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On Generalized d'Alembert Functional Equation

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Abstract

Let G be a locally compact group. Let σ be a continuous involution of G and let μ be a complex bounded measure. In this paper we study the generalized d'Alembert functional equation

$$\mathrm{D}(\mu) \qquad \int_G f(xty)d\mu(t) + \int_G f(xt\sigma(y))d\mu(t) = 2f(x)f(y) \;\; x, y \in G,$$

where $f : G \to \mathbb{C}$ to be determined is a measurable and essentially bounded function. We give some conditions under which all solutions are of the form $\frac{\prec\pi(x)\xi,\zeta\succ + \prec\pi(\sigma(x))\xi,\zeta\succ}{2}$, where $(\pi, \mathcal{H})$ is a continuous unitary representation of G such that $\pi(\mu)$ is of rank one and $\xi, \zeta \in \mathcal{H}$.
Furthermore, we also consider the case when f is an integrable solution.
In the particular case where G is a connected Lie group, we reduce the solution of $\mathrm{D}(\mu)$ to a certain problem in operator theory. We prove that the solutions of $\mathrm{D}(\mu)$ are exactly the common eigenfunctions of some operators associated to a left invariant differential operators on G.

Key words: Functional equation, Gelfand measure, μ-spherical function, positive deninite function, representation theory, Lie group, invariant differential operator.

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