(k) \otimes x_0)(gf) = \lim(\delta(k_\alpha) \otimes x_\alpha)(gf) = g(\phi(k))\Phi^*(\delta(k) \otimes x_0)(f)$. ∎

Using the ideas from the previous section, our next result establishes the relationship between the algebraic reflexivity of an M-embedded space and its bidual. We recall from [25] that $\mathcal{G}(\mathcal{K}(H))$ is algebraically reflexive. Also $\mathcal{K}(H)$ being a closed two sided ideal is an M-ideal in its bidual $\mathcal{L}(H)$.

Proposition 18. *For an M-embedded space X, if $\mathcal{G}(X)$ is algebraically reflexive then so is $\mathcal{G}(X^{**})$.*

Proof. Let $\Phi : X^{**} \to X^{**}$ be a local isometry. Since any element of $\mathcal{G}(X^{**})$ is the bi-transpose of an element of $\mathcal{G}(X)$, clearly $\Phi|X$ is a local isometry. Thus by our hypothesis $\Phi|X$ is onto. Again using the M-embeddness of X it follows from *(iii)* of Lemma III.2.4 of [12] that $((\Phi)|X)^{**} = \Phi$. In particular Φ is onto. ∎

Remark 19. We recall from [9] that any into isometry of the little Bloch space β_0 is surjective. Thus $\mathcal{G}(\beta_0)$ is algebraically reflexive. As this is an M-embedded space we get that for its bidual, the Bloch space, $\mathcal{G}(\beta)$ is also algebraically reflexive. Similarly since $\mathcal{G}(c_0)$ is algebraically reflexive and as it is an M-ideal in its bidual ℓ^∞, we get a different proof of the algebraic reflexivity of $\mathcal{G}(\ell^\infty)$, see [31], [5] and [25].

Remark 20. The above result gives an alternative proof of the algebraic reflexivity of $\mathcal{G}(\mathcal{L}(\ell^2))$ established in [24]. It was shown in [5] that for any Banach space with a symmetric basis which is not isomorphic to ℓ^2, $\mathcal{G}(X)$ is algebraically reflexive. [37] contains several examples of function spaces X with symmetric basis which are M-embedded. Thus in all these cases $\mathcal{G}(X^{**})$ is again algebraically reflexive.

Remark 21. We recall that $\mathcal{G}(X)$ is said to be topologically reflexive, if a linear map Φ is such that $\phi(x) \in \overline{\mathcal{G}(X)(x)}$ (here $\mathcal{G}(X)(x)$ denotes the orbit at x and the closure is taken in the strong operator topology) for all $x \in X$, is in $\mathcal{G}(X)$. Clearly Φ is an into isometry. Our arguments also show that for a M-embedded space X, if $\mathcal{G}(X)$ is topologically reflexive then so is $\mathcal{G}(X^{**})$. Since $\mathcal{G}(\mathcal{L}(\ell^2))$ is topologically reflexive but $\mathcal{G}(\mathcal{K}(\ell^2))$ is not topologically reflexive (see [24]) we get that the converse does not hold for the M-embedded space $\mathcal{K}(\ell^2)$. We also note from [25] that the group of isometries of $\mathcal{K}(\ell^2)^*$, the space of trace class operators, is algebraically reflexive.

Remark 22. See [5] for examples to show that in general algebraic reflexivity of $\mathcal{G}(X)$ and that of $\mathcal{G}(X^*)$ are not related.

4. Nice surjections

In this section we study the class $M = \{\text{all nice surjections}\}$. Note that any surjective isometry is in M. Also note that if a $T \in M$ is one-one and $T^{-1} \in M$ then T is an isometry. Thus this is a more general class than the earlier ones.

Our first result is an easy Proposition (perhaps folklore) that describes the class M when the underlying space is $C(X)$.

PROPOSITION 23. *Let X be a compact Hausdorff space. Let $\phi : X \to X$ be a continuous one-one map. Then $\Phi : C(X) \to C(X)$ defined by $\Phi(f) = \tau f \circ \phi$, where $\tau \in \partial_e C(X)_1$, is a linear surjection whose adjoint preserves the extreme points. Conversely any surjective such map is of the above form*

Proof. We only need to check the surjectivity. Given $g \in C(X)$, consider

$g \circ \phi^{-1} : \phi(X) \to \mathbb{C}$ extend this by Tietzs extension theorem to a $h \in C(X)$. Now $\Phi(h) = h \circ \phi = g$.

For the converse we first note that Φ maps extreme points of $C(X)_1$ to extreme points. Thus by multiplying with $\overline{\Phi(1)}$ if necessary, we can assume w.l.o.g that $\Phi(1) = 1$. Thus ϕ is the restriction of Φ^* to the Dirac measures and $\Phi(f) = f \circ \phi$. That ϕ is one-to-one follows from surjectivity of Φ. ∎

The proof of the following theorem proceeds along the same lines as the results on isometries from [25]. However instead of using the Russo-Dye theorem as was done in [25] we prefer to use the well known Gleason-Kahane-Żelazko Theorem (GKZ).

Theorem 24. *Let X be a first countable compact space. M is algebraically reflexive.*

Proof. Let Φ be in the algebraic closure of M. From the above proposition it follows that we can again assume w.l.o.g that $\Phi(1) = 1$. Thus for any $x \in X$, $\Phi^*(\delta(x))$ is a probability measure. Let $N = ker\Phi^*(\delta(x)) = \{f \in C(X) : \Phi(f)(x) = 0\}$. As Φ is in the algebraic closure of M, keeping in view the description of M we see that every element of N has a zero in X and hence is noninvertible. Therefore by the GKZ Theorem it follows that $N = ker\delta(x')$ for some $x' \in X$. We thus get a function $\phi : X \to X$ such that $\Phi(f) = f \circ \phi$. The continuity of ϕ is an easy consequence of the weak*-continuity of Φ^*. If $\phi(x) = \phi(y)$ let $f \in C(X)$ be such that $0 \leq f \leq 1$ and $f^{-1}(0) = \{\phi(x)\}$. By the above proposition again we have that $\Phi(f) = \tau f \circ \psi = f \circ \phi$ for some weight function τ and a continuous one-one map $\psi : X \to X$. Now $0 = f(\phi(x)) = f(\phi(y)) = f(\psi(x)) = f(\psi(y))$. Thus as ψ is one-one we get that $x = y$. Therefore we get that ϕ is one-one and Φ is onto. ∎

Next we are interested in formulating and proving similar results for $A(K)$ for a metrizable Choquet simplex K. It is well known that for any simplex K with $\partial_e K$ closed, the space $A(K)$ is isometric to $C(\partial_e K)$. Thus what follows is an extension of the preceding set-up. Since K is a simplex it follows from [28] that any weight function $\tau \in \partial_e A(K)_1$ is in the center of $A(K)$ (see Section II.7 in [3]). Thus for any $a \in A(K)$ there exists a $b \in A(K)$ such that $\tau(k)a(k) = b(k)$ for all $k \in \partial_e K$. In what follows this is the interpretation of 'multiplication' in $A(K)$. These results are applicable to either scalar field.

Theorem 25. *Let K be a Choquet simplex. Suppose $\phi : K \to K$ is a one-one continuous affine map such that $\phi(\partial_e K) \subset \partial_e K$. Let $\tau \in \partial_e A(K)_1$. Define*

$\Phi : A(K) \to A(K)$ *by* $\Phi(a) = \tau a \circ \phi$. *Then* Φ *is a nice surjection.Conversely any nice surjection* Φ *of* $A(K)$ *is of this form.*

Proof. We recall that $\partial_e A(K)_1^* = \{t\epsilon(k) : k \in \partial_e K,\ t \in \Gamma\}$. Since K is a simplex, we have $\tau(\partial_e K) \subset \Gamma$. Thus we see that Φ^* preserves extreme points. Since $\phi(K) \subset K$ is a closed convex with $\partial_e \phi(K) = \phi(\partial_e K) \subset \partial_e K$, we get from Lemma 3.1.6 in [2] that $\phi(K)$ is a face of K. Now for any $a \in A(K)$, let $b = a \circ \phi^{-1} \in A(\phi(K))$. Since K is a simplex by Theorem II.6.22 of [1] we have that $\phi(K)$ is a split face of K and thus by Theorem II.6.15 of [1] we have that there is a $c \in A(K)$ such that $c = b$ on $\phi(K)$. It is easy to see that $\phi(\overline{\tau} c) = a$. Therefore Φ is onto.

Conversely since Φ^* preserves extreme points it is easy to see that $|\Phi(1)(\partial_e K)| = 1$. Since K is a simplex it follows from [28] that the weight function $\tau = \Phi(1)$ is in the center of $A(K)$ (see Section II.7 in [3]). Thus by 'multiplying' with $\overline{\Phi(1)}$ if necessary we may assume that $\Phi(1) = 1$.

Let $\phi = \Phi^*|K$. We now have that ϕ is weak*-continuous, affine, $\phi(\partial_e K) \subset \partial_e K$ and $\Phi(a) = a \circ \phi$. We next claim that ϕ is one-one. If $\phi(x) = \phi(y)$ for some $x \neq y \in K$, there exists a $a \in A(K)$, with $a(x) \neq a(y)$. Now since Φ is onto, let $\Phi(b) = b \circ \phi = a$ so that $b(\phi(x)) = b(\phi(y)) = a(x) = a(y)$. Therefore $x = y$. ∎

Corollary 26. *Let* K *be a metrizable Choquet simplex. Let* Φ *be in the algebraic closure of* M. *Assume further that* Φ *is a nice operator. Then* $\Phi \in M$.

Proof. In view of the above theorem and the discussion preceding it, it is easy to see that Φ maps $\partial_e A(K)_1$ to itself. Thus we assume w.l.o.g that $\Phi(1) = 1$. Now since Φ is a nice operator, it follows from the proof of the above theorem that there exists an extreme point preserving affine continuous map $\phi : K \to K$ such that$\Phi(a) = a \circ \phi$ for all $a \in A(K)$. That ϕ is one-one follows from arguments similar to those given during the proof of Theorem in [31]. Therefore $\Phi \in M$. ∎

Remark 27. It is still an open question (even for the group of isometries) if the additional assumption of niceness on Φ can be removed.

For a Banach space E let $A(K, E)$ denote the space of E-valued affine continuous functions on K, equipped with the supremum norm. We now assume that the scalar field is real. When K is a Choquet simplex, using the canonical identification of the injective tensor product space $A(K) \otimes_\epsilon E$

with $A(K,E)$, it is possible to formulate vector-valued analogues of the above results. In this situation it is easy to see that for a closed face F of K, any element of $A(F,E)$ has an extension to $A(K,E)$. If one further assumes that $Z(E)$, the centralizer of E is trivial (see [4] Chapter 3), since $A(K)$ has a centralizer-norming system (see Example 4 on page 154 of [4]) one gets from the remarks on page 129 of [4] that $Z(A(K,E)) = Z(A(K) \otimes_\epsilon E)$ is the norm closure of $Z(A(K)) \otimes Z(E)$. Therefore one has that $Z(A(K,E)) = \{M_a : a \in Z(A(K))\}$, where $M_a : A(K,E) \to A(K,E)$ is defined by $M_a(b)(k) = a(k)b(k)$ for $k \in \partial_e K$. Now note that when $|a| = 1$ on $\partial_e K$, one has that M_a is an isometry. The following proposition is now easy to prove.

Proposition 28. *Let K be a simplex. Let $\phi : K \to K$ be an affine one-one map such that $\phi(\partial_e K) \subset K$. Let $a \in \partial_e A(K)_1$. Define $\Phi : A(K,E) \to A(K,E)$ by $\Phi(b)(k) = a(\phi(k))b(\phi(k))$ for $k \in \partial_e K$. Then Φ is a nice surjection.*

Remark 29. We do not know how to formulate the result with an operator-valued weight function. Since we are interested in formulating a necessary and sufficient condition it may be reasonable to assume that E^* is strictly convex (in this case $Z(E)$ is trivial, see Section 6 of [15]). Thus nice operators coincides with the class of co-isometries. Now to achieve surjectivity if one were also to assume invertibility then the situation reduces to the case of weight function which takes isometries as values.

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