EXTRACTA MATHEMATICAE Vol. **21**, Núm. 1, 51 – 65 (2006)

Generalized a-Weyl's Theorem and the Single-Valued Extension Property

Mohamed Amouch

Department of Mathematics, Faculty of Science, Semlalia
B.O. 2390 Marrakesh, Morocco
e-mail: meamouch@hotmail.com

(Presented by M. Mbekhta)

AMS *Subject Class.* (2000): 47A10, 47A20, 47A53 *Received October* 22, 2005

1. Introduction and definitions

Throughout this paper, $\mathcal{L}(X)$ denote the algebra of all bounded linear operators acting on a Banach space X. For $T \in \mathcal{L}(X)$, let T^*, $N(T)$, $R(T)$, $\sigma(T)$, $\sigma_p(T)$ and $\sigma_{ap}(T)$ denote respectively the adjoint, the null space, the range, the spectrum, the point spectrum and the approximate point spectrum of T. Let $\alpha(T)$ and $\beta(T)$ be the nullity and the deficiency of T defined by

$$\alpha(T) = \dim N(T) \qquad \text{and} \qquad \beta(T) = \operatorname{codim} R(T)\,.$$

If the range $R(T)$ of T is closed and $\alpha(T) < \infty$ (resp. $\beta(T) < \infty$), then T is called an *upper semi-Fredholm* (resp. a *lower semi-Fredholm*) operator. In the sequel $SF_+(X)$ (resp. $SF_-(X)$) will denote the set of all upper (resp. lower) semi-Fredholm operators. If $T \in \mathcal{L}(X)$ is either upper or lower semi-Fredholm, then T is called a *semi-Fredholm* operator, and the *index* of T is defined by $\operatorname{ind}(T) = \alpha(T) - \beta(T)$. If both $\alpha(T)$ and $\beta(T)$ are finite, then T is a *Fredholm* operator. An operator T is called *Weyl* if it is Fredholm of index zero. For $T \in \mathcal{L}(X)$ and $n \in \mathbb{N}$ define $c_n(T)$ and $c'_n(T)$ as follows $c_n(T) = \dim R(T^n)/R(T^{n+1})$ and $c'_n(T) = \dim N(T^{n+1})/N(T^n)$. The *descent* $q(T)$ and the *ascent* $p(T)$ are given by

$$q(T) = \inf\big\{n \,:\, c_n(T) = 0\big\} = \inf\big\{n \,:\, R\big(T^n\big) = R\big(T^{n+1}\big)\big\}\,,$$
$$p(T) = \inf\big\{n \,:\, c'_n(T) = 0\big\} = \inf\big\{n \,:\, N\big(T^n\big) = N\big(T^{n+1}\big)\big\}\,.$$

Key words: semi-B-Fredholm operator, generalized a-Weyl's theorem, single-valued extension property.

(We shall, henceforth, shorten $T-\lambda I$ to $T-\lambda$). A bounded linear operator T is called *Browder* if it is Fredholm of finite ascent and descent. The essential spectrum $\sigma_e(T)$, Weyl spectrum $\sigma_w(T)$, and Browder spectrum $\sigma_b(T)$ of $T \in \mathcal{L}(X)$ are defined by

$$\sigma_e(T) = \{\lambda \in \mathbb{C} : T-\lambda \text{ is not Fredholm}\},$$
$$\sigma_w(T) = \{\lambda \in \mathbb{C} : T-\lambda \text{ is not Weyl}\},$$
$$\sigma_b(T) = \{\lambda \in \mathbb{C} : T-\lambda \text{ is not Browder}\}.$$

Evidently

$$\sigma_e(T) \subseteq \sigma_w(T) \subseteq \sigma_b(T).$$

For a subset $K \subseteq \mathbb{C}$, we write $\operatorname{acc} K$ (resp. $\operatorname{iso} K$) for the accumulation (resp. isolated) points of K.

We say that *Weyl's theorem* holds for $T \in \mathcal{L}(X)$ if

$$\sigma(T) \setminus \sigma_w(T) = E_0(T),$$

where $E_0(T)$ is the set of isolated point of $\sigma(T)$ which are eigenvalues of finite multiplicity, and that *Browder's theorem* holds for $T \in \mathcal{L}(X)$ if

$$\sigma_w(T) = \sigma_b(T).$$

For $T \in \mathcal{L}(X)$, let $SF_+^-(X)$ be the class of all $T \in SF_+(X)$ with $\operatorname{ind} T \leq 0$. The *essential approximate point spectrum* $\sigma_{SF_+^-}(T)$ and the *Browder essential approximate point spectrum* $\sigma_{ab}(T)$ are defined by

$$\sigma_{SF_+^-}(T) = \left\{\lambda \in \mathbb{C} : T-\lambda \text{ is not in } SF_+^-(X)\right\},$$
$$\sigma_{ab}(T) = \bigcap\{\sigma_{ap}(T+K) : TK = KT \text{ and } K \in \mathcal{K}(X)\},$$

where $\mathcal{K}(X)$ is the ideal of compact operators on X. Recall that [25] a complex number λ is not in $\sigma_{ab}(T)$ if and only if $T-\lambda \in SF_+^-(X)$ and $p(T-\lambda) < \infty$. We say that *a-Wey's theorem* holds for $T \in \mathcal{L}(X)$ if

$$\sigma_{ap}(T) \setminus \sigma_{SF_+^-}(T) = E_0^a(T),$$

where $E_0^a(T)$ is the set of isolated points of $\sigma_{ap}(T)$ which are eigenvalues of finite multiplicity, and that *a-Browder's theorem* holds for $T \in \mathcal{L}(X)$ if

$$\sigma_{SF_+^-}(T) = \sigma_{ab}(T).$$

In [9, 26], it is shown that for any $T \in \mathcal{L}(X)$ we have the implications:

$$\text{a-Weyl's theorem} \;\Rightarrow\; \text{Weyl's theorem} \;\Rightarrow\; \text{Browder's theorem}\,,$$

$$\text{a-Weyl's theorem} \;\Rightarrow\; \text{a-Browder's theorem} \;\Rightarrow\; \text{Browder's theorem}\,.$$

For a bounded linear operator T and a nonnegative integer n define T_n to be the restriction of T to $R(T^n)$ viewed as a map from $R(T^n)$ into $R(T^n)$ (in particular $T_0 = T$). If for some integer n the range space $R(T^n)$ is closed and T_n is an upper (resp. a lower) semi-Fredholm operator, then T is called an *upper* (resp. *lower*) *semi-B-Fredholm* operator, see [7]. In this case the *index* of T is defined as the index of the semi-Fredholm operator T_n, see [6]. Moreover if T_n is a Fredholm operator, then T is called a *B-Fredholm* operator. A *semi-B-Fredholm* operator is an upper or a lower semi-B-Fredholm operator. An operator $T \in \mathcal{L}(X)$ is said to be a *B-Weyl operator* if it is a B-Fredholm operator of index zero. The *semi-B-Fredholm spectrum* $\sigma_{SBF}(T)$ and the *B-Weyl spectrum* $\sigma_{BW}(T)$ of T are defined by

$$\sigma_{SBF}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not a semi-B-Fredholm operator}\}\,,$$
$$\sigma_{BW}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not a B-Weyl operator}\}\,.$$

We say that *generalized Weyl's theorem* holds for T if

$$\sigma(T) \setminus \sigma_{BW}(T) = E(T)\,,$$

where $E(T)$ is the set of all isolated eigenvalues of T, and *generalized Browder's theorem* holds for T if

$$\sigma(T) \setminus \sigma_{BW}(T) = \pi(T)\,,$$

where $\pi(T)$ is the set of all poles of T(see [6, Definition 2.13]). Generalized Weyl's theorem and generalized Browder's theorem has been studied in [5, 6]. Similarly, let $SBF_+(X)$ be the class of all upper semi-B-Fredholm operators, and $SBF_+^-(X)$ the class of all $T \in SBF_+(X)$ such that $\text{ind}(T) \le 0$. Also let

$$\sigma_{SBF_+^-}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not in } SBF_+^-(X)\}\,,$$

called the *semi-essential approximate point spectrum*, see [6]. We say that T obeys *generalized a-Weyl's theorem* if

$$\sigma_{SBF_+^-}(T) = \sigma_{ap}(T) \setminus E^a(T)\,,$$

where $E^a(T)$ is the set of all eigenvalues of T which are isolated in $\sigma_{ap}(T)$ ([6, Definition 2.13]). From [6], we know that

$$\text{generalized a-Weyl's theorem} \;\Rightarrow\; \text{generalized Weyl's theorem} \;\Rightarrow\; \text{Weyl's theorem}\,,$$

$$\text{generalized a-Weyl's theorem} \;\Rightarrow\; \text{a-Weyl's theorem}\,.$$

For $T \in \mathcal{L}(X)$ we say that T is *Drazin invertible*, if there exists $B, U \in \mathcal{L}(X)$ such that U is nilpotent and $TB = BT$, $BTB = B$ and $TBT = T + U$. It is known that T is Drazin invertible if and only if it has finite ascent and descent, which is also equivalent to the fact that $T = T_0 \oplus T_1$, where T_0 is invertible and T_1 is nilpotent, see [14, Proposition A] and [17, Corollary 2.2]. The Drazin spectrum is defined by

$$\sigma_D(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Drazin invertible } \}\,.$$

As in [21], define the set $LD(X)$ by

$$LD(X) = \Big\{T \in \mathcal{L}(X) \,:\, p(T) < \infty \text{ and } R\Big(T^{p(T)+1}\Big) \text{ is closed}\Big\}\,.$$

An operator $T \in \mathcal{L}(X)$ is said to be *left Drazin invertible* if $T \in LD(X)$. The left Drazin spectrum $\sigma_{LD}(T)$ of T is defined by

$$\sigma_{LD}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not in } LD(X)\}\,.$$

It is known, see [6, Lemma 2.12], that

$$\sigma_{SBF_+^-}(T) \subseteq \sigma_{LD}(T) \subseteq \sigma_{ap}(T)\,.$$

We say that $\lambda \in \sigma_{ap}(T)$ is a *left pole* of T if $T - \lambda \in LD(X)$, and that $\lambda \in \sigma_{ap}(T)$ is a left pole of T of finite rank if λ is a left pole of T and $\alpha(T - \lambda) < \infty$. We denote by $\pi^a(T)$ the set of all left poles of T, and by $\pi_0^a(T)$ the set of all left poles of finite rank. We say that T obeys *generalized a-Browder's theorem* if

$$\sigma_{SBF_+^-}(T) = \sigma_{ap}(T) \setminus \pi^a(T)\,.$$

It is known [6], that

$$\text{generalized a-Browder's theorem} \;\Rightarrow\; \text{a-Browder's theorem}\,,$$

$$\text{generalized a-Browder's theorem} \;\Rightarrow\; \text{generalized Browder's theorem}\,.$$

Generalized a-Weyl's theorem has been studied in [6]. In particular it is shown that generalized a-Weyl's theorem implies generalized a-Browder's theorem. It has been established for operator T on a Hilbert space for which the adjoint T^* is p-hyponormal or M-hyponormal [8]. In this paper, we study generalized a-Weyl's theorem and generalized a-Browder's theorem for operator T acting on a Banach space such that T or T^* has the SVEP. In section 2, we prove that the spectral mapping theorem holds for the semi-essential approximate point spectrum $\sigma_{SBF_+^-}(T)$, for every $f \in H(\sigma(T))$, where $H(\sigma(T))$ denotes the set of all analytic functions defined on an open neighbourhood U of $\sigma(T)$. In section 3, we show that if T is a bounded linear operator such that T^* has the SVEP, then T satisfies generalized a-Weyl's theorem if and only if it satisfies generalized Weyl's theorem, and we show that generalized a-Browder's theorem holds for $f(T)$ for every $f \in H(\sigma(T))$, also we give a necessary and sufficient condition for T to obey generalized Weyl's theorem. One class of operators which was introduced in [23], is the class $\mathcal{P}(X)$ of all operators $T \in \mathcal{L}(X)$ for which for every complex number λ there exists an integer $d_\lambda \geq 1$ such that the following condition holds

$$H_0(T-\lambda) = N(T-\lambda)^{d_\lambda}.$$

In section 4, we give an application for the class $\mathcal{P}(X)$.

2. Spectral mapping theorem for the semi-essential approximate point spectrum

We say that $T \in \mathcal{L}(X)$ has the *single-valued extension property* at λ_0, (SVEP for short) if for every open neighbourhood U of λ_0, the only analytic function $f : U \to X$ which satisfies the equation

$$(T-\lambda)f(\lambda) = 0$$

for all $\lambda \in U$ is the function $f \equiv 0$. $T \in \mathcal{L}(X)$ is said to have the SVEP if T has the SVEP at every point $\lambda \in \mathbb{C}$ (see [16]).

Recall that the Drazin spectrum $\sigma_{LD}(T)$, $T \in \mathcal{L}(X)$ satisfies the spectral mapping theorem for analytic functions on an open neighbourhood of $\sigma(T)$ which is non-constant on each component of its domain of definition, see [22] page 194. In this section we will show that under the hypothesis T or T^* has the SVEP; the spectral mapping theorem holds for the semi-essential approximate point spectrum $\sigma_{SBF_+^-}(T)$, for every analytic functions on an open neighbourhood of $\sigma(T)$.

We start with the following:

Proposition 2.1. *Let $S, T, A, B \in \mathcal{L}(X)$ be mutually commuting operators, satisfying $TA + BS = I$. Then $TS \in SBF_+(X)$ if and only if $T, S \in SBF_+(X)$.*

Proof. The result follows from [21, Lemma 1] and [21, Lemma 8]. ∎

As an immediate consequence of the previous proposition we have the following:

Corollary 2.1. *Let $P(X) = (X - \lambda_1)^{n_1} \cdots (X - \lambda_m)^{n_m}$ be a polynomial with complex coefficients. Then $P(T) = (T - \lambda_1)^{n_1} \cdots (T - \lambda_m)^{n_m}$ is a upper semi-B-Fredholm operator if and only if $T - \lambda_i$ is a upper semi-B-Fredholm operator for all $i \in \{1, \ldots, m\}$.*

Proof. Since for all relatively prime polynomials P, Q there exist polynomials P_1, Q_1 such that $PP_1 + QQ_1 = 1$, we have $P(T)P_1(T) + Q(T)Q_1(T) = I$. From Proposition 2.1 applied inductively for a relatively prime polynomials $(X - \lambda_i)$ and $(X - \lambda_j)$ where $i, j \in \{1, \ldots, m\}$, we get the desired result. ∎

Let $A(X)$ be the set of all $T \in \mathcal{L}(X)$ such that

$$\mathrm{ind}(T - \lambda)\ \mathrm{ind}(T - \mu) \geq 0 \qquad \text{for all } \lambda, \mu \in \mathbb{C} \setminus \sigma_{SBF_+}(T)\,.$$

Theorem 2.1. *If $T \in A(X)$, then*

$$f(\sigma_{SBF_+^-}(T)) = \sigma_{SBF_+^-}(f(T)) \qquad \textit{for every } f \in H(\sigma(T))\,.$$

Proof. Let $\mu \in \sigma_{SBF_+^-}(f(T))$. Since $f - \mu$ has only a finite number of zeros $\lambda_1, \lambda_2, \ldots, \lambda_m$ in $\sigma(T)$, it can be written as $f(z) - \mu = (z - \lambda_1)^{n_1} \cdots (z - \lambda_m)^{n_m} g(z)$, where g is a function analytic on a neighbourhood of $\sigma(T)$ and $g(z) \neq 0$ for $z \in \sigma(T)$. Then by the spectral mapping theorem for the ordinary spectrum we have $f(T) - \mu = (T - \lambda_1)^{n_1} \cdots (T - \lambda_m)^{n_m} g(T)$, and $g(T)$ is invertible. If $\mu \notin f(\sigma_{SBF_+^-}(T))$, then $\lambda_i \notin \sigma_{SBF_+^-}(T)$ for all $i \in \{1, \ldots, m\}$, because if there exists $\lambda_j \in \sigma_{SBF_+^-}(T)$, $j \in \{1, \ldots, \}$, then $f(\lambda_j) - \mu = 0$, hence $\mu = f(\lambda_j) \in f(\sigma_{SBF_+^-}(T))$. By Corollary 2.1, this implies that $(T - \lambda_1)^{n_1} \cdots (T - \lambda_m)^{n_m}$ is upper semi-B-Fredholm. Hence, from Proposition 2.1 applied for $(T - \lambda_1)^{n_1} \cdots (T - \lambda_m)^{n_m}$ and $g(T)$, we get that $f(T) - \mu$ is upper semi-B-Fredholm. Since $\mathrm{ind}(f(T) - \mu) = \sum_{i=1}^{m} \mathrm{ind}(T - \lambda_i)^{n_i} \leq 0$, then μ is not in $\sigma_{SBF_+^-}(f(T))$. Which is a contradiction. Thus

$$\sigma_{SBF_+^-}(f(T)) \subseteq f(\sigma_{SBF_+^-}(T))\,.$$

Conversely, suppose that $\mu \notin \sigma_{SBF_+^-}(f(T))$. That is $f(T)-\mu$ is upper semi-B-Fredholm and $\operatorname{ind}(f(T)-\mu) \leq 0$. By Proposition 2.1 applied for $f(T)-\mu$ and $g(T)^{-1}$, we conclude that $(T-\lambda_1)^{n_1}\cdots(T-\lambda_m)^{n_m}$ is upper semi-B-Fredholm. Hence, by Corollary 2.1, we get that $T-\lambda_i$ is upper semi-B-Fredholm for $i \in \{1,\dots,m\}$. Since $\operatorname{ind}(f(T)-\mu) = \sum_{i=1}^m \operatorname{ind}(T-\lambda_i)^{n_i} \leq 0$ and $T \in A(X)$, then $\operatorname{ind}(T-\lambda_i) \leq 0$ for $i \in \{1,\dots,m\}$. So $T-\lambda_i \in SBF_+^-(X)$. Thus $\mu \notin f(\sigma_{SBF_+^-}(T))$. ∎

For $T \in \mathcal{L}(X)$, let $\rho_{SBF}(T) = \mathbb{C} \setminus \sigma_{SBF}(T)$. In the following proposition we prove that if T or T^* has the SVEP, then $T \in A(X)$.

Proposition 2.2. *Let T be a bounded linear operator on X.*

(i) *If T has the SVEP, then* $\operatorname{ind}(T-\lambda) \leq 0$ *for all* $\lambda \in \rho_{SBF}(T)$.

(ii) *If T^* has the SVEP, then* $\operatorname{ind}(T-\lambda) \geq 0$ *for all* $\lambda \in \rho_{SBF}(T)$.

Proof. (i) Let $\lambda \in \rho_{SBF}(T)$, then $T-\lambda$ is semi-B-Fredholm. By [7, Corollary 3.2], for $\mu \in \mathbb{C}$ such that $|\lambda-\mu|$ is small enough we have $T-\mu$ is semi-Fredholm and $\operatorname{ind}(T-\lambda) = \operatorname{ind}(T-\mu)$. If T has the SVEP, then from [3, Corollary 2.7] we deduce that $\operatorname{ind}(T-\mu) \leq 0$, and hence $\operatorname{ind}(T-\lambda) \leq 0$. Which proves (i).

(ii) Suppose that T^* has the SVEP, then from [3, Corollary 2.7] we get that $\operatorname{ind}(T-\mu) \geq 0$, and hence $\operatorname{ind}(T-\lambda) \geq 0$. ∎

As an immediate consequence of Theorem 2.1 and Proposition 2.2 we obtain the following result.

Corollary 2.2. *Let $T \in \mathcal{L}(X)$. If T or T^* has the SVEP, then*

$$f(\sigma_{SBF_+^-}(T)) = \sigma_{SBF_+^-}(f(T)) \qquad \text{for every } f \in H(\sigma(T))\,.$$

3. Generalized a-Weyl's theorem and the SVEP.

Let $T \in \mathcal{L}(X)$ and $d \in \mathbb{N}$. Then T has a *uniform descent* for $n \geq d$ if

$$R(T) + N(T^n) = R(T) + N(T^d) \qquad \text{for all } n \geq d\,.$$

If in addition, $R(T)+N(T^d)$ is closed then T is said to have a topological uniform descent for $n \geq d$, see [12].

For an operator $T \in \mathcal{L}(X)$, we denote by $F^a(T)$ the set of all isolated points λ of $\sigma_{ap}(T)$ for which $T-\lambda$ is semi-B-Fredholm.

The following Proposition was established in [6], however the arguments used are different.

Proposition 3.1. *Let $T \in \mathcal{L}(X)$. The following assertions hold.*

(i) $F^a(T) = \pi^a(T)$, *and hence* $\sigma_{LD}(T) = \sigma_{ap}(T) \setminus F^a(T) = \operatorname{acc}\sigma_{ap}(T) \cup \sigma_{SBF}(T)$.

(ii) *If generalized a-Weyl's theorem holds for T, then so does generalized a-Browder's theorem.*

(iii) *If T satisfies generalized a-Browder's theorem, then T satisfies generalized a-Weyl's theorem if and only if $F^a(T) = E^a(T)$.*

Proof. (i) If $\lambda \in F^a(T)$, then $T - \lambda$ is semi-B-Fredholm and λ is isolated in $\sigma_{ap}(T)$, in particular $T - \lambda$ is an operator of topological uniform descent for $n \geq d$. Hence from [12, Theorem 4.7], if $|\beta - \lambda|$ is sufficiently small, then $c'_n(T - \beta) = c'_d(T - \lambda)$ for $n \geq d$. Since λ is isolated in $\sigma_{ap}(T)$, then we can choose β such that $\beta \notin \sigma_{ap}(T)$, and hence $T - \beta$ is injective. So $c'_d(T - \lambda) = 0$, that is $p(T - \lambda)$ is finite. On the other hand, by [21, Lemma 12], $R((T - \lambda)^{p(T-\lambda)+1})$ is closed. This implies that $\lambda \notin \sigma_{LD}(T)$. That is $\lambda \in \pi^a(T)$. Thus $F^a(T) \subseteq \pi^a(T)$. For the reverse inclusion suppose that $\lambda \in \pi^a(T)$, then λ is isolated in $\sigma_{ap}(T)$, see [6, Remark 2.7]. Also, $\lambda \in \pi^a(T)$ implies that $\lambda \notin \sigma_{LD}(T)$, and hence $\lambda \notin \sigma_{SBF_+^-}(T)$. So $T - \lambda$ is semi-B-Fredholm. This implies that, $\lambda \in F^a(T)$. So $\pi^a(T) \subseteq F^a(T)$, and hence $\pi^a(T) = F^a(T)$. Since $\sigma_{LD}(T) = \sigma_{ap}(T) \setminus \pi^a(T)$, then $\sigma_{LD}(T) = \sigma_{ap}(T) \setminus F^a(T) = \operatorname{acc}\sigma_{ap}(T) \cup \sigma_{SBF}(T)$. This gives the proof of (i).

(ii) Suppose that generalized a-Weyl's theorem holds for T, that is $\sigma_{SBF_+^-}(T) = \sigma_{ap}(T) \setminus E^a(T)$. Since

$$E^a(T) \cap \sigma_{SBF}(T) \subseteq E^a(T) \cap \sigma_{SBF_+^-}(T) = \emptyset\,,$$

then

$$E^a(T) \subseteq \operatorname{iso}\sigma_{ap}(T) \cap \rho_{SBF}(T) = F^a(T)\,.$$

Thus $E^a(T) \subseteq F^a(T)$. Since we have always $F^a(T) \subseteq E^a(T)$, then $E^a(T) = F^a(T)$ and $\sigma_{SBF_+^-}(T) = \sigma_{ap}(T) \setminus F^a(T)$, hence by (i) we conclude that T satisfies generalized a-Browder's theorem.

(iii) Suppose that T satisfies generalized a-Weyl's theorem. If $\lambda \in E^a(T)$, then $\lambda \in \sigma_{ap}(T) \setminus \sigma_{SBF_+^-}(T)$. Since T satisfies generalized a-Browder's thoerem then $\lambda \in \pi^a(T)$. Hence by (i) $\lambda \in F^a(T)$. Thus $E^a(T) \subseteq F^a(T)$ and therefore

$F^a(T) = E^a(T)$. Conversely Assume that $E^a(T) = F^a(T)$. Since T satisfies generalized a-Browder's theorem, then

$$\begin{aligned}\sigma_{SBF_+^-}(T) &= \sigma_{ap}(T) \setminus \pi^a(T) \\ &= \sigma_{ap}(T) \setminus F^a(T) \quad \text{(by (i))} \\ &= \sigma_{ap}(T) \setminus E^a(T)\,.\end{aligned}$$

Hence T satisfies generalized a-Weyl's theorem. ∎

As mentioned above, generalized a-Weyl's theorem implies generalized Weyl's theorem ([6]). In the following we give a sufficient condition to get the reverse implication.

THEOREM 3.1. *Let $T \in \mathcal{L}(X)$.*

(i) *If T^* has the SVEP, then T satisfies generalized a-Weyl's theorem if and only if it satisfies generalized Weyl's theorem.*

(ii) *If T has the SVEP, then T^* satisfies generalized a-Weyl's theorem if and only if it satisfies generalized Weyl's theorem.*

Proof. (i) Suppose that T^* has the SVEP, then by [16, Proposition 1.3.2], we have $\sigma_{ap}(T) = \sigma(T)$, and hence $E(T) = E^a(T)$. If T satisfies generalized Weyl's theorem, then $\sigma_{BW}(T) = \sigma(T) \setminus E(T)$. To prove that generalized a-Weyl theorem holds for T, it suffices to show that $\sigma_{BW}(T) = \sigma_{SBF_+^-}(T)$. For this, suppose that $\lambda \notin \sigma_{SBF_+^-}(T)$, then $T - \lambda$ is an upper semi-B-Fredholm operator and $\text{ind}(T - \lambda) \leq 0$. By Proposition 2.2, we have that $\text{ind}(T - \lambda) \geq 0$. So $\text{ind}(T - \lambda) = 0$. Which implies that $T - \lambda$ is semi-B-Fredholm of index 0, hence $T - \lambda$ is B-Fredholm of index 0, that is $\lambda \notin \sigma_{BW}(T)$. This gives $\sigma_{SBF_+^-}(T) \supseteq \sigma_{BW}(T)$. The other inclusion is always true. So $\sigma_{SBF_+^-}(T) = \sigma_{BW}(T)$. Since the revers implication is known[6], then the equivalence between Weyl's theorem and a-Weyl's theorem holds for T.

(ii) Outlines the proof of the first statement. ∎

In general, we cannot expect that generalized a-Weyl's theorem holds for operators satisfying the SVEP. Let T defined on l_2 by

$$T(x_1, x_2, \dots) = \Big(\frac{1}{2}\, x_2, \frac{1}{3}\, x_3, \dots\Big)\,.$$

Then T has the SVEP and $\sigma(T) = \sigma_{SBF_+^-}(T) = E^a(T) = \{0\}$. Thus T does not obey generalized a-Weyl's theorem. However, generalized a-Browder's theorem holds for T whenever T or T^* has the SVEP as shown by the followingf:

Theorem 3.2. *If T or T^* has the SVEP, then generalized a-Browder's theorem holds for $f(T)$ for every $f \in H(\sigma(T))$.*

Proof. Suppose that $\lambda \in \sigma_{ap}(T) \setminus \sigma_{SBF_+^-}(T)$, then $T - \lambda$ is upper semi-B-Fredholm and $\operatorname{ind}(T-\lambda) \leq 0$. The operator $T - \lambda$ has a topological uniform descent, so from [7, Corollary 3.2], if β is in $\sigma_{ap}(T)$ such that $|\beta - \lambda|$ is sufficiently small, then $T-\beta$ is upper semi-Fredholm operator and $\operatorname{ind}(T-\beta) \leq 0$. Hence $\beta \in \sigma_{ap}(T) \setminus \sigma_{SF_+^-}(T)$. By [24, Proposition 2.4], The SVEP for T or T^* implies that a-Browder's theorem holds for T that is $\sigma_{SF_+^-}(T) = \sigma_{ap}(T) \setminus \pi_0^a(T)$, hence $\beta \in \pi_0^a(T)$. This implies that $p(T-\beta) < \infty$, by [12, Theorem 4.7] $p(T-\lambda) < \infty$. Now, since $T-\lambda$ is semi-B-Fredholm, then there exists an integer n such that $R((T-\lambda)^n)$ is closed and $(T-\lambda) \mid_{R((T-\lambda)^n)}$ is Fredholm. We can assume that $n \geq p(T-\lambda)$, see the proof of Proposition 2.1 of [6]. Since we have $R(T-\lambda) + N((T-\lambda)^{i+1}) = R(T-\lambda) + N((T-\lambda)^i)$ for every $i \geq p(T-\lambda)$ and $R((T-\lambda)^n)$ is closed, then by [22, Lemma 17], we get that $R((T-\lambda)^{p(T-\lambda)+1})$ is closed. So $\lambda \in \pi^a(T)$. Thus

$$\sigma_{ap}(T) \setminus \sigma_{SBF_+^-}(T) \subseteq \pi^a(T)\,.$$

For the reverse inclusion. If $\lambda \in \pi^a(T)$, then by Proposition 3.1 (i), $\lambda \in F^a(T)$, that is λ is isolated in $\sigma_{ap}(T)$ and $T-\lambda$ is semi-B-Fredholm. From [7, Corollary 3.2], if we choose $\beta \in \mathbb{C}$ such that $|\lambda - \beta|$ is small enough and $\beta \notin \sigma_{ap}(T)$, then $T-\beta$ is upper semi-Fredholm with $\operatorname{ind}(T-\beta) \leq 0$. So $T-\lambda$ is an upper semi-B-Fredholm operator and $\operatorname{ind}(T-\lambda) \leq 0$, that is $\lambda \notin \sigma_{ap}(T) \setminus \sigma_{SBF_+^-}(T)$. Finally, $\sigma_{ap}(T) \setminus \sigma_{SBF_+^-} = \pi^a(T)$. Thus generalized a-Browder's theorem holds for T. To complete the proof, if $f \in H(\sigma(T))$, then by [16, Theorem 3.3.6], $f(T)$ or $f(T^*)$ has the SVEP. Similarly we get the result. ∎

Corollary 3.1. *If T or T^* has the SVEP, then generalized a-Weyl's theorem holds for T if and only if $F^a(T) = E^a(T)$.*

Proof. If T or T^* has the SVEP, then by the preceding theorem generalized a-Browder's theorem holds for T, and by (iii) of Proposition 3.1, we deduce the result. ∎

We have noted that $\sigma_{LD}(T)$ satisfies the spectral mapping theorem for analytic functions f on an open neighbourhood of $\sigma(T)$ which is non-constant on each component of its domain of definition. Following we show that under the hypothesis T or T^* has the SVEP; the condition that f is non-constant on each component of its domain of definition can be left out.

THEOREM 3.3. *If T or its adjoint T^* has the SVEP, then*

$$f(\sigma_{LD}(T)) = \sigma_{LD}(f(T)), \text{ for every } f \in H(\sigma(T)).$$

Proof. By Theorem 3.2, generalized a-Browder's theorem holds for T and $f(T)$, for every $f \in H(\sigma(T))$. So $\sigma_{LD}(T) = \sigma_{SBF_+^-}(T)$ and $\sigma_{LD}(f(T)) = \sigma_{SBF_+^-}(f(T))$. If T or T^* has the SVEP then by Corollary 2.2, we have $f(\sigma_{SBF_+^-}(T)) = \sigma_{SBF_+^-}(f(T))$. Hence

$$\begin{aligned} f(\sigma_{LD}(T)) &= f(\sigma_{SBF_+^-}(T)) \\ &= \sigma_{SBF_+^-}(f(T)) \\ &= \sigma_{LD}(f(T)). \end{aligned}$$

∎

4. APPLICATIONS

In this section we will study generalized a-Weyl's theorem and generalized a-Browder's theorem for some classes of operators. For this let us introduce some basic notions which will be used later.

The analytic core of an operator $T \in \mathcal{L}(X)$ is the subspace

$$\begin{aligned} K(T) := \Big\{ x \in X : \quad & Tx_{n+1} = x_n, \ Tx_1 = x, \ \|x_n\| \le c^n \|x\| \\ & \text{for some } c > 0 \ (n = 1, 2, \dots), \ x_n \in X \Big\}. \end{aligned}$$

The quasi-nilpotent part of T is the subspace

$$H_0(T) := \Big\{ x \in X : \lim_{n \to \infty} \|T^n x\|^{\frac{1}{n}} = 0 \Big\}.$$

The spaces $K(T)$ and $H_0(T)$ are hyperinvariant under T and satisfy $T^{-n}(0) \subset H_0(T)$, $K(T) \subset T^n(X)$ for all $n \in \mathbb{N}$, $TK(T) = K(T)$. For their further properties, see [18, 19].

The class of operators $T \in \mathcal{L}(X)$ for which $K(T) = \{0\}$ was introduced and studied by M. Mbekhta in [19]. It was shown that for such operators, the spectrum is connected and the SVEP holds.

THEOREM 4.1. *If there exists λ such that $K(T-\lambda) = \{0\}$, then $f(T)$ satisfies generalized a-Browder's theorem, for every $f \in H(\sigma(T))$. Moreover, if in addition $N(T-\lambda) = 0$, then generalized a-Weyl's theorem holds for $f(T)$.*

Proof. Since T has the SVEP, then by Theorem 3.2, generalized a- Browder's theorem holds for $f(T)$. Let $\alpha \in \sigma(f(T))$, then $f(z) - \alpha = \prod_{i=1}^{n}(z - \lambda_i)g(z)$, where $\lambda_1, \lambda_2, \ldots, \lambda_n \in \sigma(T)$ and g is an analytic function on an open neighbourhood of $\sigma(T)$, without zeros in $\sigma(T)$. Since $g(T)$ is invertible, then we deduce that

$$N(f(T) - \alpha) = N\left(\prod_{i=1}^{n}(T - \lambda_i)\right) = \bigoplus_{i=1}^{n} N(T - \lambda_i)\,.$$

On the other hand, from [19, Proposition 2.1], we get that $\sigma_p(T) \subseteq \{\lambda\}$. If we suppose that $N(T-\lambda) = \{0\}$, then $\sigma_p(T) = \emptyset$. Which implies that

$$N(f(T) - \lambda) = \{0\}.$$

That is $\sigma_p(f(T)) = \emptyset$. Thus, $F^a(f(T)) = E^a(f(T)) = \emptyset$. Since $f(T)$ satisfies generalized a-Browder's theorem, then by (iii) of Proposition 3.1, generalized a-Weyl's theorem holds for $f(T)$. Which completes the proof. ∎

Let $\mathcal{P}(X)$ be the set of all operators $T \in \mathcal{L}(X)$ such that for every complex number λ there exists an integer $d_\lambda \geq 1$ for which the following condition holds

$$H_0(T - \lambda) = N(T - \lambda)^{d_\lambda}.$$

It is known that if $H_0(T - \lambda)$ is closed for every complex number λ, then T has the SVEP, see [2, 15]. So that, the SVEP is shared by all the operators of $\mathcal{P}(X)$. The class of operators $\mathcal{P}(X)$ is considerably large, it contains, in particular, the classes consisting of generalized scalar, subscalar and algebraically totally paranormal operators on a Banach space, hyponormal, p-hyponormal $(0 < p < 1)$ and M-hyponormal operators on a Hilbert space (see [23]).

For p-hyponormal and M-hyponormal operators in Hilbert space, it is shown in [8] that generalized a-Weyl's theorem holds for $f(T)$ for every $f \in H(\sigma(T))$. In the following we will give more for Banach space operators.

THEOREM 4.2. *Let T a bounded operator on X. If there exists a function $h \in H(\sigma(T))$ non constant in any connected component of its domain, and such that $h(T^*) \in \mathcal{P}(X^*)$, then generalized a-Weyl's theorem holds for $f(T)$, for every $f \in H(\sigma(T))$.*

Proof. Suppose that $h(T^*) \in \mathcal{P}(X^*)$, then by [23, Theorem 3.4], we have $T^* \in \mathcal{P}(X^*)$. First, we will show that generalized a-Weyl's theorem holds for T. Since T^* has the SVEP, then by Corollary 3.1, it suffices to show that $F^a(T) = E^a(T)$. For this let $\lambda \in E^a(T)$, then λ is isolated eigenvalue of $\sigma_{ap}(T)$. Since T^* has the SVEP, then $\sigma_{ap}(T) = \sigma(T)$, see [16]. So $X^* = H_0(T^*-\lambda) \oplus K(T^*-\lambda)$, where the direct sum is topological. Since $T^* \in \mathcal{P}(X^*)$, then $H_0(T^* - \lambda) = N(T^* - \lambda)^d$ for some integer d, and hence $X^* = N(T^* - \lambda)^d \oplus K(T^* - \lambda)$. Since

$$(T^* - \lambda)^d(X^*) = (T^* - \lambda)^d K(T^* - \lambda) = K(T^* - \lambda)\,,$$

then $K(T^* - \lambda) = R(T^* - \lambda)^d$, and hence

$$X^* = N(T^* - \lambda)^d \oplus R(T^* - \lambda)^d.$$

So $T - \lambda \mid R(T-\lambda)^d$ is surjective. This implies that $T-\lambda$ is semi-B-Fredholm. So $E^a(T) \subseteq F^a(T)$. Since we have always that $F^a(T) \subseteq E^a(T)$, then $F^a(T) = E^a(T)$. Now, from [23, Theorem 3.4], if $T^* \in \mathcal{P}(X^*)$, then $f(T^*) \in \mathcal{P}(X^*)$ for every $f \in H(\sigma(T))$. Hence, by the same argument we conclude that generalized a-Weyl's theorem holds for $f(T)$ for every $f \in H(\sigma(T))$. ∎

As an easy consequence of the previous theorem, we have the following corollary.

COROLLARY 4.1. *If $T^* \in \mathcal{P}(X^*)$, then generalized a-Weyl's theorem holds for $f(T)$, for every $f \in H(\sigma(T)$.*

Following we give condition for $T \in \mathcal{P}(X)$ which forces $f(T)$ to obey generalized Weyl's theorem for $f \in H(\sigma(T).)$

THEOREM 4.3. *If $T \in \mathcal{P}(X)$ be such that $\sigma(T) = \sigma_{ap}(T)$, then generalized a-Weyl's theorem holds for $f(T)$, for every $f \in H(\sigma(T))$.*

Proof. Suppose that $T \in \mathcal{P}(X)$ and $\sigma(T) = \sigma_{ap}(T)$. First we will prove that generalized a-Weyl's theorem holds for T. Since T has the SVEP, then by Corollary 3.1, it suffices to show that $F^a(T) = E^a(T)$. For this let $\lambda \in E^a(T)$.

Then λ is isolated in $\sigma_{ap}(T) = \sigma(T)$. By [18, Theorem 1.6] $X = H_0(T-\lambda) \oplus K(T-\lambda)$, where the direct sum is topological. Since there exist an integer n such that $H_0(T-\lambda) = N(T-\lambda)^n$, then $X = N((T-\lambda)^n) \oplus K(T-\lambda)$. This implies that $(T-\lambda)^n(X) = (T-\lambda)K(T-\lambda) = K(T-\lambda)$. Thus

$$X = N((T-\lambda)^n) \oplus R((T-\lambda)^n)\,.$$

So $(T-\lambda)^n$ is Fredholm of index 0, and so is $T-\lambda$, see [13]. Hence $T-\lambda$ is B-Fredholm. Finally, $E^a(T) \subseteq F^a(T)$. The other inclusion is clear. Thus $E^a(T) = F^a(T)$. Similarly, we prove that $f(T)$ satisfies generalized Weyl's theorem, because $f(T) \in \mathcal{P}(X)$ and

$$\sigma(f(T)) = f(\sigma(T)) = f(\sigma_{ap}(T)) = \sigma_{ap}(f(T))\,.$$

∎

Acknowledgements

This work was done during the author's visit at the UFR of Mathmatics , Lille1 university, Villeneuve d'Ascq France. He would like to thank for the warm hospitality and perfect working conditions there. He is grateful to the referee for helpful suggestions concerning this paper

References

[1] Aiena, P., "Fredholm and Local Spectral Theory, with Applications to Multipliers", Kluwer Academic Publishers Dordrecht, 2004.

[2] Aiena, P., Colasante, M.L., González, M., Operators which have a closed quasi-nilpotent part, *Proc. Amer. Math. Soc.* **130** (2002), 2701–2710.

[3] Aiena, P., Monsalve, O., Operators which do not have the single valued extension property, *J. Math. Anal. Appl.* **250** (2000), 435–448.

[4] Berkani, M., On a class of quasi-Fredholm operators, *Integral Equations Operator Theory* **34** (1999), 244–249.

[5] Berkani, M., Arroud, A., Generalized Weyl's theorem and hyponormal operators, *J. Austral. Math. Soc.* **76** (2004), 291–302.

[6] Berkani, M., Koliha, J.J., Weyl type theorems for bounded linear operators, *Acta Sci. Math. (Szeged)* **69** (2003), 359–376.

[7] Berkani, M., Sarih, M., On semi B-Fredholm operators, *Glasg. Math. J.* **43** (2001), 457–465.

[8] Cao, X., Guo, M., Meng, B., Weyl type for p-hyponormal operators and M-hyponormal operators, *Studia Math.* **163** (2004), 177–187.

[9] Djordjević, S.V., Han, Y.M., Browder's theorems and spectral continuity, *Glasg. Math. J.* **42** (2000), 479–486.

[10] Dunford, N., Spectral theory I. Resolution of the identity, *Pacific J. Math.* **2** (1952), 559 – 614.

[11] Dunford, N., Spectral operators, *Pacific J. Math.* **4** (1954), 321 – 354.

[12] Grabiner, S., Uniform ascent and descent of bounded operators, *J. Math. Soc. Japan* **34** (1982), 317 – 337.

[13] Heuser, H., "Functional Analysis", John Wiley & Sons, Ltd., Chichester, 1982.

[14] Koliha, J.J., isolated spectral points, *Proc. Amer. Math. Soc.* **124** (1996), 3417 – 3424.

[15] Laursen, K.B., Operators with finite ascent, *Pacific J. Math.* **152** (1992), 323 – 336.

[16] Laursen, K.B., Neumann, M.M., "An Introduction to Local Spectral Theory", London Math. Soc. Monogr. (N.S.) 20, The Clarendon Press, Oxford University Press, New York, 2000.

[17] Lay, D.C., Spectral analysis using ascent, descent, nullity and defect, *Math. Ann.* **184** (1970), 197 – 214.

[18] Mbekhta, M., Généralisation de la décomposition de Kato aux opérateurs paranormaux et spectraux, *Glasg. Math. J.* **29** (1987), 159 – 175.

[19] Mbekhta, M., Sur la théorie spectrale locale et limite de nilpotents, *Proc. Amer. Math. Soc.* **3** (1990), 621 – 631.

[20] Mbekhta, M., Ouahab, A., Opérateurs s-regulier dans un espace de Banach et théorie spectrale, *Acta Sci. Math. (Szeged)* **59** (1994), 525 – 543.

[21] Mbekhta, M., Müler, V., On the axiomatic theory of the spectrum II, *Studia Math.* **119** (1996), 129 – 147.

[22] Müler, V., "Spectral Theory of Linear Operators", Oper. Theory Adv. Appl. 139, Birkhäuser Verlag, Basel, 2003.

[23] Oudghiri, M., Weyl's and Browder's theorem for operators satisfying the SVEP, *Studia Math.* **163** (1) (2004), 85 – 101.

[24] Oudghiri, M., a-Weyl's theorem and the single valued extension property, to appear in *Extracta Math.*

[25] Rakočević, V., Approximate point spectrum and commuting compact perturbations, *Glasg. Math. J.* **28** (1986), 193 – 198.

[26] Rakočević, V., Operators obeying a-Weyl's theorem, *Rev. Roumaine Math. Pures Appl.* **34** (1989), 915 – 919.