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L-sets and the Pełczyński-Pitt Theorem

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1. Introduction and statement of results

The classical Pitt's lemma [32] asserts that $\mathcal{L}(\ell_p, \ell_q) = \mathcal{K}(\ell_p, \ell_q)$ for $1 \leq q < p < +\infty$ (for $p \leq q$ the canonical inclusion ℓ_p en ℓ_q is not compact). Pełczyński [31] extends this result to N-linear forms to show that the norm $\|\cdot\|$ cannot be uniformly approximated by polynomials in a class of spaces that includes the ℓ_p for $1 < p < +\infty$ and c_0. However, the core of Pełczyński's ideas seems to be the so-called τ_α continuity (see below) of multilinear forms, and from that their weak sequential continuity.

Several authors have obtained different extensions of the results of Pitt and Pełczyński: Emmanuele [17], Aron-Globenik [4], Gonzalo-Jaramillo [22], Dimant-Zalduendo [14], Alencar-Floret [2], Ausekle-Oja [6]; and there are many other papers dealing with different aspects of the result [11, 15, 16, 29, 34, 35, 36]... Apparently, the most general form of the result, from now on called Pełczyński-Pitt theorem, was obtained by Alencar and Floret in [2], and it connects the three main topics involved in the problem:

Proposition 1. *Let $1 \leq p_i, q < +\infty$. The following are equivalent:*

(1) *Every N-linear map $\ell_{p_1} \times \cdots \times \ell_{p_N} \to \ell_q$ is sequentially weak to norm continuous.*
(2) *Every N-linear map $\ell_{p_1} \times \cdots \times \ell_{p_N} \to \ell_q$ is compact.*
(3) $\frac{1}{p_1} + \cdots + \frac{1}{p_N} < \frac{1}{q}$.
(4) *The space of multilinear forms $\mathcal{L}^N(\ell_{p_1}, \cdots, \ell_{p_N}; \ell_q)$ is reflexive.*

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The weak-to-norm sequential continuity of multilinear forms is in fact the common point in most of the previous papers, although not always explicitly considered. Moreover, the weak-to-norm sequential continuity of multilinear forms is connected with several problems in Banach space theory such as:

- The impossibility of uniformly approximating the norm by polynomials (Pełczyński [31], see also Kurzwell [27]).
- The existence of bases of monomials in spaces of polynomials or multilinear forms (Alencar [1], Dimant-Zalduendo [14], Dimant-Dineen [13] and Ryan [34]).
- Interpolation with polynomials in infinite dimensional spaces -namely, given a bounded sequence $(x_n) \in X$ and $(a_n) \in \ell_\infty$ does there exist a polynomial $P \in \mathcal{P}(^m X)$ such that $P(x_n) = a_n$? (Valdivia [35] and Dineen [15, 16]). See also Aron-Globevnik [4] and Gómez-Jaramillo [19] for other types of interpolation.
- Several approximation problems in infinite dimension (Aron-Prolla [5] and Llavona [30]).
- The embedding of ℓ_∞ in spaces of polynomials or multilinear forms (Dineen [16] y [36]).
- The reflexivity of spaces of polynomials and operators (Alencar [1], Gonzalo-Jaramillo [22], see also [2, 6, 14, 18, 34, 35]).

The weak-strong continuity of N-linear mappings was studied in [2] rediscovering Pełcyński's notion of τ_α-convergence. Recall that for $0 \leq \alpha \leq 1$ a sequence (x_n) is sad to be τ_α-convergent to 0 if there is a constant $C > 0$ such that for all finite subsets $B \subset \mathbb{N}$ one has

$$\Big\| \sum_{n \in B} x_n \Big\| \leq C |B|^\alpha .$$

A different tool introduced by Gonzalo [20] were the lower index $l(X)$ and the upper index $u(X)$ of a space X: A sequence (x_n) is said to admit an upper p-estimate (resp. a lower p-estimate) if for some constant C and all finite sequences of scalars (r_n) one has $\|\sum r_n x_n\| \leq C \|(r_n)\|_{l_p}$ (resp. $\geq$). A Banach space X is said to admit an upper p-estimate (resp. a lower q-estimate) if every normalized weakly null sequence contains a subsequence admitting an upper p-estimate (resp. a lower q-estimate). The definitions of the indices $l(X)$ and $u(X)$ are as follows:

$$l(X) = \sup\{p \geq 1 : X \text{ admits upper } p\text{-estimates}\}$$

$$u(X) = \inf\{q \leq +\infty : X \text{ admits lower } q\text{-estimates}\}.$$

The relationships between the τ_α-convergence, upper and lower index can be seen in [2, 21]. Several authors, such as Gonzalo-Jaramillo [22], Dimant-Zalduendo [14] and others [21, 23, 35] use the lower and upper estimates to study the weak-to-norm continuity of N-linear forms. Finally, other authors [6, 11, 26, 29] use implicitly in their papers these notions.

Let us denote by $\mathcal{L}^N(X)$ the space of all N-linear forms on X, by $\mathcal{L}^N_{wsc}(X)$ the space of all weakly sequentially continuous N-linear forms on X and by $\mathcal{K}^N(X)$ the space of all compact N-linear forms on X. Recall that a multilinear form is said to be weakly sequentially continuous when it transforms weakly Cauchy sequences into convergent sequences; and compact if the image of the unit ball is a relatively compact set. Probably the basic fact connecting estimates and weakly sequentially continuous polynomials is: If X admits an upper p-estimate then $\mathcal{L}^N(X) = \mathcal{L}^N_{wsc}(X)$, for all $N < p$ ([2] or [22]). Nevertheless, the following example was obtained in [8, Theorem 4.1].

Example. There exists a space X such that

$$\mathcal{L}^2(X) = \mathcal{L}^2_{wsc}(X) = \mathcal{K}(X, X^*)$$

and X admits no upper 2-estimate. Moreover, $\mathcal{L}^2(X^*) = \mathcal{L}^2_{wsc}(X^*)$ and X^* does not admit an upper or lower 2-estimate.

Thus, neither the upper or lower indices, nor the τ_α-convergence give necessary conditions to have the weak-strong continuity of N-linear maps. Consequently, to obtain a characterization one needs to look for something else.

A close inspection of the papers of Pitt and Pełczyński, Alencar-Floret [2], Dimant-Zalduendo [14], Gonzalo-Jaramillo [22] and [6, 11, 29], shows that the results there obtained just using the estimates of certain sequences inside the ambient space (it is, on the other hand, clear that if U is a subspace of X then $l(U) \geq l(X)$ and $u(U) \leq u(X)$); and this "heredity assumption" is precisely what fails in the previous example. In other words, if one attempts to give a characterization involving the indices of the space, one must necessarily take into account the subspaces. This is what we will do, obtaining Theorem 1. This result is the most natural, and maybe general, form of the Pełcyński-Pitt theorem. We obtain in passing a unified treatment of the results of several authors: Alencar-Floret [2], Ausekle-Oja [6], Defant-López Molina-Rivera [11, 29].

Throughout the paper $X_1, \ldots, X_N, Y$ shall denote Banach spaces. The space of all N-linear forms defined on $X_1 \times \cdots \times X_N$ with values in Y shall be denoted $\mathcal{L}^N(X_1, \ldots, X_N; Y)$; the space of weakly sequentially continuous N-linear forms shall be denoted $\mathcal{L}_{wsc}{}^N(X_1, \ldots, X_N; Y)$, while the space of compact N-linear forms shall be denoted $\mathcal{K}_{wsc}{}^N(X_1, \ldots, X_N; Y)$. We shall indicate the absence of an space by putting it inside brackets; so, $\mathcal{L}^{N-1}(X_1, \ldots, [X_j], \ldots, X_N; Y)$ denotes the space of all $N-1$-linear forms defined on $X_1 \times \cdots \times X_{j-1} \times X_{j+1} \times \cdots \times X_N$ with values in Y. An element $A \in \mathcal{L}^N(X_1, \ldots, X_N; Y)$ is said to be compact if the image of the unit ball is a relatively compact set. The space of all compact Y-valued N-linear forms shall be denoted by $\mathcal{L}^N(X_1, \ldots, X_N; Y)$. When $Y = \mathbb{R}$ we simply write $\mathcal{L}^N(X_1, \ldots, X_N)$ instead of $\mathcal{L}^N(X_1, \ldots, X_N; \mathbb{R})$.

Theorem 1. *Let $X_1, \cdots, X_N$ be Banach spaces not containing ℓ_1. The following are equivalent.*

(1) *For all subspaces $U_i \subset X_i$ and all $1 \leq j \leq N$*

$$\mathcal{L}^{N-1}(U_1, \ldots, [U_j], \ldots, U_N; U_j^*) = \mathcal{K}^{N-1}(U_1, \ldots, [U_j], \ldots, U_N; U_j^*)$$

(2) *For all subspaces $U_i \subset X_i$*

$$\mathcal{L}^N(U_1, \ldots, U_N) = \mathcal{L}^N_{wsc}(U_1, \ldots, U_N)$$

(3)

$$\frac{1}{l(X_1)} + \cdots + \frac{1}{l(X_N)} < 1.$$

(4) *For all choices of subspaces $U_i \subset X_i$ the space $U_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi U_N$ does not contain ℓ_1.*

If, moreover, the spaces are reflexive then conditions (1)-(4) *are also equivalent to:*

(5) *For all subspaces or quotients U_i of X_i*

$$\mathcal{L}^{N-1}(U_1, \ldots, [U_j], \ldots, U_N; U_j^*) = \mathcal{K}^{N-1}(U_1, \ldots, [U_j], \ldots, U_N; U_j^*)$$

(6) *For all choices of subspaces or quotients U_i of X_i the space $\mathcal{L}^N(U_1, \ldots, U_N)$ is reflexive.*

It is clear that if U is a subspace of X then $l(U) \geq l(X)$ and $u(U) \leq u(X)$. Thus, condition (3) is equivalent to

(3') *For all subspaces U_i of X_i one has*

$$\frac{1}{l(U_1)} + \cdots + \frac{1}{l(U_N)} < 1.$$

Let us observe that the class of spaces to which Theorem 1 applies is certainly not empty; assuming for simplicity $X_1 = \cdots = X_N$ then condition (2) means that X must be what was called in [8] an "hereditarily $\mathcal{M}_N$-space" (see next section). Examples of such spaces are all the Banach spaces with the hereditary Dunford-Pettis property and not containing l_1; the Lorentz sequences spaces $d(w,p)$ for $N < p$ (which include, of course, the ℓ_p spaces, and thus Theorem 1 contains the result of Alencar and Floret); certain Orlicz sequence spaces, Tsirelson's original space, James's space, Tsirelson-James spaces, as well as their vector sums.

The next polynomial version of the previous result are immediate once one is aware, after Gonzalo's work [20, 21], that the estimates control the behaviour of multilinear forms as well as that of polynomials.

PROPOSITION 2. *Let X be Banach space not containing ℓ_1. The following are equivalent.*

(1) *For all subspaces $U \subset X$, and all $1 \leq j \leq N$*

$$\mathcal{P}(^{N-1}U; U^*) = \mathcal{P}_{\mathcal{K}}(^{N-1}U; U^*)$$

(2) *For all subspace $U \subset X$, $\mathcal{P}(^N U) = \mathcal{P}_{wsc}(^N U)$.*

(3) $N < l(X)$.

(4) *For all subspace $U \subset X$ the space $U \widehat{\otimes}_{s,\pi} \cdots \widehat{\otimes}_{s,\pi} U$ does not contain ℓ_1.*

If, moreover, the spaces are reflexive then conditions (1)-(4) *are also equivalent to:*

(5) *For all subspace or quotient U of X the space $\mathcal{P}(^N U)$ is reflexive*

It is perhaps worth to remark the problem mentioned in [2]: if the reflexivity of the N-fold tensor product of E is equivalent to the reflexivity of N-fold symmetric tensor product. The previous results yield:

Corollary 1. *The statements* (1) *and* (2) *are equivalent; the statements* (3) *and* (4) *are equivalent.*

(1) *For all closed subspaces U of E the space $\oplus_\pi^N U$ is reflexive.*
(2) *For all closed subspaces U of E the space $\oplus_{\pi,s}^N U$ is reflexive.*
(3) *For all closed subspaces U of E the space $\oplus_\pi^N U$ does not contain ℓ_1.*
(4) *For all closed subspaces U of E the space $\oplus_{\pi,s}^N U$ does not contain ℓ_1.*

To prove Theorem 1 we will first reduce the problem to the study of the space $\mathcal{L}_{wsc}^N(X)$ of weakly sequentially continuous N-linear forms on X

Proposition 3. *Let $X_1, \dots, X_N$ be Banach spaces not containing ℓ_1. The following are equivalent:*

(1) *For all $j \in \{1, \dots, N\}$ one has*

$$\mathcal{L}^{N-1}(X_1, \dots, [X_j], \dots, X_N; X_j^*) = \mathcal{K}^{N-1}(X_1, \dots, [X_j], \dots, X_N; X_j^*)$$

(2) $\mathcal{L}^N(X_1, \dots, X_N) = \mathcal{L}_{wsc}^N(X_1, \dots, X_N)$.

A couple of remarks before passing to the proof of Proposition 3 and Theorem 1. About the inductive statement, let us recall that Jiménez and Payá [26] showed that there exist Banach spaces such that all $N-1$ linear, but not all N-linear, forms between them are compact. Second, that the restriction "not containing ℓ_1" is in some sense necessary if one is trying to characterize when vector valued multilinear forms on Banach spaces are compact. Indeed, as we show next, when a Banach space X contains ℓ_1, for each infinite dimensional separable Banach space Y there exists an homogeneous polynomial $P : X \to Y$ of degree 2 which is a surjection (in particular, it is not compact): since X contains ℓ_1 then there exists a quotient map $q : X \to \ell_2$ such that $q(e_{2n}) = e_n$. Let (y_n) be a dense sequence in the unit ball of Y. The continuous bilinear form $B : \ell_2 \times \ell_2 \to Y$ given by $B(\sum_n \lambda_n e_n, \sum_n \mu_n e_n) = \sum_n \lambda_n \mu_n y_n$ yields the bilinear surjection $B(q(\cdot), q(\cdot))$ from $X \times X$ onto Y. This observation should be compared with the results in [20, 21] about the compactness of polynomials and with the results in [25] about the existence of nonlinear smooth surjections between Banach spaces.

In a different line, although important for us since that is the place where the crucial notion of L-set was introduced, Emmanuele shows in [17] that if X and Y do not contain ℓ_1 and $\mathcal{L}(X, Y^*) = \mathcal{K}(X, Y^*)$ then $X \widehat{\otimes}_\pi Y$ cannot contain

ℓ_1. To obtain the converse Emmanuele needs the Approximation Property. Theorem 1 yields an extension of Emmanuele's result to spaces of multilinear forms and shows a way to circumvent the using of the AP (see the final remark at the end of Section 4).

2. L-sets and multilinear forms

Our approach is based on a generalization of Emmanuele's notion of L-set (see [17]). Recall that a subset $A \subset X^*$ is said to be an L-set if for every weakly null sequence $(x_n) \subset X$ one has

$$\lim_{n\to\infty} \sup_{x^* \in A} | < x^*, x_n > | = 0.$$

Emmanuele shows in [17] that a Banach space X does not contain ℓ_1 if and only if L-sets of X^* are relatively compact.

Definition 1. Let X be a Banach space and let $N \geq 1$. A bounded set $A \subset \mathcal{L}^N(X)$ is said to be an L_N-set if for every weakly null sequence $(x_n^1, \cdots, x_n^N) \subset X^N$ one has

$$\lim_{n\to\infty} \sup_{\eta \in A} |\eta(x_n^1, \cdots, x_n^N)| = 0.$$

When $N = 1$ we just get the notion of L-set. Let us observe that L_N-sets are not necessarily L-sets with respect to the natural predual $\left(\widehat{\otimes}_{N,\pi} X\right)$ of $\mathcal{L}^N(X)$; while one of the main results in [8] implies that not all L-sets in $\mathcal{L}^N(X)$ are L_N-sets. The simplest examples of L_N-sets are provided by norm null sequences. Recall from [8] that Banach spaces in which all continuous N-linear (resp. all continuous multilinear) forms are weakly sequentially continuous have been called $\mathcal{M}_N$-spaces (resp. -spaces). It is implicit in [8] that a Banach space X is an $\mathcal{M}_N$-space if and only if every operator $X \to \mathcal{L}^{N-1}(X)$ transforms weakly null sequences into L_{N-1}-sets.

The hypothesis "X does not contain l_1" shall be used via Rosenthal's theorem (see [12, 33]): *An infinite dimensional Banach space X does not contain ℓ_1 if and only if every bounded sequence admits a weakly Cauchy subsequence*. It can be seen following the arguments in [3] that it is possible to replace the condition "the sequence $(x_n^1, \ldots, x_n^N)$ is weakly null" in the definition of L_N-set by "one of the sequences (x_n^i) is weakly null and the other are weakly Cauchy". We are thus ready to obtain the multilinear version of Emmanuele's characterization.

Lemma 1. *A Banach space X does not contain ℓ_1 if and only if, for every N, each L_N-set of $\mathcal{L}^N(X)$ is relatively compact.*

Proof. The proof of the "only if" part, which is the only that needs proof, goes by induction: if X is a Banach space that does not contain l_1, Emmanuele's result provides the case $N = 1$ of our assertion. Assume that the case $N-1$ has already been proved, and let (B_n) be a sequence of N-linear forms on X that form an L_N-set which is not compact. From now on we shall pass to subsequences without further warning or relabelling. So, we will assume that $\|B_{n+1} - B_n\| \geq \varepsilon$ for all n. Let (x_n) be a bounded sequence of X such that $\|B_{n+1}(x_{n+1}) - B_n(x_{n+1})\| \geq \varepsilon$. The boundedness of (x_n) allows us to assume that it is weakly Cauchy and therefore $\{B_n(x_n)\}$ is an L_{N-1}-set of $\mathcal{L}^{N-1}(X)$; by induction, it is relatively compact. Hence we can assume that

$$\lim_n \|B_{n+1}(x_{n+1}) - B_n(x_n)\| = 0.$$

This will lead us to a contradiction after proving that

$$\lim_n \|B_n(x_{n+1}) - B_n(x_n)\| = 0.$$

To this end, take $(y_n^1, \ldots, y_n^{N-1}) \in X^{N-1}$ a bounded sequence of points such that $\|B_n(x_{n+1} - x_n)\| \geq \|B_n(x_{n+1} - x_n, y_n^1, \ldots, y_n^{N-1})\| - \frac{1}{n}$; since (B_n) is an L_N-set and the sequences (x_n) and (y_n^j) can be assumed weakly Cauchy (which, in particular, means that the sequence $(x_{n+1} - x_n)_n$ is weakly null) then

$$\lim_n \|B_n(x_{n+1} - x_n, y_n^1, \ldots, y_n^{N-1})\| = 0.$$

The contradiction now appears since

$$\lim_n \|B_{n+1}(x_{n+1}) - B_n(x_{n+1}) + B_n(x_{n+1} - B_n(x_n)\| = 0,$$

and

$$\lim_n \|B_n(x_{n+1}) - B_n(x_n)\| = 0.$$

while

$$\|B_{n+1}(x_{n+1}) - B_n(x_{n+1})\| \geq \varepsilon.$$

∎

A related additional information is that Gutiérrez [24] shows that if $\mathcal{C}_\infty$ denotes the class of completely continuous operators then a Banach space X does not contain ℓ_1 if and only if for all (some) $N \geq 2$, $\mathcal{C}_\infty(X, \mathcal{L}^{N-1}(X)) = \mathcal{L}(X, \mathcal{L}^{N-1}(X))$.

3. The Pelczynski-Pitt theorem revisited

Proof of Proposition 3. We shall make the proof when $X_i = X$ for all $1 \le i \le N$; in this case condition (2) can be reformulated as

(2') X is an $\mathcal{M}_N$-space.

Condition (1) is $\mathcal{L}^{N-1}(X; X^*) = \mathcal{K}^{N-1}(X; X^*)$ and we will show that the two are equivalent to

(1') $\mathcal{L}(X, \mathcal{L}^{N-1}(X)) = \mathcal{K}(X, \mathcal{L}^{N-1}(X))$.

That (2') is equivalent to (1') is now easy: X is an $\mathcal{M}_N$-space if and only if every operator $X \to \mathcal{L}^{N-1}(X)$ transforms weakly null sequences into L_{N-1}-sets, which have to be relatively compact by Lemma 1. When X does not contain l_1 this means that all those operators are compact. The equivalence between (1) and (1') follows from the well-known fact (see [31, 34]) that the natural isomorphism between $\mathcal{L}^N(X; Y)$ and $\mathcal{L}\left(\widehat{\otimes}_{N,\pi} X, Y\right)$ transforms compact N-linear forms into compact operators; and taking into account the natural isomorphism between $\mathcal{L}(X, Y^*)$ and $\mathcal{L}(Y, X^*)$ given by transposition. ∎

Proof of Theorem 1. The equivalence between (1) and (2) has been proved in Proposition 1.
(2) ⇒ (3) Assume that $\frac{1}{l(X_1)} + \cdots + \frac{1}{l(X_N)} \ge 1$ and let us show the existence of a non-weakly sequentially continuous N-linear form defined on some subspaces of the X_i. There is no loss of generality assuming that for each $1 \le k \le N$ there exists a basic normalized weakly null sequence $(x_n^k)_n \subset X_k$ so that $(x_n^k)_n$ admits an upper $l(X_k)$-estimate. It is possible to find for each k a sequence $(f_n^k)_n \subset X_k^*$ biorthogonal to $(x_n^k)_n$ such that the sequence of its restrictions $(f_n^k|[(x_n^k)_n])_n$ admits a lower $l(X_k)^*$-estimate. Using Hölder's inequality (reasoning as in [22]) one obtains that $(f^1_{n|[(x_n^1)_n]})_n, \ldots, (f^N_{n\ |[(x_n^N)_n]})_n$ admits a lower 1-estimate. Thus, the following N-linear form

$$A(u_1, \ldots, u_N) = \sum_{n=1}^{\infty} f_n^1(u_1) \cdots f_n^N(u_N)$$

is well defined and continuous on the closed linear span $[(x_n^1)_n] \times \cdots \times [(x_n^N)_n]$, and it is not weakly sequentially continuous.
(3) ⇒ (2) It can be deduced from [21] -or found, explicit, in [14]- that if

$$\frac{1}{l(X_1)} + \cdots + \frac{1}{l(X_N)} < 1$$

then $\mathcal{L}^N(X_1, \cdots, X_N) = \mathcal{L}^N_{wsc}(X_1, \cdots, X_N)$.

(2) $\Rightarrow$ (4): Following Choi and Kim (see [10, Th. 3.1], where the proof is made for the symmetric tensor product), it is not hard to show that $X_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_N$ does not contain l_1 if and only if for every bounded sequence $(x_n^1, \ldots, x_n^N)$ the sequence $(x_n^1 \otimes \cdots \otimes x_n^N)$ contains a weakly Cauchy subsequence. Now, if X_i does not contain ℓ_1 every sequence (x_n^i) contains a weakly Cauchy subsequence (say, itself). The sequence $(x_n^1 \otimes \cdots \otimes x_n^N)$ must be weakly Cauchy since every scalar N-linear form is weakly sequentially continuous.

The implication (4) $\Rightarrow$ (1) has been essentially proved by Valdivia, and can be found in [35, Prop. 7]. The word "essentially" here means that Valdivia's result was obtained for reflexive spaces, which in practice means "working with weakly convergent sequences". Getting the result for Banach spaces not containing l_1 means to work with weakly Cauchy sequences; something that, as we remarked already, presents no further difficulty in the context of this paper.

Since the upper estimates pass to quotients in the case of reflexive spaces (see [20]), the last two equivalences can be obtained without difficulty.

It is worth to remark that a reflexive Banach space X with the approximation property is an $\mathcal{M}_N$-space if and only if the space $\mathcal{L}^N(X)$ is reflexive. Since c_0 and the ℓ_p spaces, $p \neq 2$, admit a subspace without the approximation property it follows the existence, for each N, of $\mathcal{M}_N$-spaces Z, without the approximation property, and such that $\mathcal{L}^N(Z)$ is reflexive. In particular, for some subspace H of c_0 the space $\widehat{\oplus}_\pi^N H$ does not contain l_1, while $\mathcal{L}(H, H^*) = \mathcal{K}(H, H^*)$. ∎

4. Multilinear forms between ℓ_p, Orlicz and Lorentz sequence spaces

For this section it is only required from the reader a nodding acquaintance with the basic definitions of Lorentz sequence spaces $d(w,p)$ constructed with a suitable sequence w and a parameter p, and Orlicz sequence spaces l_M constructed with a suitable function M. For the convenience of the reader, let us recall the results of Ausekle and Oja [6]:

Proposition 4. *Let X be a subspace of ℓ_p and let Y be a subspace of $d(w,q)$. If $p > q$ and $w \notin \ell_{p/(p-q)}$ then $\mathcal{L}(X,Y) = \mathcal{K}(X,Y)$.*

Proposition 5. *Let X be a subspace of $d(w,p)$ and let Y be a subspace of ℓ_M. If $p > \beta_M$ then $\mathcal{L}(X,Y) = \mathcal{K}(X,Y)$.*

(in this proposition β_M denotes the upper Boyd index of the Orlicz function M.) These results can be extended as follows:

PROPOSITION 6. *Let X be a subspace of ℓ_p and let Y be a subspace of $d(w,q)$. Let $N \in \mathbb{N}$. If $p > Nq$ and $w \notin \ell_{p/(p-Nq)}$ then $\mathcal{L}^N(X,Y) = \mathcal{K}^N(X,Y)$.*

Proof. In [22, Th. 2.5] it is proved (for polynomials, but the proof easily extends to the multilinear case) that $Nu(d(w,q)) < l(l_p)$ implies $\mathcal{L}^N(X^N;Y) = \mathcal{L}^N_{wsc}(X^N;Y)$ for all the subspaces $X \subset l_p$ and $Y \subset d(w,q)$; in our case, that is enough. On the other hand, a combination of either [20] or [22] and [26] yields that $l(l_p) = p$ and that if we set $r = \inf\{s \in [1,\infty] : w \in l_s\}$ then $u(d(w,q)) = r^*q$. Hence, what we want to obtain is the inequality $Nu(d(w,q)) < p$, or else $Nr^*q < p$. Since $p > Nq$ one has

$$Nr^*q < p \ \Leftrightarrow\ r^* < \frac{p}{Nq} \ \Leftrightarrow\ rnq < (r-1)p \ \Leftrightarrow\ r > \frac{p}{p-Nq}.$$

∎

Observe that with this approach l_p can be replaced by suitable $d(\eta,p)$ since $l(d(\eta,p)) = p$. Also, observe that the result is optimal: since the space $d(w,q)$ contains complemented copies of l_q (see [28, p.177]), when $p \le Nq$ then it is enough to apply Proposition 3 to obtain that $\mathcal{L}^N(l_p,\dots,l_p;d(w,q)) \neq \mathcal{K}^N(l_p,\dots,l_p;d(w,q))$.

PROPOSITION 7. *Let X be a subspace of $d(w,p)$ and let Y be a subspace of ℓ_M. If $p > N\beta_M$ then $\mathcal{L}^N(X,Y) = \mathcal{K}^N(X,Y)$.*

Proof. It is enough to prove that $Nu(Y) < l(X)$. In this case one has $u(l_M) = \beta_M$ and $l(d(w,p)) = p$; therefore

$$N\beta_M < p \ \Leftrightarrow\ Nu(Y) < l(X) = p.$$

∎

Again, the result is optimal: since l_M contains complemented copies of ℓ_q for $q = \beta_M$ (see [28, p.143]), when $p \le N\beta_M$ then the Alencar-Floret Proposition 1 to obtain that

$$\mathcal{L}^N(d(w,q),\dots,d(w,p);l_M) \neq \mathcal{L}^N(d(w,p),\dots,d(w,q);l_M).$$

The polynomial versions are obtained with the same techniques:

PROPOSITION 8. *Let X be a subspace of ℓ_p and let Y be a subspace of $d(w,q)$. let $N \in \mathbb{N}$. If $p > Nq$ and $w \notin \ell_{p/(p-Nq)}$ then $\mathcal{P}(^N X, Y) = \mathcal{P}_{\mathcal{K}}(^N X, Y)$.*

PROPOSITION 9. *Let X be a subspace of $d(w,p)$ and let Y be a subspace of ℓ_M. If $p > N\beta_M$ then $\mathcal{P}(^N X, Y) = \mathcal{P}_{\mathcal{K}}(^N X, Y)$.*

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