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Hereditarily Normaloid Operators

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1. Introduction

A Banach space operator T, $T \in B(\mathcal{X})$, is said to be *hereditarily normaloid*, $T \in \mathcal{H}N$, if every *part* of T (i.e., every restriction of T to an invariant subspace) is normaloid; a $T \in \mathcal{H}N$ is *totally hereditarily normaloid*, $T \in \mathcal{T}HN$, if every invertible part of T is also normaloid, and a $T \in B(\mathcal{X})$ is *completely (totally) hereditarily normaloid*, $T \in \mathcal{C}HN$, if either $T \in \mathcal{T}HN$ or $T - \lambda I \in \mathcal{H}N$ for every complex number λ. The class $\mathcal{C}HN$ is large. In particular, Hilbert space operators T, $T \in B(\mathcal{H})$, which are either *hyponormal* ($|T^*|^2 \leq |T|^2$) or *p-hyponormal* ($|T^*|^{2p} \leq |T|^{2p}$ for some $0 < p < 1$) or *w-hyponormal* (if T has the polar decomposition $T = U|T|$, and $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ denotes the Aluthge transform of T, then $|\tilde{T}^*| \leq |T| \leq |\tilde{T}|$) are $\mathcal{T}HN$ operators. Again, *totally $*$-paranormal* Hilbert space operators ($||(T - \lambda I)^*x||^2 \leq ||(T - \lambda I)x||^2$ for every unit vector x) are $\mathcal{H}N$ operators, and *paranormal operators* $T \in B(\mathcal{X})$ ($||Tx||^2 \leq ||T^2x||$ for all unit vectors $x \in \mathcal{X}$) are $\mathcal{T}HN$ operators. (We refer the reader to the monograph [15] for information on these classes of operators.) $\mathcal{T}HN$ operators were introduced in [10], and have since been studied in [13] and [12]. In this note we study operators $T \in \mathcal{C}HN$, and prove (amongst other things) that the Riesz projection associated with a $\lambda \in \operatorname{iso}\sigma(T)$, $T \in \mathcal{C}HN \cap B(\mathcal{H})$, is self-adjoint if and only if $(T - \lambda I)^{-1}(0) \subseteq (T^* - \overline{\lambda}I)^{-1}(0)$. Operators $T \in \mathcal{C}HN$ have the important

* To Professor Carl Pearcy on his seventieth birthday.

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property that both T and the conjugate operator T^* have the *single-valued extension property* at points λ which are not in the *Weyl spectrum* of T; we exploit this property to prove *a-Browder* and *a-Weyl theorems* for operators $T \in \mathcal{CHN}$. Studies of this type for individual classes of operators in $T \in \mathcal{CHN}$ have been carried out by a number of authors in the recent past (see, for example, [4, 6, 7, 8, 9, 10, 17, 20, 23, 26, 31, 34]; see also [1, Chapter 3] for a state of the art introduction to Browder-Weyl theorems).

Recall that an operator $T \in B(\mathcal{X})$ is said to be Fredholm, $T \in \Phi(\mathcal{X})$, if $T(\mathcal{X})$ is closed and both the *deficiency indices* $\alpha(T) = \dim(T^{-1}(0))$ and $\beta(T) = \dim(\mathcal{X}/T(\mathcal{X}))$ are finite, and then the *index* of T, $\mathrm{ind}(T)$, is defined to be $\mathrm{ind}(T) = \alpha(T) - \beta(T)$. The ascent of T, $\mathrm{asc}(T)$, is the least non-negative integer n such that $T^{-n}(0) = T^{-(n+1)}(0)$ and the descent of T, $\mathrm{dsc}(T)$, is the least non-negative integer n such that $T^n(\mathcal{X}) = T^{n+1}(\mathcal{X})$. We say that T is of *finite ascent* (resp., *finite descent*) if $\mathrm{asc}(T - \lambda I) < \infty$ (resp., $\mathrm{dsc}(T - \lambda I) < \infty$) for all complex numbers λ. We shall, henceforth, shorten $(T - \lambda I)$ to $(T - \lambda)$. The operator T is *Weyl* if it is Fredholm of zero index, and T is said to be *Browder* if it is Fredholm "of finite ascent and descent". Let $\mathbb{C}$ denote the set of complex numbers. The Browder spectrum $\sigma_b(T)$ and the Weyl spectrum $\sigma_w(T)$ of T are the sets $\sigma_b(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Browder}\}$ and $\sigma_w(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Weyl}\}$. Let $\pi_0(T)$ denote the set of *Riesz points* of T (i.e., the set of $\lambda \in \mathbb{C}$ such that $T - \lambda$ is Fredholm of finite ascent and descent [5]), and let $\pi_{00}(T)$ denote the set of eigenvalues of T of finite geometric multiplicity. The operator $T \in B(\mathcal{X})$ is said to satisfy *Browder's theorem* if $\sigma(T) \setminus \sigma_w(T) = \pi_0(T)$, and T is said to satisfy *Weyl's theorem* if $\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T)$. Recall [18] that Weyl's theorem for T implies Browder's theorem for T, and Browder's theorem for T is equivalent to Browder's theorem for T^*.

The (Fredholm) essential spectrum $\sigma_e(T)$ of $T \in B(\mathcal{X})$ is the set $\sigma_e(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Fredholm}\}$. If we let $\mathrm{acc}\,\sigma(T)$ denote the set of accumulation points of $\sigma(T)$, then $\sigma_e(T) \subseteq \sigma_w(T) \subseteq \sigma_b(T) \subseteq \sigma_e(T) \cup \mathrm{acc}\,\sigma(T)$. Let $\pi_{a0}(T)$ be the set of $\lambda \in \mathbb{C}$ such that λ is an isolated point of $\sigma_a(T)$ and $0 < \alpha(T - \lambda) < \infty$, where $\sigma_a(T)$ denotes the approximate point spectrum of the operator T. Then $\pi_0(T) \subseteq \pi_{00}(T) \subseteq \pi_{a0}(T)$. We say that *a-Weyl's theorem holds for T* if

$$\sigma_{aw}(T) = \sigma_a(T) \setminus \pi_{a0}(T),$$

where $\sigma_{aw}(T)$ denotes *the essential approximate point spectrum* of T (i.e., $\sigma_{aw}(T) = \cap\{\sigma_a(T + K) : K \in K(\mathcal{X})\}$ with $K(\mathcal{X})$ denoting the ideal of compact operators on $\mathcal{X}$). Let $\Phi_+(\mathcal{X}) = \{T \in B(\mathcal{X}) : \alpha(T) < \infty$ and $T(\mathcal{X})$ is closed$\}$ (resp., $\Phi_-(\mathcal{X}) = \{T \in B(\mathcal{X}) : \beta(T) < \infty\}$) denote the semi-group of *upper semi-Fredholm* (resp., *lower semi-Fredholm*) operators in $B(\mathcal{X})$ and let $\Phi_+^-(\mathcal{X}) = \{T \in \Phi_+(\mathcal{X}) : \operatorname{ind}(T) \leq 0\}$. Then $\sigma_{aw}(T)$ is the complement in $\mathbb{C}$ of all those λ for which $(T - \lambda) \in \Phi_+^-(\mathcal{X})$ [27]. The concept of a-Weyl's theorem was introduced by Rakočvić: a-Weyl's theorem for $T \Rightarrow$ Weyl's theorem for T, but the converse is generally false [29]. If we let $\sigma_{ab}(T)$ denote the *Browder essential approximate point spectrum of* T,

$$\begin{aligned}\sigma_{ab}(T) &= \cap\{\sigma_a(T+K) \,:\, TK = KT \text{ and } K \in K(\mathcal{X})\}\\ &= \{\lambda \in \mathbb{C} \,:\, T - \lambda \notin \Phi_+^-(\mathcal{X}) \text{ or } \operatorname{asc}(T-\lambda) = \infty\},\end{aligned}$$

then $\sigma_{aw}(T) \subseteq \sigma_{ab}(T)$. We say that T satisfies *a-Browder's theorem* if $\sigma_{ab}(T) = \sigma_{aw}(T)$ [27].

An operator $T \in B(\mathcal{X})$ has *the single-valued extension property* at $\lambda_0 \in \mathbb{C}$, SVEP at λ_0 for short, if for every open disc $\mathcal{D}_{\lambda_0}$ centered at λ_0 the only analytic function $f : \mathcal{D}_{\lambda_0} \to \mathcal{X}$ which satisfies

$$(T - \lambda)f(\lambda) = 0 \quad \text{for all } \lambda \in \mathcal{D}_{\lambda_0}$$

is the function $f \equiv 0$. Trivially, every operator T has SVEP at points of the resolvent $\rho(T) = \mathbb{C} \setminus \sigma(T)$; also T has SVEP at $\lambda \in \operatorname{iso}\sigma(T)$. *We say that T has SVEP if it has SVEP at every $\lambda \in \mathbb{C}$.* It is known that a Banach space operator T with SVEP satisfies Browder's theorem [3, Corollary 2.12].

The *quasinilpotent part* $H_0(T - \lambda)$ and the *analytic core* $K(T - \lambda)$ of $(T - \lambda)$ are defined by

$$H_0(T-\lambda) = \left\{x \in \mathcal{X} : \lim_{n \longrightarrow \infty} ||(T-\lambda)^n x||^{\frac{1}{n}} = 0\right\}$$

and

$$K(T-\lambda) = \left\{ x \in \mathcal{X} \;\;:\;\; \begin{array}{c} \text{there exists a sequence } \{x_n\} \subset \mathcal{X} \text{ and} \\ \delta > 0 \text{ for which } x = x_0, (T-\lambda)(x_{n+1}) = x_n \\ \text{and } \|x_n\| \leq \delta^n \|x\| \text{ for all } n = 1, 2, \ldots \end{array} \right\}.$$

We note that $H_0(T-\lambda)$ and $K(T-\lambda)$ are (generally) non-closed hyperinvariant subspaces of $(T-\lambda)$ such that $(T-\lambda)^{-q}(0) \subseteq H_0(T-\lambda)$ for all $q = 0, 1, 2, \ldots$ and $(T - \lambda)K(T - \lambda) = K(T - \lambda)$ [24]. The operator $T \in B(\mathcal{X})$ is said to be *semi-regular* if $T(\mathcal{X})$ is closed and $T^{-1}(0) \subset T^{\infty}(\mathcal{X}) = \cap_{n \in \mathbb{N}} T^n(\mathcal{X})$; T admits a *generalized Kato decomposition*, *GKD* for short, if there exists a pair of

T-invariant closed subspaces (M, N) such that $\mathcal{X} = M \oplus N$, the restriction $T|_M$ is quasinilpotent and $T|_N$ is semi-regular. An operator $T \in B(\mathcal{X})$ has a GKD at every $\lambda \in \operatorname{iso}\sigma(T)$, namely $\mathcal{X} = H_0(T-\lambda) \oplus K(T-\lambda)$. We say that T is of *Kato type* at a point λ if $(T-\lambda)|_M$ is nilpotent in the GKD for $(T-\lambda)$. If $T-\lambda$ is Kato type, then $K(T-\lambda) = (T-\lambda)^\infty(\mathcal{X})$ [3]. Fredholm (also, semi-Fredholm) operators are Kato type [21, Theorem 4]. (For more information on semi-Fredholm operators, semi-regular operators and Kato type operators, see [1] and [25].)

2. Results

The following theorem characterizes isolated points of the spectrum of an operator $T \in \mathcal{C}HN$ as the poles of the resolvent of T. The proposition is proved in [10, Lemma 2.1] for the case in which $T \in \mathcal{T}HN$, but we include the proof here for completeness.

Proposition 2.1. *If $T \in \mathcal{C}HN$ and $\lambda \in \operatorname{iso}\sigma(T)$, then λ is a simple pole of the resolvent of T.*

Proof. If $\lambda \in \operatorname{iso}\sigma(T)$, then

$$\mathcal{X} = H_0(T-\lambda) \oplus K(T-\lambda)\,,$$

where $H_0(T-\lambda) \neq \{0\}$ and $(T-\lambda)K(T-\lambda) = K(T-\lambda)$ [24]. Set $T_1 = T|_{H_0(T-\lambda)}$. Then $T_1 \in \mathcal{C}HN$, $\sigma(T_1) = \{\lambda\}$ and $\sigma(T|_{K(T-\lambda)}) = \sigma(T) \setminus \{\lambda\}$. If $\lambda = 0$, then (T_1 being normaloid, it follows that) $T_1 = 0$, which implies that $H_0(T) = T^{-1}(0)$. Let $\lambda \neq 0$. Then, since $\mathcal{C}HN$ operators are closed under multiplication by non-zero scalars, we may assume that $\lambda = 1$. If $T-\mu \in \mathcal{H}N$ for all $\mu \in \mathbb{C}$, then $\sigma(T_1 - 1) = \{0\} \Rightarrow ||T_1 - 1|| = 0 \Rightarrow T_1 = I|_{H_0(T-1)}$. If, instead, $T \in \mathcal{T}HN$, then $T_1 \in \mathcal{T}HN$ and $\sup ||T_1^n|| \leq 1$, where the supremum is taken over all integers n. Applying [22, Theorem 1.5.14], it follows that $T_1 = I|_{H_0(T-1)} \Rightarrow H_0(T-1) = (T-1)^{-1}(0)$. Thus $H_0(T-\lambda) = (T-\lambda)^{-1}(0)$, which implies that $(T-\lambda)\mathcal{X} = 0 \oplus (T-\lambda)K(T-\lambda) = K(T-\lambda) \Rightarrow \mathcal{X} = (T-\lambda)^{-1}(0) \oplus (T-\lambda)\mathcal{X}$. Hence λ is a simple pole of the resolvent of T. ∎

An operator $T \in B(\mathcal{X})$ is said to be *isoloid* (resp., *reguloid*) if $\lambda \in \operatorname{iso}\sigma(T) \Rightarrow \lambda$ is an eigenvalue of T (resp., $\lambda \in \operatorname{iso}\sigma(T) \Rightarrow (T-\lambda)^{-1}(0)$ and $(T-\lambda)\mathcal{X}$ are complemented in $\mathcal{X}$). Evidently, the reguloid property implies the isoloid property. The following corollary is an immediate consequence of Proposition 2.1.

Corollary 2.2. *Operators $T \in \mathcal{C}HN$ are reguloid.*

Corollary 2.3. *$\pi_0(T) = \pi_{00}(T)$ for operators $T \in \mathcal{C}HN$.*

Proof. Evidently, $\pi_0(T) \subseteq \pi_{00}(T)$. For the reverse inclusion, apply Proposition 2.1 to $\lambda \in \operatorname{iso}\sigma(T)$. ∎

Recall, [22, p. 42], that a $T \in B(\mathcal{X})$ is said to be a *Riesz operator* if, for each $\lambda \in \mathbb{C} \setminus \{0\}$, both the deficiency indices are finite.

Corollary 2.4. *An operator $T \in \mathcal{C}HN$ such that $\sigma(T)$ is countable with 0 as its only limit point is a Riesz operator if and only if $\alpha(T-\lambda) < \infty$ for all non-zero $\lambda \in \sigma(T)$.*

Proof. Let $0 \neq \lambda \in \sigma(T)$. Then $\lambda \in \operatorname{iso}\sigma(T)$, and it follows from Proposition 2.1 that $\operatorname{asc}(T-\lambda) = \operatorname{dsc}(T-\lambda) \leq 1 \Rightarrow \alpha(T-\lambda) = \beta(T-\lambda) < \infty$ [19, Proposition 38.5] $\Rightarrow T$ is a Riesz operator. The necessity being obvious, the proof is complete. ∎

Observe that as a Riesz operator, the operator T of Corollary 2.4 is a *decomposable operator* such that the local spectral subspaces $\mathcal{X}_T(F) = \{x \in \mathcal{X} : \sigma_T(x) \subseteq F\}$ are closed and finite dimensional for every $F \subseteq \mathbb{C} \setminus \{0\}$ [22, Theorem 1.4.7]. (Here $\sigma_T(x)$ denotes the *local spectrum* of T at x [22, p. 16].) If $\mathcal{X} = \mathcal{H}$ is a Hilbert space, then (by a well known result of T.T. West - see [5, Theorem 3.52.]) T is the sum of a compact and a quasi-nilpotent operator: more is true in this case, as we shall see below.

A subspace M of $\mathcal{X}$ is said to be orthogonal to a subspace N of $\mathcal{X}$, $M \perp N$, in the sense of G. Birkhoff, if $||m|| \leq ||m+n||$ for every $m \in M$ and $n \in N$ [14, p. 93]. This asymmetric definition of orthogonality coincides with the usual concept of orthogonality in the case in which $\mathcal{X}$ is a Hilbert space. Let $\sigma_\pi(T) = \{\lambda \in \sigma(T) : |\lambda| = ||T||\}$ denote the *peripheral spectrum* of $T \in B(\mathcal{X})$.

Proposition 2.5. *Let α and β be eigen-values of an operator $T \in \mathcal{C}HN$, with corresponding eigen-spaces N and M (respectively), such that $|\alpha| < |\beta|$. If $\alpha = 0$, then $M \perp N$, and if $\alpha \neq 0$, then M and N are mutually orthogonal.*

Proof. The proposition is proved in [10] for the case in which $T \in \mathcal{T}HN$, but here is a short proof. Let $\mathcal{L}$ denote the subspace generated by M and N. Then $T_1 = T|_{\mathcal{L}} \in \mathcal{C}HN$, and $\beta \in \sigma_\pi(T_1)$. Applying [19, Proposition 54.4] it follows that $M \perp N$. If $\alpha \neq 0$ and $T - \lambda \in \mathcal{H}N$ for every $\lambda \in \mathbb{C}$, then

$A = T_1 - \beta \in \mathcal{H}N$, $A^{-1}(0) = M$ and, for $\mu = \alpha - \beta$, $(A-\mu)^{-1}(0) = N$; since $|\mu| > 0$ we have $N \perp M$. Again, if $\alpha \neq 0$ and $T \in \mathcal{T}HN$, then $A = T_1^{-1} \in \mathcal{T}HN$, $(A - \alpha^{-1})^{-1}(0) = N$ and $(A - \beta^{-1})^{-1}(0) = M$; since $|\beta^{-1}| < |\alpha^{-1}|$ we have $N \perp M$. ∎

PROPOSITION 2.6. *If an operator $T \in \mathcal{C}HN \cap B(\mathcal{H})$ has countable spectrum with* 0 *as its only limit point, then T is normal.*

Proof. The isolated points of a $\mathcal{C}HN$ operator being poles of the resolvent of T, the operator T is *spectrally normaloid* (in the terminology of [19, p. 227]). Enumerating the points of $\sigma(T)$, allowing for repetition, according to their absolute values by $|\lambda_1| \geq |\lambda_2| \geq \ldots$, it follows that there exists a non-empty finite set $S = \{\lambda_{m_1}, \ldots, \lambda_{m_t}\}$ such that $\sigma_\pi(T) = S$. Let $\lambda_\nu \in S$. Then the spectral projection P_ν corresponding to the simple pole λ_ν has norm 1 and $(T-\lambda_\nu)^{-1}(0) \perp (T-\lambda_\nu)\mathcal{H}$ (see [19, Proposition 54.4]). In the setting of our Hilbert space $\mathcal{H}$, this implies that $(T-\lambda_\nu)^{-1}(0)$ reduces T and $T_1 = T|_{(T-\lambda_\nu)\mathcal{H}} \in \mathcal{C}HN$. Repeating this argument, starting with T_1, it follows that $\mathcal{H}_\lambda = \bigvee_{0 \neq \lambda \in \sigma(T)} (T-\lambda)^{-1}(0)$ reduces T and $T_0 = T|_{\mathcal{H} \ominus \mathcal{H}_\lambda} \in \mathcal{C}HN$. We claim that $\sigma(T_0) = \{0\}$. For if $\sigma(T_0) \neq \{0\}$, then there exists a $0 \neq \mu \in \sigma(T_0)$ such that $\mu \in \sigma_\pi(T_0)$, and hence that $(T-\mu)^{-1}(0)$ reduces T_0. But then $(T_0 - \mu)^{-1}(0) \in \mathcal{H}_\lambda$. Our claim having been established, it follows that T_0 is a quasi-nilpotent operator. Since $T_0 \in \mathcal{C}HN$ implies $||T_0|| = 0$, we conclude that T is a diagonal operator. ∎

Proposition 2.6 implies in particular that compact paranormal operators in $B(\mathcal{H})$ are normal. Staying for the moment with Hilbert spaces, it is apparent from Proposition 2.1 that the Riesz projection P_λ associated with a $\lambda \in \operatorname{iso}\sigma(T)$, $T \in \mathcal{C}HN$, is in general not self-adjoint: the following theorem gives a necessary and sufficient condition for P_λ to be self-adjoint.

THEOREM 2.7. *If $T \in B(\mathcal{H}) \cap \mathcal{C}HN$ and $\lambda \in \operatorname{iso}\sigma(T)$, then P_λ is self-adjoint if and only if $(T-\lambda)^{-1}(0) \subseteq (T^* - \overline{\lambda})^{-1}(0)$.*

Proof. If $\lambda \in \operatorname{iso}\sigma(T)$ and $T \in \mathcal{C}HN$, then

$$\mathcal{H} = H_0(T-\lambda) \oplus K(T-\lambda) = (T-\lambda)^{-1}(0) \oplus (T-\lambda)\mathcal{H}$$

as a topological direct sum and T has an upper triangular matrix decomposition

$$T = \begin{bmatrix} T_1 & T_2 \\ 0 & T_3 \end{bmatrix} \begin{pmatrix} (T-\lambda)^{-1}(0) \\ (T-\lambda)\mathcal{H} \end{pmatrix},$$

where $\sigma(T_1) = \{\lambda\}$ and $\sigma(T_3) = \sigma(T) \setminus \{\lambda\}$. Observe that $P_\lambda\mathcal{H} = H_0(T - \lambda) = (T-\lambda)^{-1}(0)$ and $P_\lambda^{-1}(0) = (T-\lambda)\mathcal{H}$. Thus, if P_λ is self-adjoint, then $P_\lambda\mathcal{H}^\perp = P_\lambda^{-1}(0)$. Since $P_\lambda\mathcal{H}^\perp = (T-\lambda)\mathcal{H}^\perp = (T^* - \overline{\lambda})^{-1}(0)$, the condition $(T-\lambda)^{-1}(0) \subseteq (T^* - \overline{\lambda})^{-1}(0)$ is necessary. Conversely, assume that $(T-\lambda)^{-1}(0) \subseteq (T^* - \overline{\lambda})^{-1}(0)$. Since $x = x_1 \oplus x_2 \in (T-\lambda)^{-1}(0)$ if and only if $x_2 = 0$ and $x_1 \in (T_1 - \lambda)^{-1}(0)$, and since $(T-\lambda)^{-1}(0) \subseteq (T^* - \overline{\lambda})^{-1}(0)$, $T_2^* x_1 = 0 \Rightarrow T_2^*$, and so also T_2, is the 0 operator. Consequently, $(T-\lambda)^{-1}(0)$ reduces $T \Rightarrow P_\lambda^{-1}(0)^\perp = P_\lambda\mathcal{H} \Rightarrow P_\lambda$ is self-adjoint. ∎

Theorem 2.7 has been proved for hyponormal operators by Stampfli [33], for w-hyponormal operators by Han, Lee and Wang [17], for paranormal operators by Uchiyama [34], and for totally $*$-paranormal operators by Han and Kim [16]. Observe that points $\lambda \in \operatorname{iso}\sigma(T)$ (resp., $0 \neq \lambda \in \operatorname{iso}\sigma(T)$) of a hyponormal or totally $*$-paranormal operator T (resp., w-hyponormal or paranormal operator T) are normal eigen-values of the operator. Evidently, Theorem 2.7 holds for M-hyponormal operators T for all $\lambda \in \operatorname{iso}\sigma(T)$, and for quasihyponormal operators for all $0 \neq \lambda \in \operatorname{iso}\sigma(T)$. An extension of Theorem 2.7 is given by the following corollary.

COROLLARY 2.8. *Suppose that an operator $T \in B(\mathcal{H})$ has a triangulation*

$$T = \begin{bmatrix} T_1 & T_2 \\ 0 & T_3 \end{bmatrix} \begin{pmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{pmatrix}$$

such that $T_1 \in \mathcal{CHN}$, T_3 is nilpotent and $\sigma(T_1) \subseteq \sigma(T) \subseteq \sigma(T_1) \cup \{0\}$. If the non-zero isolated eigen-values of T_1 are normal, and if $(T_1 - \lambda)^{-1}(0) \oplus 0 \subseteq (T^* - \overline{\lambda})^{-1}(0)$ for a $0 \neq \lambda \in \operatorname{iso}\sigma(T)$, then the Riesz projection P_λ associated with λ is self-adjoint.

Proof. If $0 \neq \lambda \in \operatorname{iso}\sigma(T)$, then $\lambda \in \operatorname{iso}\sigma(T_1)$, and it follows from the proof of Theorem 2.7 that $T_1 - \lambda$ has a triangulation

$$T_1 = \begin{bmatrix} 0 & 0 \\ 0 & T_{11} - \lambda \end{bmatrix} \begin{pmatrix} (T_1 - \lambda)^{-1}(0) \\ \mathcal{H}_1 \ominus (T_1 - \lambda)^{-1}(0) \end{pmatrix}$$

and $T - \lambda$ has a triangulation

$$T - \lambda = \begin{bmatrix} 0 & 0 & T_{21} \\ 0 & T_{11} - \lambda & T_{22} \\ 0 & 0 & T_3 - \lambda \end{bmatrix} = \begin{bmatrix} 0 & A \\ 0 & B \end{bmatrix} \begin{pmatrix} \mathcal{H}_1' \\ \mathcal{H}_2' \end{pmatrix},$$

where $A = [0\ T_{21}]$, $B = \begin{bmatrix} T_{11} & T_{22} \\ 0 & T_3 - \lambda \end{bmatrix}$ is invertible, $\mathcal{H}_1' = (T_1 - \lambda)^{-1}(0)$ and $\mathcal{H}_2' = (\mathcal{H}_1 \ominus \mathcal{H}_1') \oplus \mathcal{H}_2$. Evidently,

$$P_\lambda \mathcal{H} = H_0(T - \lambda) = \left\{ x = x_1 \oplus x_2 \in \mathcal{H}_1' \oplus \mathcal{H}_2' \ :\ \lim_{n\to\infty} ||T^n x||^{\frac{1}{n}} = 0 \right\}$$

$$= \left\{ x = x_1 \oplus x_2 \in \mathcal{H}_1' \oplus \mathcal{H}_2' \ :\ \lim_{n\to\infty} \left|\left| \begin{bmatrix} AB^{n-1}x_1 \\ B^n x_2 \end{bmatrix} \right|\right|^{\frac{1}{n}} = 0 \right\}.$$

Since B is invertible,

$$||x_2||^{\frac{1}{n}} \le ||B^{-1}|| ||B^n x_2||^{\frac{1}{n}} \longrightarrow 0 \qquad \text{as } n \to \infty,$$

which implies that $x_2 = 0$. Hence $P_\lambda \mathcal{H} = (T_1 - \lambda)^{-1}(0) \oplus 0 = (T - \lambda)^{-1}(0) \subseteq (T^* - \overline{\lambda})^{-1}(0)$. But then $A = 0 \Rightarrow P_\lambda \mathcal{H} = (T^* - \overline{\lambda})^{-1}(0) = P_\lambda^{-1}(0)^\perp \Rightarrow P_\lambda$ is self-adjoint. ∎

Let $\gamma(T)$,

$$\gamma(T) = \inf \left\{ \frac{||Tx||}{\operatorname{dist}(x, T^{-1}(0))} \ :\ x \in \mathcal{X}/T^{-1}(0) \right\},$$

denote the *reduced minimum modulus* of T (with the convention that $\gamma(T) = \infty$ if $T = 0$). Then $\gamma(T^*) = \gamma(T)$, and $T(\mathcal{X})$ is closed if and only if $\gamma(T) > 0$ [22, p. 203]. The following theorem will play an important part in our considerations below.

THEOREM 2.9. *If $T \in \mathcal{C}HN$, then T and T^* have SVEP at points $\lambda \in \sigma(T) \setminus \sigma_w(T)$.*

Proof. Let $\lambda \in \sigma(T) \setminus \sigma_w(T)$; then $T - \lambda \in \Phi(\mathcal{X})$ and $\operatorname{ind}(T - \lambda) = 0$. Suppose to the contrary that T does not have SVEP at λ. Then $\lambda \in \operatorname{acc} \sigma_p(T)$ [3, Theorem 2.6], and so there exists a sequence $\{\lambda_n\}$ of non-zero eigen-values of T converging to λ. Choose $\lambda_m \in \{\lambda_n\}$. Recall from Proposition 2.5 that the eigenspaces corresponding to non-zero eigenvalues of T are mutually orthogonal, and if $\lambda = 0$ then the eigenspace corresponding to the eigenvalue λ_m is orthogonal to the eigenspace corresponding to the eigenvalue 0. Thus, $\operatorname{dist}(x, (T - \lambda)^{-1}(0)) \ge 1$ for every unit vector $x \in (T - \lambda_m)^{-1}(0)$. Since

$$\delta(\lambda_m, \lambda) \ = \ \sup\{\operatorname{dist}(x, (T - \lambda)^{-1}(0)) : x \in (T - \lambda_m)^{-1}(0), ||x|| = 1\} \ \ge \ 1$$

for all m,

$$\frac{|\lambda_m - \lambda|}{\delta(\lambda_m, \lambda)} \longrightarrow 0 \qquad \text{as } m \to \infty\,.$$

But then

$$\gamma(T-\lambda) = \frac{|\lambda_m - \lambda|}{\delta(\lambda_m, \lambda)} \longrightarrow 0 \qquad \text{as } m \to \infty\,,$$

i.e., $(T-\lambda)\mathcal{X}$ is not closed. Since $T-\lambda \in \Phi(\mathcal{X})$, we conclude that T has SVEP at λ. Applying [3, Corollary 2.10], it now follows that T^* also has SVEP at λ. ∎

Theorem 2.9 has a number of consequences, amongst them the following. Let $\sigma_{sw}(T)$ denote the *Weyl surjectivity spectrum* $\{\lambda \in \mathbb{C} : T-\lambda \notin \Phi_-(\mathcal{X})$ or $\operatorname{ind}(T-\lambda) \not\geq 0\}$, and let $\Phi_\pm(T) = \{\lambda \in \mathbb{C} : T-\lambda \in \Phi_\pm(\mathcal{X})\}$, where $\Phi_\pm(\mathcal{X})$ denotes the set of T which are either upper semi-Fredholm or lower semi-Fredholm, denote the semi-Fredholm region of $T \in B(\mathcal{X})$.

Corollary 2.10. *Operators $T \in \mathcal{CHN}$ have SVEP at points $\lambda \in \Phi_\pm(T)$. In particular, operators $T \in \mathcal{CHN}$ have SVEP at points $\lambda \notin \sigma_{aw}(T)$ and $\lambda \notin \sigma_{sw}(T)$.*

Proof. If $T-\lambda \in \Phi_\pm(\mathcal{X})$, then $T-\lambda$ is Kato type [21, Theorem 4]. Hence, if T does not have SVEP at λ, then $\lambda \in \operatorname{acc}\sigma_p(T)$ [3, Theorem 2.6]. But then, by the argument of the proof of Theorem 2.9, $(T-\lambda)\mathcal{X}$ is not closed. Since $T-\lambda \in \Phi_\pm(\mathcal{X})$, we have a contradiction. Hence T has SVEP at λ. ∎

Corollary 2.11. *$T-\lambda \in \Phi(\mathcal{X})$ for an operator $T \in \mathcal{CHN}$ if and only if $T-\lambda \in \Phi_-(\mathcal{X})$. Consequently, $\sigma_e(T) = \{\lambda \in \mathbb{C} : \beta(T-\lambda) = \infty\}$.*

Proof. The following implications hold:

$$\begin{aligned}
T-\lambda \in \Phi_-(\mathcal{X}) &\implies T \text{ has SVEP at } \lambda \ \ (\text{see Theorem 2.9})\\
&\implies \operatorname{asc}(T-\lambda < \infty \ \ (\text{see [3, Theorem 2.6]})\\
&\implies \operatorname{ind}(T-\lambda) \leq 0 \ \ (\text{see [19, Proposition 38.5]})\\
&\implies \alpha(T-\lambda) \leq \beta(T-\lambda) < \infty\\
&\implies T-\lambda \in \Phi(\mathcal{X}).
\end{aligned}$$

This completes the proof. ∎

COROLLARY 2.12. *If Ω is a connected component of $\Phi(T) = \{\lambda \in \mathbb{C} : T - \lambda \in \Phi(\mathcal{X})\}$ for a $T \in \mathcal{C}HN$, then either*

(I) $\operatorname{ind}(T-\lambda) = 0$ *and* $\operatorname{asc}(T-\lambda) = \operatorname{dsc}(T-\lambda) < \infty$ *for all* $\lambda \in \Omega$, *or*

(II) $\operatorname{ind}(T-\lambda) < 0$, $\operatorname{asc}(T-\lambda) < \infty$ *and* $\operatorname{dsc}(T-\lambda) = \infty$ *for all* $\lambda \in \Omega$.

Proof. Since T has SVEP at points $\lambda \in \Phi(T)$, $\operatorname{asc}(T-\lambda) < \infty$ [3, Theorem 2.6] $\Rightarrow \operatorname{ind}(T-\lambda) \le 0$. If also $\operatorname{dsc}(T-\lambda) < \infty$, then we have (I); otherwise, we have (II) (see [19, Theorem 51.1]). ∎

COROLLARY 2.13. *If $T \in \mathcal{C}HN$, then $\rho(T) = \{\lambda \in \mathbb{C} : T-\lambda$ is surjective$\}$.*

Proof. If $T - \lambda$ is surjective, then $\operatorname{dsc}(T-\lambda) = 0 = \beta(T-\lambda)$. This, since $\sigma_e(T) = \{\lambda \in \mathbb{C} : \beta(T-\lambda) = \infty\}$ (see Corollary 2.11) implies that $T - \lambda \in \Phi(\mathcal{X}) \Rightarrow \operatorname{asc}(T-\lambda) = 0$ (by Corollary 2.12 (I)) $\Rightarrow T-\lambda$ is injective. ∎

Corollaries 2.11, 2.12 and 2.13 have been proved by Schmoeger [32] for paranormal operators T.

Recall from [18] and [28] that whereas the *spectral mapping theorem* holds for the Browder and the Browder essential approximate point spectra, it fails (in general) for the Weyl and the Weyl essential approximate point spectra. The following corollary shows that the operators in the class $\mathcal{C}HN$ are much better behaved. Let $\mathcal{H}(T)$ denote the set of functions f which are defined and analytic in an open neighbourhood of $\sigma(T)$.

COROLLARY 2.14. *If $T \in \mathcal{C}HN$, then $f(\sigma_w(T)) = \sigma_w(f(T)) = f(\sigma_w(T^*)) = \sigma_w(f(T^*))$, $f(\sigma_{aw}(T)) = \sigma_{aw}(f(T))$ and $f(\sigma_{sw}(T)) = \sigma_{sw}(f(T))$ for every $f \in \mathcal{H}(T)$.*

Proof. As observed in Corollary 2.10, T has SVEP at points λ such that $T - \lambda \in \Phi_{\pm}(\mathcal{X})$. Thus, if $T - \lambda \in \Phi_+(\mathcal{X})$, then $\operatorname{asc}(T-\lambda) < \infty \Rightarrow \operatorname{ind}(T-\lambda) \le 0$; again, if $T - \lambda \in \Phi_-(\mathcal{X})$, then $T - \lambda \in \Phi(\mathcal{X})$ (see Corollary 2.11) $\Rightarrow \operatorname{ind}(T-\lambda) \le 0$. Now apply [32, Theorem 2] to obtain $f(\sigma_w(T)) = \sigma_w(f(T))$, and [30, Theorems 2 and 4] to obtain $f(\sigma_{aw}(T)) = \sigma_{aw}(f(T))$ and $f(\sigma_{sw}(T)) = \sigma_{sw}(f(T))$. To complete the proof, we observe that $\sigma(T) = \sigma_w(T^*)$, and $\sigma_w(f(T^*)) = \sigma_w(f(T)) = f(\sigma_w(T)) = f(\sigma_w(T^*))$. ∎

Parts of Corollary 2.14 have been observed for M-hyponormal operators by Hou and Zhang [20], for paranormal operators (on a Hilbert space) by Curto and Han [9], and for $*$-paranormal operators by Han and Kim [16].

(We remark that M-hyponormal operators and $*$-paranormal operators have the finite ascent property [16]; hence they have SVEP.) It is apparent from the proof of Corollary 2.14 that a sufficient condition for $\sigma_{aw}(T)$ (or, $\sigma_{sw}(T)$), for a general $T \in B(\mathcal{X})$, to satisfy the spectral mapping theorem is that T has SVEP at points $\lambda \notin \sigma_{aw}(T)$ (resp., $\lambda \notin \sigma_{sw}(T)$); cf. [8, Theorem 3.1].

COROLLARY 2.15. *If $T \in \mathcal{CHN}$, then $f(T)$ and $f(T^*)$ satisfy a-Browder's theorem for every $f \in \mathcal{H}(T)$.*

Proof. In view of Corollary 2.14, it will suffice to prove that T and T^* satisfy a-Browder's theorem. Observe that if $\lambda \notin \sigma_{aw}(T)$, then (as seen above) $\mathrm{asc}(T-\lambda) < \infty \Rightarrow \lambda \notin \sigma_{ab}(T)$. Since $\sigma_{aw}(T) \subseteq \sigma_{ab}(T)$ for every operator T, it follows that $\sigma_{aw}(T) = \sigma_{ab}(T) \Rightarrow T$ satisfies a-Browder's theorem. Now let $\lambda \notin \sigma_{sw}(T)$. Then $T - \lambda \in \Phi(\mathcal{X})$ (see Corollary 2.11) and $\mathrm{ind}(T-\lambda) \geq 0$. Hence $\mathrm{ind}(T-\lambda) = 0 \Rightarrow \mathrm{asc}(T-\lambda) = \mathrm{dsc}(T-\lambda) < \infty$ [3, Corollary 2.10] $\Rightarrow$ $\lambda \notin \sigma_{ab}(T^*) \Rightarrow \sigma_{ab}(T^*) \subseteq \sigma_{sw}(T) = \sigma_{aw}(T^*)$. But then $\sigma_{aw}(T^*) = \sigma_{ab}(T^*) \Rightarrow$ T^* satisfies a-Browder's theorem. ∎

WEYL'S THEOREM. An operator $T \in B(\mathcal{X})$ satisfies Weyl's theorem if and only if T has SVEP at points $\lambda \notin \sigma_w(T)$ and $\pi_0(T) = \pi_{00}(T)$ [11, Theorem 2.3]. Combining this with Theorem 2.9 and Corollary 2.3, it follows that operators $T \in \mathcal{CHN}$ satisfy Weyl's theorem. Furthermore, since $\sigma(T^*) = \sigma(T)$, $\sigma_w(T^*) = \sigma_w(T)$, T^* has SVEP at points $\lambda \notin \sigma_w(T^*)$ (by Theorem 2.9), and since $\lambda \in \pi_{00}(T^*) \Rightarrow \lambda \in \mathrm{iso}\,\sigma(T) \Rightarrow \lambda \in \pi_0(T) = \pi_0(T^*) = \pi_{00}(T) \Rightarrow \pi_{00}(T^*) = \pi_{00}(T)$, T^* satisfies Weyl's theorem. More is true:

COROLLARY 2.16. *If $T \in \mathcal{CHN}$, then $f(T)$ and $f(T^*)$ satisfy Weyl's theorem for every $f \in \mathcal{H}(T)$.*

Proof. Both T and T^* being isoloid (by Proposition 2.1), $f(T)$ and $f(T^*)$ satisfy Weyl's theorem by [31, Theorem2]. ∎

As earlier observed, a-Weyl's theorem for a $T \in B(\mathcal{X})$ implies Weyl's theorem for T, but the reverse implication fails in general. Even the hypothesis $T \in \mathcal{CHN}$ has SVEP is not sufficient for T to satisfy a-Weyl's theorem. The (forward) unilateral shift $T \in B(\ell^2) \cap \mathcal{CHN}$ has SVEP and satisfies $\sigma_a(T) =$ the boundary of the unit disc $\mathbb{D}$, $\sigma_{aw}(T) = \mathbb{D}$ and $\pi_{a0}(T) = \emptyset$; evidently, $\sigma_a(T) \setminus \sigma_{aw}(T) \neq \pi_{a0}(T)$. Observe however that T^* satisfies a-Weyl's theorem: the following theorem shows that this phenomenon persists for operators $T \in \mathcal{CHN}$.

THEOREM 2.17. *Let $T \in \mathcal{CHN}$. If T (resp. T^*) has SVEP at points $\lambda \in \sigma_{aw}(T)$, then $f(T^*)$ (resp., $f(T)$) satisfies a-Weyl's theorem.*

Proof. Evidently, both $f(T)$ and $f(T^*)$ satisfy Weyl's theorem. The hypothesis on T (resp., T^*) implies that T (resp., T^*) has SVEP (see Theorem 2.9), and this in turn implies that $f(T)$ (resp., $f(T^*) = f(T^*)$) has SVEP [22, Theorem 3.3.6]. The proof now follows from [2, Theorem 3.6], which says that if $f(T)$ (resp., $f(T^*)$) has SVEP, then Weyl's theorem holds for $f(T^*)$ (resp., $f(T)$) if and only if a-Weyl's theorem holds for $f(T^*)$ (resp., $f(T)$). ∎

We end this paper with a necessary and sufficient condition for operators $T \in B(\mathcal{X})$, which satisfy a-Browder's theorem, to satisfy a-Weyl's theorem. (Recall from Corollary 2.15 that $\mathcal{CHN}$ operators satisfy a-Browder's theorem.) The following lemma will be required.

LEMMA 2.18. *$T \in B(\mathcal{X})$ satisfies a-Browder's theorem if and only if T has SVEP at all points $\lambda \in \sigma_a(T) \setminus \sigma_{aw}(T)$.*

Proof. The necessity is trivial: if T satisfies a-Browder's theorem, then $\sigma_{aw}(T) = \sigma_{ab}(T)$, and if $\lambda \in \sigma_a(T) \setminus \sigma_{ab}(T)$, then $\operatorname{asc}(T - \lambda) < \infty \Rightarrow T$ has SVEP at λ. For the sufficiency, assume that T has SVEP at every $\lambda \in \sigma(T) \setminus \sigma_{aw}(T)$. We prove that $\sigma_{ab}(T) \subseteq \sigma_{aw}(T)$. This, since $\sigma_{aw}(T) \subseteq \sigma_{ab}(T)$ for every $T \in B(\mathcal{X})$, would then imply $\sigma_{ab}(T) = \sigma_{aw}(T)$ (and hence that a-Browder's theorem holds for T). Suppose that $\lambda \notin \sigma_{aw}(T)$. Then $T - \lambda \in \Phi_+(\mathcal{X})$ and $\operatorname{ind}(T - \lambda) \leq 0$. Since $T - \lambda \in \Phi_+(\mathcal{X})$ implies $T - \lambda$ is Kato type, and since T has SVEP at λ, it follows from an application of [3, Theorem 2.6] that $\operatorname{asc}(T - \lambda) < \infty$. Hence $\lambda \notin \sigma_{ab}(T)$. ∎

THEOREM 2.19. *$T \in B(\mathcal{X})$ satisfies a-Weyl's theorem if and only if T satisfies a-Browder's theorem and $T - \lambda$ is Kato type at every $\lambda \in \pi_{a0}(T)$.*

Proof. If T satisfies a-Weyl's theorem, then T satisfies a-Browder's theorem and $\sigma_a(T) \setminus \sigma_{ab}(T) = \sigma_a(T) \setminus \sigma_{aw}(T) = \pi_{a0}(T)$. Again, since $\pi_{a0}(T) = \sigma_a(T) \setminus \sigma_{ab}(T) \Rightarrow T - \lambda \in \Phi_+(\mathcal{X})$ at points $\lambda \in \pi_{a0}(T)$, $T - \lambda$ is Kato type at points $\lambda \in \pi_{a0}(T)$. For the sufficiency, we start by observing that $\sigma_{ab}(T) = \sigma_{aw}(T)$ and T has SVEP at points $\lambda \in \sigma_a(T) \setminus \sigma_{aw}(T)$ (by Lemma 2.18). We prove next that $\sigma_a(T) \setminus \sigma_{ab}(T) = \pi_{a0}(T)$. Let $\lambda \in \sigma_a(T) \setminus \sigma_{ab}(T)$. Then $T - \lambda \in \Phi_+(\mathcal{X})$ and $\operatorname{asc}(T - \lambda) < \infty$, so that $T - \lambda$ is Kato type and $\operatorname{asc}(T - \lambda) < \infty$. Hence $\lambda \notin \operatorname{acc} \sigma_a(T)$ [2, Theorem 2.3]. Obviously, $0 < \alpha(T - \lambda) < \infty$; hence $\lambda \in \pi_{a0}(T) \Rightarrow \sigma_a(T) \setminus \sigma_{ab}(T) \subseteq \pi_{a0}(T)$. For the reverse inclusion, let $\lambda \in \pi_{a0}(T)$. Then (by hypothesis) $T - \lambda$ is Kato type, i.e., $\mathcal{X} = (T-\lambda)^{-p}(0) \oplus M$ for some integer $p \geq 1$ and some subspace M of $\mathcal{X}$ such that $T|_M$ is semi-regular. Since $\alpha(T-\lambda) < \infty$, $\dim(T-\lambda)^{-p}(0) < \infty$ (which implies that $T - \lambda$ is *essentially semi-regular*). Since $\sigma_a(T)$ does not cluster at λ, T has SVEP at λ (and $\operatorname{asc}(T-\lambda) < \infty$) [3]. Hence $T-\lambda \in \Phi_+(\mathcal{X})$ and $\operatorname{asc}(T-\lambda) < \infty$. Conclusion: $\lambda \notin \sigma_{ab}(T) \Rightarrow \lambda \in \sigma_a(T) \setminus \sigma_{ab}(T)$. ∎

Corollary 2.20. *$T \in \mathcal{CHN}$ satisfy a-Weyl's theorem if and only if $T - \lambda$ is Kato type at points $\lambda \in \pi_{a0}(T)$.*

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