EXTRACTA MATHEMATICAE Vol. **22**, Núm. 1, 83 – 91 (2007)

Sylow 2-Subgroups of Solvable $\mathbb{Q}$-Groups

M.R. Darafsheh, H. Sharifi

Department of Mathematics, Statistics and Computer Science, Faculty of Science
University of Tehran, Tehran, Iran, darafsheh@ut.ac.ir

Presented by Avinoam Mann   Received February 3, 2007

Abstract: A finite group whose irreducible characters are rational valued is called a rational or a $\mathbb{Q}$-group. In this paper we obtain various results concerning the structure of a Sylow 2-subgroup of a solvable $\mathbb{Q}$-group.

Key words: $\mathbb{Q}$-groups, Sylow subgroups, extraspecial 2-groups.

AMS *Subject Class.* (2000): 20C15, 20D20, 20D10.

1. Introduction and preliminary results

Let G be a finite group and χ be a complex character of G. Let $\mathbb{Q}(\chi)$ denote the subfield of the complex number $\mathbb{C}$ generated by all the values $\chi(x)$, $x \in G$. By definition χ is called rational if $\mathbb{Q}(\chi) = \mathbb{Q}$, where $\mathbb{Q}$ denotes the field of rational numbers. A finite group G is called a rational group or a $\mathbb{Q}$-group, if every complex irreducible character of G is rational. Equivalently G is a $\mathbb{Q}$-group if and only if every x in G is conjugate to x^m where $m \in \mathbb{N}$ is prime to the order of x. This will imply that for every $x \in G$ of order n we have $\frac{N_G(\langle x \rangle)}{C_G(\langle x \rangle)} \cong \mathrm{Aut}(\langle x \rangle)$ a group of order $\varphi(n)$ where φ is the Euler φ-function. The symmetric group S_n and the Weyl group of the complex Lie algebras are examples of $\mathbb{Q}$-groups. Elementary abelian 2-groups and extra-special 2-groups are also $\mathbb{Q}$-groups. Rational groups have been studied extensively, but their classification is far from being complete. It is proved in [5] that if G is a solvable $\mathbb{Q}$-group, then $\pi(G) \subseteq \{2, 3, 5\}$. Also by [4] non-abelian composition factors of any finite $\mathbb{Q}$-group can only be either $SP_6(2)$ or $O_8^+(2)$. In [2] the structure of Frobenius $\mathbb{Q}$-groups has been found.

An important problem concerning $\mathbb{Q}$-groups is to classify them through the structure of a Sylow 2-subgroup. Any non-trivial $\mathbb{Q}$-group is of even order and there is a long standing conjecture that a Sylow 2-subgroup of a $\mathbb{Q}$-group is also a $\mathbb{Q}$-group [9, page 13]. The following results determine the structure of $\mathbb{Q}$-groups with a specified Sylow 2-subgroup.

Result 1. ([9, page 21] and [1, page 60]) *Let G be a $\mathbb{Q}$-group with an abelian Sylow 2-subgroup P. Then P is an elementary abelian 2-group and G is a supersolvable $\{2,3\}$-group. Moreover the commutator subgroup G' of G is a normal Sylow 3-subgroup of G and G splits over G' with P as a complement. In other words G is a 2-nilpotent group.*

Let D_{2n} denote the dihedral group of order $2n$. It is proved in [9, page 25] that if G is a $\mathbb{Q}$-group with a Sylow 2-subgroup P isomorphic to D_{2n}, then $n = 1, 2$ or 4. If $n = 1$ or 2, then P is abelian and the structure of G follows from Result 1. But as far as the authors know the structure of G in case $n = 4$ is not mentioned anywhere. However in [1, page 61] it is proved that if G is a solvable group with a Sylow 2-subgroup isomorphic to D_8, then $\pi(G) \subseteq \{2,3\}$.

Result 2. ([9, page 35] and [1, page 62]) *Let G be a $\mathbb{Q}$-group with a Sylow 2-subgroup P isomorphic to the quaternion group Q_8. Then G contains a normal elementary abelian p-group E_p, where $p = 3$ or 5, and $G = E_p : P$, where ":" denotes semi-direct product. In other words G is a 2-nilpotent group. Moreover if G is non-nilpotent, then G is isomorphic to a Frobenius group with complement isomorphic to Q_8, the quaternion group of order 8.*

Motivated by the above results in this paper we obtain some properties of $\mathbb{Q}$-groups having certain Sylow 2-subgroups. We also determine when extensions of certain groups is a $\mathbb{Q}$-group. Finally we find conditions on solvable $\mathbb{Q}$-groups having an extra-special Sylow 2-subgroup.

2. Sylow 2-subgroups of $\mathbb{Q}$-groups

In this section we study $\mathbb{Q}$-groups with certain conditions on their Sylow 2-subgroups.

Lemma 1. *If a generalized quaternion group P is the Sylow 2-subgroup of a $\mathbb{Q}$-group G, then P is isomorphic to the quaternion group of order 8.*

Proof. The generalized quaternion group of order 2^{n+1}, $n \geq 2$, has the following presentation: $P = \left\langle x, y : x^{2^n} = 1,\, y^2 = x^{2^{n-1}},\, y^{-1}xy = x^{-1} \right\rangle$. Suppose G is a $\mathbb{Q}$-group and P is a Sylow 2-subgroup of G. By definition we have $\left[N_G(\langle x \rangle) : C_G(\langle x \rangle)\right] = \varphi(2^n) = 2^{n-1}$. Therefore $|N_G(\langle x \rangle)| = 2^{n-1} \times |C_G(\langle x \rangle)|$, hence the 2-part of $|N_G(\langle x \rangle)|$ is at least $2^{n-1} \times 2^n = 2^{2n-1}$. Since a Sylow 2-subgroup of G has order 2^{n+1}, we must have $2^{2n-1} \leq 2^{n+1}$, hence $n \leq 2$. Thus $|P| = 8$ and $P \cong Q_8$ is the quaternion group of order 8. ∎

PROPOSITION 1. *Let G be a solvable $\mathbb{Q}$-group of even order with exactly one conjugacy class of involutions. Then a Sylow 2-subgroup of G is either elementary abelian or isomorphic to the quaternion group of order* 8.

Proof. Let S be a Sylow 2-subgroup of G. By [9] the center $Z(S)$ of S is a non-trivial elementary abelian 2-group. If x and y are involutions in $Z(S)$, then by assumption x and y are conjugate in G. By a well-known result [10, page 137], x and y are conjugate in $N_G(S)$ the normalizer of S in G. But by [9] we have $N_G(S) = S$. Therefore x and y are conjugate in S implying $x = y$. Hence $|Z(S)| = 2$. Now assume $|S| > 2$. By a result of J. Thompson cited in [8, page 511], S is isomorphic to a homocyclic or a Suzuki 2-group. If S is homocyclic then S is isomorphic to the direct product of cyclic groups of the same order, hence $Z(S) = S$ must be an elementary abelian 2-group. Otherwise if S is a Suzuki 2-group, then by [8, page 311], $S' = \phi(S) = Z(S) = \{x : x \in S, x^2 = 1\}$, implying that S has only one involution. Therefore S must be isomorphic to a generalized quaternion group. Since G is assumed to be a $\mathbb{Q}$-group, hence, by Lemma 1, S is isomorphic to the quaternion group of order 8 and the proposition is proved. ∎

PROPOSITION 2. *Let G be a supersolvable $\mathbb{Q}$-group. Then Sylow 2-subgroups of G are $\mathbb{Q}$-groups.*

Proof. Let G be a non-trivial supersolvable $\mathbb{Q}$-group. Then there is a cyclic normal subgroup $\langle x \rangle$ of prime order p in G where p is the largest prime in $\pi(G)$. Now $\frac{N_G(\langle x \rangle)}{C_G(\langle x \rangle)} = \frac{G}{C_G(\langle x \rangle)} \cong \mathbb{Z}_{p-1}$ is a $\mathbb{Q}$-group, hence $p-1 \leq 2$. Therefore $\pi(G) \subseteq \{2, 3\}$. By [10, page 158], if $3 \mid |G|$ then a Sylow 3-subgroup P of G is normal in G. Hence $\frac{G}{P}$ is a Sylow 2-subgroup of G which must be a $\mathbb{Q}$-group. ∎

3. EXTENSIONS OF ABELIAN GROUPS AS $\mathbb{Q}$-GROUPS

In this section we will consider split extensions of groups and determine when they are $\mathbb{Q}$-groups. Let a group G act on a group H. The Cartesian product $H \times G$ endowed with the following law of composition: $(g, h)(g', h') = \left(gg', h^{g'}h'\right)$, $g, g' \in G$, $h, h' \in H$, is a group called the semi-direct product of H with G and is denoted by $H \rtimes G$ or $H : G$. The group $L = H \rtimes G$ is also called a split extension of H by G and we may regard H as a normal subgroup of L such that $\frac{L}{H} \cong G$.

LEMMA 2. *Split Extension of an elementary abelian 2-group by another elementary abelian 2-group is a $\mathbb{Q}$-group.*

Proof. Let E_1 and E_2 be elementary abelian 2-groups and $G = E_1 \rtimes E_2$ be their semi-direct product. Operations of E_1 and E_2 will be written additively. Since $\frac{G}{E_1} \cong E_2$, every non-identity element of G is of order 2 or 4. To prove that G is a $\mathbb{Q}$-group it is enough to prove that every element of order 4 in G is conjugate to its inverse. Let $x = (g, v) \in G$, where $g \in E_2$ and $v \in E_1$. If x is of order 4, then $v + v^g \neq 0$ and $(g, v)^{-1} = (g, v^g)$. Now $(1, v)^{-1}(g, v)(1, v) = (g, v^g)$, proving that x and x^{-1} are conjugate in G and the lemma is proved. ∎

Let V be a vector space over a finite field on which the group G acts. Then we can form the usual semi-direct product $V \rtimes G$ with the operation $(g, v)(h, u) = (gh, v^h + u)$, where $g, h \in G$ and $u, v \in V$. In the following we will assume G is a certain group and find necessary and sufficient conditions such that $V \rtimes G$ is a $\mathbb{Q}$-group.

Let p be an odd prime and V be a 2-dimensional vector space over the Galois field $GF(p)$. It is a well-known fact that there are $a, b \in GF(p)$ such that $a^2 + b^2 = -1$. If we set

$$i = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}, \qquad j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad k = \begin{pmatrix} -b & a \\ a & b \end{pmatrix},$$

then it is easy to see that $Q_8 = \langle i, j, k \rangle$ is isomorphic to the quaternion group of order 8. Therefore V is an irreducible module for Q_8 and we can form the semi-direct product $V \rtimes Q_8$. Our next result is the following.

PROPOSITION 3. *Let V be a 2-dimensional irreducible module over the field $GF(p)$, p an odd prime, for the quaternion group Q_8. Then $V \rtimes Q_8$ is a $\mathbb{Q}$-group if and only if $p = 3$ or 5.*

Proof. First we will prove that the order of elements of the group $V \rtimes Q_8$ is one of the numbers $1, 2, 4$ or p. Elements of $V \rtimes Q_8$ are of the forms (g, v) where $g \in Q_8$ and $v \in V$. It is obvious that for $n \in N$ we have $(g, v)^n = (g^n, vg^{n-1} + vg^{n-1} + \cdots + vg + v)$ and hence $O(I, o) = 1$, $O(-I, o) = 2$, $O(x, v) = 4$ for all $x \in Q_8 - \{\pm I\}$ and $v \in V$, finally $O(I, v) = p$ for all $v \in V - \{0\}$. Now for elements (g, v) and (h, u) of $V \rtimes Q_8$ it can be verified that $(h, u)^{-1}(g, v)(h, u) = (h^{-1}gh, -uh^{-1}gh + vh + u)$.

Now if we consider (x, v), $v \in Q_8 - \{\pm I\}$, $v \in V$, then x and $x^3 = -x$ are conjugate in Q_8 and hence there exists $y \in Q_8$ such that $y^{-1}xy = -x$.

Therefore from $(y,u)^{-1}(x,v)(y,u) = (y^{-1}xy, -uy^{-1}xy + vy + u) = (-x, ux + vy+u) = (x,v)^3 = (x^3, vx^2+vx+v) = (-x, vx)$ we will obtain $ux+vy+u = vx$, thus $u(x+I) = v(x-y)$ from which we will obtain $u = \frac{1}{2}v(I + x + yx - y)$. Hence (x,v) and $(x,v)^3$ for all $x \in Q_8 - \{\pm I\}$ and $v \in V$ are conjugate in $V \rtimes Q_8$.

Now we will consider elements of order p, say (I,v), where $v \in V - \{0\}$. Let m be an integer such that $0 < m < p$ and $(x,u)^{-1}(I,v)(x,u) = (I,v)^m$, then $vx = mv$. Hence m is an eigenvalue of $x \in G$. But it is easy to see that eigenvalues of elements of Q_8 are either ± 1 or roots of the equation $t^2+1 = 0$ in $GF(p)$. If the only eigenvalues occurring are ± 1, then $p = 3$ and if rots of $t^2 + 1 = 0$ occur we must have $p = 5$. The converse is obviously true, i.e., if $p = 3$ or 5, then (I,v) is conjugate to $(I,v)^m$ for all $0 < m < p$. The proposition is proved now. ∎

Next we consider the symmetric group S_n of degree n. In this case we assume V is a vector space of dimension n over the Galois field $GF(q)$ where q is a power of the prime p. We assume S_n as the symmetric group of the set $\{1, 2, \dots, n\}$ and V has basis $\{e_1, \dots, e_n\}$. Therefore the action of S_n on V is as follows: $e_i\pi = e_{(i)\pi}$ for all $1 \le i \le n$ and $\pi \in S_n$. We consider the semi-direct product $V \rtimes S_n$ called the hyperoctahedral group and prove the following result.

PROPOSITION 4. *$V \rtimes S_n$ is a $\mathbb{Q}$-group if and only if $p = 2$.*

Proof. With regard to the above explanation we consider the element $(1, e_i)$, $1 \le i \le n$, of order p in $V \rtimes S_n$. This element must be conjugate to $(1,e_i)^m$, where $0 < m < p$. Therefore there exists $(\pi, v) \in V \rtimes S_n$ such that $(\pi,v)^{-1}(1,e_i)(\pi,v) = (1,e_i)^m$ from which we obtain $e_i\pi = me_i$ and therefore $e_{(i)\pi} = me_i$ which implies $m = 1$. Therefore $p = 2$. By [9, Corollary 96A, page 96] the hyperoctahedral group B_n is a $\mathbb{Q}$-group and this is the group $V \rtimes S_n$ in the case $p = 2$, the proof is complete now. ∎

Now let V be a vector space of dimension n over the Galois field $GF(q)$, q a power of the prime p. Let $G = GL_n(q)$ be the group of automorphisms of V. Then G acts on V and we can form the semi-direct product $V \rtimes G$. Our next result is concerned with the above consideration.

LEMMA 3. *Let q and n be positive integers. Then $\varphi(q^n - 1) = n$ if and only if $(n,q) = (1,2)$, $(1,3)$ or $(2,2)$, where φ denotes the Euler φ-function.*

Proof. If $n = 1$, then $\varphi(q-1) = 1$ and obviously $q-1 = 1$ or 2 implying $q = 2$ or 3. Therefore we will assume $n \geq 2$. It can be proved that for any positive integer m if $q \geq 3$, then $q^m \geq m^2$ and in the case of $m \geq 4$ we have $2^m \geq m^2$. Now for any integer t it is easy to prove that $\varphi(t) \geq \frac{1}{2}\sqrt{t}$. Hence if $\varphi(q^n - 1) = n$, then $n \geq \frac{1}{2}\sqrt{q^n - 1}$ which implies $q^{\frac{n}{2}} < 2n + 1$. First we assume $q \geq 3$. Since $n \geq 2$ we obtain $2n+1 > q^{\frac{n}{2}} \geq \frac{n^2}{4}$ implying $n^2 < 8n+4$, hence $n \leq 8$. If $n \geq 4$, then from $q^{\frac{n}{2}} < 2n+1$ we obtain $q = 2$ which is not the case. Therefore $n = 3$ or 2. If $n = 3$, then $q = 3$ and if $n = 2$, then $q = 3$ or 4, and in both cases $\varphi(q^n - 1) \neq n$. Now we will assume $q = 2$. If $\frac{n}{2} \geq 4$, then $2n + 1 > 2^{\frac{n}{2}} \geq \frac{n^2}{4}$ implies $n \leq 8$, hence $n = 8$. But $\varphi(2^8 - 1) \neq 8$, so we assume $n < 8$. Now case by case examination of the Euler φ-function yields $\varphi(2^2 - 1) = 2$ as the only possibility. The Lemma is proved now. ∎

Proposition 5. *$V \rtimes GL_n(q)$, $n \geq 2$, is a $\mathbb{Q}$-group if and only if $(n, q) = (2, 2)$.*

Proof. If $H = V \rtimes GL_n(q)$ is a $\mathbb{Q}$-group, then by [9] the group $\frac{H}{N} \cong GL_n(q)$ is also a $\mathbb{Q}$-group. Now for any $\lambda \in GF(q)^*$ the matrices λI and $\lambda^{-1} I$ must be conjugate in $GL_n(q)$ from which we will obtain $\lambda^2 = 1$ or $\lambda = \pm 1$. Therefore $q = 2$ or 3. Now by [7, page 187] the group $GL_n(q)$ has an element h of order $q^n - 1$ such that $\frac{N(\langle h \rangle)}{G(\langle h \rangle)} \cong Z_n$. Therefore $\varphi(q^n - 1) = n$. Now by Lemma 3 we obtain $(n, q) = (2, 2)$. The converse of the proposition is obvious and the Proposition is proved now. ∎

4. Solvable $\mathbb{Q}-$groups with extraspecial Sylow 2-subgroup

As we mentioned in the introduction an extraspecial 2-group is a $\mathbb{Q}$-group and it may appear as a Sylow 2-subgroup of a $\mathbb{Q}$-group. In of [1, problem 83, page 301] part 2 asks to classify rational $\mathbb{Q}$-groups with an extra-special Sylow 2-subgroup. Now we recall the definition of an extra-special p-group and its structure from [3].

Definition 1. A finite p-group P is called extra-special if $P' = Z(P) \cong \mathbb{Z}_p$ and $\frac{P}{P'}$ is an elementary abelian p-group.

Every extra-special p-group is the central product of non-abelian p-groups of order p^3. The dihedral group D_8 and the quaternion group Q_8 are extra-special 2-groups of order 8. If P is an extra-special 2-group, then there is an $m \in \mathbb{N}$ such that $|P| = 2^{2m+1}$. Moreover either $P \cong D_8 \circ D_8 \circ \cdots \circ D_8$ or

$P \cong Q_8 \circ D_8 \circ \cdots \circ D_8$, where $\circ$ denotes the central product and in both cases m different groups are involved.

First we will prove the following two results about a general $\mathbb{Q}$-group. We recall that if G is a finite group, then the largest normal subgroup of odd order in G is denoted by $O(G)$.

Lemma 4. *Let G be a $\mathbb{Q}$-group with extra-special Sylow 2-subgroup P. If G has a non-trivial center and $O(G) = 1$, then $G = P$.*

Proof. Since $Z(G) \subseteq Z(P) = \langle x \rangle$ is a group of order 2 and $Z(G)$ is assumed to be non-trivial, hence $Z(G) = \langle x \rangle$. Now $\frac{G}{\langle x \rangle}$ is a $\mathbb{Q}$-group with $\frac{P}{\langle x \rangle}$ as a Sylow 2-subgroup. But $\frac{P}{\langle x \rangle}$ is an elementary abelian 2-group, hence, by Result 1, $\frac{G}{\langle x \rangle}$ is a supersolvable $\{2, 3\}$-group. Therefore there is a normal 3-subgroup $\overline{N}$ of $\frac{G}{\langle x \rangle}$ such that $\frac{G}{\langle x \rangle} = \overline{N}\left(\frac{P}{\langle x \rangle}\right)$. Let N be the pre-image of N in G and S be a Sylow 3-subgroup of G. Then $\overline{N} = \frac{N\langle x \rangle}{\langle x \rangle}$ and since x has order 2 we have $x \notin S$. But $x \in Z(G)$, hence $x \in C_G(S)$ implying $N = S\langle x \rangle \cong S \times \langle x \rangle$. Now S is a characteristic subgroup of N and hence $S \trianglelefteq G$. Therefore $S \leq O(G) = 1$ which implies $S = 1$ and hence $N = \langle x \rangle$. Consequently $\overline{N} = 1$ which gives the result $G = P$ and the Lemma is proved. ∎

Proposition 6. *Let G be a $\mathbb{Q}$-group with an extra-special Sylow 2-subgroup P. If $Z(G) \neq 1$, then G is a solvable group and there is a normal subgroup N of G with $\pi(N) \subseteq \{3, 5\}$ such that $G = NP$ and $N \cap P = 1$.*

Proof. We use induction on $O(G)$. If $O(G) = 1$, then by Lemma 4 we have $G = P$ and $N = 1$ will work in the proposition. Therefore we may assume $O(G) \neq 1$. We know that $\frac{G}{O(G)}$ is a $\mathbb{Q}$-group with a Sylow 2-subgroup isomorphic to P. Since $Z(G)$ is always an elementary abelian 2-group we obtain $Z(G) \neq O(G)$ from which we deduce that $Z\left(\frac{G}{O(G)}\right) \neq 1$. Hence by induction we have $\frac{G}{O(G)} = \overline{N}P$ where $\overline{N} \trianglelefteq \frac{G}{O(G)}$ and $\overline{N} \cap P = 1$. But $\overline{N} = O\left(\frac{G}{O(G)}\right) = 1$ and therefore $G = O(G)P$. Now we set $N = O(G)$, hence $G = NP$. Since $\frac{G}{N}$ is a solvable group and N has odd order we deduce that G is a solvable group. Now, by [5], G is a $\{2, 3, 5\}$-group and hence $\pi(N) \subseteq \{3, 5\}$ and the proposition is proved. ∎

Next we turn to solvable $\mathbb{Q}$-groups with an extra-special Sylow 2-subgroup. First of all let us determine the structure of the solvable $\mathbb{Q}$-groups with Sylow 2-subgroups isomorphic to the dihedral group D_8.

Theorem 1. *Let G be a rational solvable group with a Sylow 2-subgroup isomorphic to D_8. Then G contains a normal 3-subgroup N such that $\frac{G}{N}$ is isomorphic to either D_8 or $\mathbb{S}_4$.*

Proof. By [1, page 61] we have $|G| = 8 \cdot 3^n$, where n is a non-negative integer. The number of Sylow 3-subgroups N_3 of G is either 1 or 4. If $N_3 = 1$, then a Sylow 3-subgroup N of G is normal in G and $\frac{G}{N} \cong D_8$. Assume that $N_3 = 4$ and $\Omega = \{Q_1, Q_2, Q_3, Q_4\}$ is the set of distinct Sylow 3-subgroups of G. If N denotes the kernel of the action of G on Ω by conjugation, then $\frac{G}{N}$ is isomorphic to a subgroup of $\mathbb{S}_4$. Since G is assumed to be a $\mathbb{Q}$-group, therefore $\frac{G}{N}$ is also a $\mathbb{Q}$-group. Since $|N_G(Q_i)| = 2 \cdot 3^n$ and $N = \cap_{i=1}^4 N_G(Q_i)$, hence $4 \mid \left|\frac{G}{N}\right|$. Now it is easy to see that the rational subgroups of $\mathbb{S}_4$ with order divisible by 4 are isomorphic to one of the groups $\mathbb{Z}_2 \times \mathbb{Z}_2$, D_8 or $\mathbb{S}_4$.

If $\frac{G}{N} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, then $|N| = |N_G(Q_i)|$, for all $1 \leq i \leq 4$, which is a contradiction because the Q_i's are distinct. If $\frac{G}{N} \cong D_8$ or S_4, then we are done and the Theorem is proved now. ∎

Theorem 2. *Let G be a solvable $\mathbb{Q}$-group with an extra-special Sylow 2-subgroup. Then one of the following possibilities holds:*

(a) *G is a 2-nilpotent group.*

(b) *There is a proper normal subgroup N of G such that $\frac{G}{N} = P : E(2)$, where P is a 3-group and $E(2)$ is an elementary abelian 2-group.*

Proof. We use induction on $|G|$. Let P be a Sylow 2-subgroup of G which by assumption is extra-special. By [5] we have $\pi(G) \subseteq \{2, 3, 5\}$. Let E be a minimal normal subgroup of G.

Case 1: $|E|$ is even. Therefore E is a proper elementary abelian 2-subgroup of G and we may assume $E \leq P$. Since $1 \neq E \trianglelefteq P$, hence $E \cap Z(P) \neq 1$. But $Z(P) = P'$ is of order 2. Therefore $Z(P) = P' \subseteq E$. Thus $\frac{P}{E}$ is an abelian group and it is a Sylow 2-subgroup of $\frac{G}{E}$. Hence $\frac{G}{E}$ is a $\mathbb{Q}$-group with an abelian Sylow 2-subgroup, hence by Result 1, $\frac{G}{E} = P : E(2)$ where P is a 3-subgroup of $\frac{G}{E}$, and hence of G, and $E(2)$ is an elementary abelian 2-group. Therefore case (b) of the theorem holds.

Case 2: $|E|$ is odd. Hence $\frac{G}{E}$ is a $\mathbb{Q}$-group with an extra-special Sylow 2-subgroup isomorphic to P.

If a minimal normal subgroup $\frac{A}{E}$ of $\frac{G}{E}$ has even order, then by Case 1, $(\frac{G}{E})/(\frac{A}{E}) \cong \frac{G}{A} = P : E(2)$ where P is a 3-group and $E(2)$ is an elementary abelian 2-group as stated on part (b) of the theorem.

If a minimal normal subgroup $\frac{A}{E}$ of $\frac{G}{E}$ has an odd order, then $(\frac{G}{E})/(\frac{A}{E}) \cong \frac{G}{A}$, $|\frac{G}{E}| < |G|$ and $|A|$ is odd. Therefore by induction we reach a point such that there is a normal subgroup N of G with $\frac{G}{N}$ isomorphic to a Sylow 2-subgroup of G. This implies that G is a 2-nilpotent group, and case (a) of the theorem holds. ∎

Acknowledgements

This research was done while the first author was awarded a visiting position during 2006-2007 by the mathematics and statistics department of the University of North Carolina at Charlotte. The first author would like to thank the department of mathematics and statistics of UNC Charlotte that made this visit possible. Both authors deeply appreciate the referee for his/her comments concerning various parts of the paper.

References

[1] Berkovich, Ya., Zhmud, E.M., "Characters of Finite Groups, Parts 1", Transl. Math. Monogr. 172, Amer. Math. Soc., Providence, RI, 1997.

[2] Darafsheh, M.R., Sharifi, H., Frobenius $\mathbb{Q}$-groups, *Arch. Math. (Basel)* **83** (2) (2004), 102 – 105.

[3] Dornhoff, L., "Group Representation Theory, Part A: Ordinary Representation Theory", Marcel Dekker Inc., New York, 1971.

[4] Feit, W., Seitz, G.M., On finite rational groups and related topics, *Illinois J. Math.* **33** (1) (1988), 103 – 131.

[5] Gow, R., Groups whose characters are rational-valued, *J. Algebra* **40** (1976), 280 – 299.

[6] Hayashi, M., On a generalization of F.M. Markel's Theorem, *Hokkaido Math. J.* **4** (1975), 278 – 280.

[7] Huppert, B., "Endliche Gruppen I", Springer-Verlag, Berlin-New York, 1967.

[8] Huppert, B., Blackburn, N., "Finite Groups II", Springer-Verlag, Berlin-New York, 1982.

[9] Kletzing, D., "Structure and Representations $\mathbb{Q}$-Groups", Lectures Notes in Math. 1087, Springer-Verlag, Berlin, 1984.

[10] Scott, W.R., "Group Theory", Second edition, Dover Publications Inc., New York, 1987.