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Continuous Multilinear Operators on $C(K)$ Spaces and Polymeasures †

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Abstract: Every continuous k-linear operator from a product $C(K_1) \times \cdots \times C(K_k)$ into a Banach space X (K_i being compact Hausdorff spaces) admits a Riesz type integral representation

$$T(f_1, \dots, f_k) := \int (f_1, \dots, f_k)\, d\gamma,$$

where γ is the *representing polymeasure* of T, i.e., a set function defined on the product of the Borel σ-algebras $\mathrm{Bo}(K_i)$ with values in X^{**} which is separately finitely additive. As in the linear case, the interplay between T and its representing polymeasure plays an important role. The aim of this paper is to survey some features of this relationship.

Key words: Multilinear operators, spaces of continuous functions, tensor products of Banach spaces, polymeasures.

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1. Introduction

F. Riesz's representation theorem establishes that any continuous linear form T on $C(K)$ can be represented as an integral with respect to a (unique) Radon measure μ_T on K:

$$T(f) = \int_K f\, \mu_T, \quad \forall f \in C(K).$$

Since its publication in 1909 ([33]; in case $K = [0,1]$), this theorem has been a fundamental tool in the study of $C(K)$ spaces and the development of abstract measure theory.

The *representing measure* μ_T of a continuous linear form $T \in C(K)^*$ can be obtained in a constructive way (see for instance [34, Theorem 2.14]). But from a functional analytic point of view, the solution is quite easy: The space

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$\mathcal{B}(K)$ of bounded Borel measurable functions (i.e., uniform limits of Borel simple functions) is a closed subspace of the bidual $C(K)^{**}$, which contains $C(K)$ isometrically. Hence $\tilde{T} := T^{**}|_{\mathcal{B}(K)}$ is a continuous linear extension of T. If A is a Borel subset of K, then $\mu_T(A) = \tilde{T}(\chi_A)$. Moreover, for every $g \in \mathcal{B}(K)$, $\int g\, d\mu_T = \tilde{T}(g)$. The same trick works for representing the general operators (i.e., continuous linear maps) T from $C(K)$ to an arbitrary Banach space X: The restriction of the bitranspose $T^{**} : C(K)^{**} \to X^{**}$ to the closed subspace $\mathcal{B}(K)$ is an extension $\tilde{T} : \mathcal{B}(K) \to X^{**}$ of T. The set function $m_T(A) := \tilde{T}(\chi_A)$ for every Borel set $A \subset K$ is a finitely additive measure with finite semivariation (the *representing measure* of T) such that

$$T(f) = \int f\, dm_T, \quad \forall f \in C(K).$$

(in fact, m_T is also the representing measure of $\tilde{T}$, in an obvious sense). This representation plays an important role in the fundamental work of Grothendieck on the space $C(K)$ ([21]) and since then the study of the relationship between T and m_T is a central topic in the theory.

A similar approach can be used to obtain suitable representation theorems for operators on spaces of vector valued continuous functions.

Soon after Riesz's paper, M. Fréchet obtained ([18]) an integral characterization of the continuous *bilinear* forms

$$B : C([a,b]) \times C([a,b]) \to \mathbb{R},$$

but the studies on "bilinear integration" (i.e., continuous bilinear forms on $C(K_1) \times C(K_2)$) had to wait till the work of Clarkson and Adams ([12]) in the mid 1930s and Morse and Transue in a series of papers appeared between the late 1940s and the mid 1950s (see, for instance, the references included in [4]), where the name *bimeasure* was coined to denote such bilinear forms. Ylinen defined in [38] bimeasures as (scalar or vector-valued) set functions on the cartesian product of two σ-algebras *"... motivated by a desire to find an analogue of the Riesz representation theorem and its vector generalization..."* ([38, Introduction]). Its natural generalization, polymeasures, were introduced by Dobrakov in [16]; in a series of papers (see [17] and the bibliography there mentioned) he developed a very general theory of integration for them, but, as we shall see, this general theory is not needed to obtain a Riesz type representation theorem for multilinear operators on the product of $C(K)$ spaces. We must mention here that, for several results, the bilinear case is essentially simpler than the k-linear case for $k > 2$.

The notation and terminology used along the paper will be the standard used in Banach space theory, as for instance in [14]. In any case, we will now fix some notation. Throughout the paper E, X will be Banach spaces and $B(E)$ will stand for the unit ball of E. K will denote a compact Hausdorff space and $C(K)$ (resp. $\mathcal{B}(K)$) the Banach space of all continuous (resp. bounded Borel measurable) scalar functions defined on it, with the usual sup norm. We will write $\mathcal{L}(E;X)$ to indicate the linear operators from E into X, and $\mathcal{L}^k(E_1,\dots,E_k;X)$ will denote the space of continuous multilinear operators from $E_1\times\cdots\times E_k$ into X. If $X=\mathbb{K}$, the scalar field, we will not write it. As usual, $E_1\hat{\otimes}\cdots\hat{\otimes}E_k$ (resp., $E_1\check{\otimes}\cdots\check{\otimes}E_k$) will stand for the complete projective (resp., injective) tensor product of $E_1,\dots,E_k$.

When $T\in\mathcal{L}^k(E_1,\dots,E_k;X)$ we shall denote by $\hat{T}: E_1\otimes\cdots\otimes E_k\to X$ its linearization. We consider to be well known that $\mathcal{L}^k(E_1,\dots,E_k;X)$ is naturally isometric to $\mathcal{L}(E_1\hat{\otimes}\cdots\hat{\otimes}E_k;X)$.

2. Polymeasures and representation theorems.

As in the linear case, the key to obtain a Riesz type representation theorem for multilinear operators on $C(K)$ spaces is an extension theorem for multilinear operators from $C(K_1)\times\cdots\times C(K_k)$ to a larger space containing $\mathcal{B}(K_1)\times\cdots\times\mathcal{B}(K_k)$. A first result in this direction was a theorem of Pełczyński ([27, Theorem 2]) establishing that under suitable hypothesis, the continuous multilinear map $T: C(K_1)\times\cdots\times C(K_k)\to X$ extends to a continuous multilinear map $\tilde{T}:\mathcal{B}^\Omega(K_1)\times\cdots\times\mathcal{B}^\Omega(K_k)\to X$, where $\mathcal{B}^\Omega(K)$ denotes the Banach space of all the bounded Baire functions on K. The limitation to the Baire functions is basic in the proof of the theorem, which uses transfinite induction to arrive from $C(K_i)$ to $\mathcal{B}^\Omega(K_i)$. This result was used by Dobrakov in [17] to obtain a representation theorem for multilinear operators on $C(K)$ spaces (there is a gap in the statement of [17, Theorem 4]). The following extension theorem was proved in [8]:

Theorem 2.1. [8, Theorem 2.3] *Let $K_1,\dots K_k$ be compact Hausdorff spaces, let X be a Banach space and let $T\in\mathcal{L}^k(C(K_1),\dots,C(K_k);X)$. Then there is a unique*

$$\tilde{T}\in\mathcal{L}^k(\mathcal{B}(K_1),\dots,\mathcal{B}(K_k);X^{**})$$

which extends T and is $\omega^*-\omega^*$ separately continuous (the ω^*-topology that we consider in $\mathcal{B}(K_i)$ is the one induced by the ω^*-topology of $C(K_i)^{**}$). Besides, we have

1.- $\|\tilde{T}\| = \|T\|$.

2.- *If* $\bar{g}_d = (g_1, \dots, g_{d-1}, g_{d+1}, \dots g_k)$ *with* $g_i \in \mathcal{B}(K_i)$*, then there is a unique* X^{**}*-valued bounded* ω^**-Radon measure* $m_{\bar{g}_d}$ *on* K_d *(i.e., an* X^{**}*-valued finitely additive bounded vector measure on the Borel subsets of* K_d*, such that for every* $y^* \in X^*$*,* $(y^* \circ m_{\bar{g}_d})$ *is a Radon measure on* K_d*), verifying*

$$\int g d\, m_{\bar{g}_d} = \tilde{T}(g_1, \dots, g_{d-1}, g, g_{d+1}, \dots g_k), \ \forall g \in \mathcal{B}(K_d).$$

3.- $\tilde{T}$ *is* $\omega^* - \omega^*$ *jointly sequentially continuous (i.e., if* $(g_i^n)_{n\in\mathbb{N}} \subset \mathcal{B}(K_i)$*,* $\forall i = 1, \dots, k$*, and* $g_i^n \xrightarrow{\omega^*} g_i$*, then*

$$\lim_{n\to\infty} \tilde{T}(g_1^n, \dots, g_k^n) = \tilde{T}(g_1, \dots, g_k)$$

in the $\sigma(X^{**}, X^*)$ *topology).*

The map $\tilde{T}$ is just the restriction to $\mathcal{B}(K_1) \times \cdots \times \mathcal{B}(K_k)$ of the so called *Aron-Berner extension* $T^{**} : C(K_i)^{**} \times \cdots \times C(K_k)^{**} \longrightarrow X^{**}$ ([2]) (although in the cited paper the existence of T^{**} is proved directly.) Part 3 is a measure-theoretic result: Suppose it is true for $k-1$ and let $(f_1, \dots, f_{k-1}) \in C(K_1) \times \dots \times C(K_{k-1})$, $g \in \mathcal{B}(K_k)$. Define

$$T_g(f_1, \dots, f_{k-1}) := \tilde{T}(f_1, \dots, f_{k-1}, g).$$

By uniqueness it is clear that $\tilde{T_g} = (\tilde{T})_g$, i.e.

$$\tilde{T_g}(g_1, \dots, g_{d-1}) = \tilde{T}(g_1, \dots, g_{k-1}, g), \ \forall g_i \in \mathcal{B}(K_i).$$

Suppose now $(g_i^n) \xrightarrow{w^*} g_i$ in $\mathcal{B}(K_i)$ $(1 \le i \le k)$ and define

$$\nu_n := m_{g_1^n, \dots, g_{k-1}^n}, \ \nu := m_{g_1, \dots, g_{k-1}}.$$

By the induction hypothesis we have in particular

$$w^* - \lim \nu_n(A) = \nu(A), \quad \text{for every Borel subset } A \subset K_k.$$

Then the Vitali-Hahn-Saks and Egoroff's theorems applied to the countably additive measures $(y^* \circ \nu_n)$ $(y^* \in X^*)$ give

$$\lim_{n\to\infty} \langle \int g_k^n\, \nu_n, y^* \rangle = \langle \int g_k d\, \nu, y^* \rangle.$$

(see [27, Proposition 1(c)].)

The well known Orlicz-Pettis theorem yields that the ω^*-countably additive measures (ν_n) are norm countably additive if they take values in X. If we repeat the above argument in this case we obtain now that

$$\|\cdot\| - \lim_{n\to\infty} \int g_k^n \, \nu_n = \int g_k d\, \nu.$$

So we have

Corollary 2.2. *With the notations of Theorem 2.1, if $\tilde{T}$ takes its values in X, it is ω^*-norm sequentially continuous.*

It is known that the Aron-Berner extension of a multilinear map $T : E_1 \times \cdots \times E_k \longrightarrow X$ takes values in X when every operator from each E_i into X is weakly compact (see [8, Corollary 2.2] for a direct proof). Then, with the notations of Theorem 2.1, we can assure that $\tilde{T}$ takes values in X in the following cases:

- X contains no copy of c_0.
- Each K_i is Stonean and X contains no copy of ℓ_∞.
- Each $C(K_i)$ is a Grothendieck space and X is separable.

Theorem 2.1 is the main ingredient to obtain our representation theorem for multilinear operators on $C(K)$ spaces. To this aim, let us begin with the following

Definition 2.3. ([16, Definition 1]) Let Σ_i $(1 \le i \le k)$ be σ-algebras of subsets on some non void sets Ω_i. A function $\gamma : \Sigma_1 \times \cdots \times \Sigma_k \longrightarrow X$ or $\gamma : \Sigma_1 \times \cdots \times \Sigma_k \longrightarrow [0, +\infty]$ is a (countably additive) *k-polymeasure* if it is separately (countably) additive.

The trivial example of a polymeasure is given by the restriction to $\Sigma_1 \times \cdots \times \Sigma_k$ of a *measure* defined on the σ-algebra $\sigma(\Sigma_1 \times \cdots \times \Sigma_k)$ generated by $\Sigma_1 \times \cdots \times \Sigma_k$. But there are *true* polymeasures which are not of this type, as we shall see in a later section.

We define the *semivariation* (called *Fréchet variation* in [4]) of a polymeasure γ as the set function

$$\|\gamma\| : \Sigma_1 \times \cdots \times \Sigma_k \longrightarrow [0, +\infty]$$

given by the formula

$$\|\gamma\|(A_1,\ldots,A_k)=\sup\left\{\left\|\sum_{j_1=1}^{n_1}\cdots\sum_{j_k=1}^{n_k}a_1^{j_1}\ldots a_k^{j_k}\gamma(A_1^{j_1},\ldots,A_k^{j_k})\right\|\right\}$$

where the supremum is taken over all the finite Σ_i-partitions $(A_i^{j_i})_{j_i=1}^{n_i}$ of A_i $(1\le i\le k)$, and all the collections $(a_i^{j_i})_{j_i=1}^{n_i}$ contained in the unit ball of the scalar field.

As in the case $k=1$, it can be proved that any *countably additive* vector valued polymeasure has finite semivariation (see [16, Theorems 2 and 3]).

Let $S(\Sigma_i)$ be the normed space of the Σ_i-simple functions with the supremum norm. If $s_i=\sum_{j_i=1}^{n_i}a_{i,j_i}\chi_{A_{i,j_i}}\in S(\Sigma_i)$, for every X-valued polymeasure γ the formula

$$T_\gamma(s_1,\ldots,s_k)=\sum_{j_1=1}^{n_1}\cdots\sum_{j_k=1}^{n_k}a_{1,j_1},\ldots a_{k,j_k}\gamma(A_{1,j_1},\ldots,A_{k,j_k})$$

defines a multilinear map $T_\gamma:S(\Sigma_1)\times\ldots\times S(\Sigma_k)\to X$ such that

$$\|T_\gamma\|=\|\gamma\|(\Omega_1,\ldots,\Omega_k)\|\ (\stackrel{\text{def}}{=}\|\gamma\|)\ \le\infty.$$

So, if $\|\gamma\|<\infty$, i.e., γ has *finite semivariation*, T_γ can be uniquely extended (with the same norm) to $\mathcal{B}(\Sigma_1)\times\ldots\mathcal{B}(\Sigma_k)$, where $\mathcal{B}(\Sigma)$ stands for the completion of $S(\Sigma)$. We will still denote this extension by T_γ and we shall write also

$$T_\gamma(g_1,\ldots,g_k)=\int(g_1,\ldots,g_k)d\gamma.$$

The correspondence $\gamma\mapsto T_\gamma$ is an isometric isomorphism between the space $bpm(\Sigma_1,\ldots,\Sigma_k;X)$ of all X-valued polymeasures of finite semivariation (with the semi-variation norm), and $\mathcal{L}^k(\mathcal{B}(\Sigma_1)\ldots\mathcal{B}(\Sigma_k);X)$.

This cheap k-linear integral is enough to obtain a Riesz type representation theorem for multilinear operators on $C(K)$ spaces:

THEOREM 2.4. [8, Theorem 2.9] *Let $K_1,\ldots K_k$ be compact Hausdorff spaces, X a Banach space and $T\in\mathcal{L}^k(C(K_1),\ldots,C(K_k);X)$.*

I.- *If, with the notations of Theorem 2.1, we define $\Gamma:Bo(K_1)\times\ldots\times Bo(K_k)\to X^{**}$ (where $Bo(K)$ stands for the Borel σ algebra of K) by*

$$\Gamma(A_1,\ldots,A_k)=\tilde{T}(\chi_{A_1},\ldots\chi_{A_k}),$$

then Γ is a polymeasure of bounded semivariation that verifies:

1. $\|T\| = \|\Gamma\|$.
2. $T(f_1, \ldots, f_k) = \int (f_1, \ldots, f_k) d\,\Gamma$ $(f_i \in C(K_i))$
3. *For every* $z^* \in X^*$, $z^* \circ \Gamma \in ((C(K_1)\hat{\otimes} \ldots \hat{\otimes} C(K_k))^*$ *and the map* $z^* \mapsto z^* \circ \Gamma$ *is continuous for the topologies* $\sigma(X^*, X)$ *and* $\sigma((C(K_1)\hat{\otimes} \ldots \hat{\otimes} C(K_k))^*, C(K_1)\hat{\otimes} \ldots \hat{\otimes} C(K_k))$

II.- *Conversely, if* $\Gamma : Bo(K_1) \times \cdots \times Bo(K_k) \to X^{**}$ *is a polymeasure which verifies* (I.3), *then it has finite semivariation and formula* (I.2) *defines a* k*-linear continuous operator from* $C(K_1) \times \cdots, \times C(K_k)$ *into* X *for which* (I.1) *holds.*

In particular, $\mathcal{L}^k(K_1, \ldots, K_k) = \big(C(K_1)\hat{\otimes} \cdots \hat{\otimes} C(K_k)\big)^*$ can be identified with the space $\mathcal{M}(K_1, \ldots, K_k)$ of all the countably additive polymeasures defined on the product of the Borel σ-algebras of the K_i's which are separately Radon measures, endowed with the norm of the semivariation.

Remark 2.5. A first version of theorem 2.4 (with only *Baire* polymeasures and weak* sequentially continuous extensions considered) appeared in [17, Theorem 5]. Unfortunately, the technical condition I.3 is not included in this version. However, this condition cannot be omitted, even in the case $k = 1$, as it is well known. For instance, the ℓ_∞-valued ω^*-Radon measure defined on the subsets of $\mathbb{N}$ be the formula $m(A) := \chi_A$ is the representing measure of *no* operator $T : \ell_\infty \to c_0$.

3. Some classes of multilinear operators.

In his fundamental paper [21], Grothendieck gave a deep insight in the study of Banach space properties of $C(K)$ spaces. The paper emphasizes the "functorial" point of view of Grothendieck, by studying the structure of a Banach space E in terms of the behavior of certain classes of operators from and into E. In this way, Grothendieck axiomatizes some of the properties studied in the space $C(K)$ and introduces the now well known Dunford-Pettis, reciprocal Dunford Pettis, Dieudonné and Grothendieck properties. The seminal ideas contained in this paper were not well appreciated by Banach space researchers for more than 10 years. But then they became tremendously influential in the development of the theory. (For more on Grothendieck's work on Functional Analysis, we refer the interested reader to [5].)

Of course, one of the main tools in Grothendieck's paper was the integral representation theorem of operators on $C(K)$ spaces in terms of their representing measures. The relationships between properties of the operator and properties of its representing vector measure were widely used there and in later work on this subject, being very fruitful for both theories: linear operators on $C(K)$ spaces (and also on vector valued function spaces $C(K,E)$) and vector measures. For instance, we have the following result (recall that a series $\sum_{n\in\mathbb{N}} x_n$ in a Banach space E is called *weakly unconditionally Cauchy* or w.u.C. in short when $\sum_{n^1}^{\infty} |x^*(x_n)| < \infty \ \forall x^* \in E^*$):

THEOREM 3.1. ([21], [26]) *For a linear operator $T : C(K) \longrightarrow X$ the following assertions are equivalent:*

1) $\tilde{T} : \mathcal{B}(K) \to X^{**}$ *takes values in* X.
2) *The representing measure of* T *takes values in* X.
3) *The representing measure of* T *is* σ*-additive.*
4) *The representing measure of* T *is regular.*
5) T *is weakly compact.*
6) T *sends weakly Cauchy sequences into norm convergent ones* (*i.e.,* T *is* completely continuous).
7) T *sends w.u.C. series into unconditionally convergent series* (*i.e.,* T *is* unconditionally converging).

Theorem 2.4 in the previous section allows us to extend immediately the equivalences (1)-(4) of the above theorem to the multilinear setting under some of the conditions stated just before Definition 2.3.

In order to see what happens with the rest of equivalences in Theorem 3.1, we have to introduce the corresponding classes of multilinear operators. But in general this is not a trivial question. In fact, several properties that are equivalent in the linear case and characterize a class of operators, are *not* equivalent in the multilinear case. For instance, a multilinear operator $T : E_1 \times \cdots \times E_k \to X$ is *weakly compact* if $T(B(E_1) \times \cdots \times B(E_k))$ is a weakly relatively compact subset of X. It is well known that a *linear* operator is weakly compact if and only if its second adjoint takes values in X. But let us consider the following

EXAMPLE 3.2. Let $E = C([0,1])$ and let $(\varphi_n)_{n=1}^{\infty}$ be a bounded sequence in E which is also an orthonormal system in the sense of the usual inner product in E (for example, $\varphi_n(t) := \sqrt{2}\sin 2\pi nt$). Let $T : E \times E \to \ell_1$ be defined by

$$T(f,g) := \left(\int_0^1 \varphi_n(t)f(t)\,dt \cdot \int_0^1 \varphi_n(t)g(t)\,dt\right)_{n=1}^{\infty} \quad (\in \ell_1).$$

Since ℓ_1 contains no copy of c_0, it follows that the Aron-Berner extension T^{**} takes its values in ℓ_1. However T is **not** weakly compact, because $T(\varphi_n, \varphi_n) = e_n$ has no weakly convergent subsequence.

Moreover, for the other classes of operators appearing in Theorem 3.1 there are not a universally accepted definition in the multilinear setting (a situation that we shall again encounter in the next Section.) Hence, we shall start with the suitable definitions:

Definition 3.3. A multilinear operator $T : E_1 \times \cdots \times E_k \to X$ is

- *completely continuous* if whenever $(x_i^n)_{n=1}^{\infty} \subset E_i$ $(1 \le i \le k)$ are weakly Cauchy sequences,
$$(T(x_1^n, \ldots, x_k^n))_{n=1}^{\infty} \subset X$$
is norm-convergent.
- *unconditionally convergent* if given w.u.C. series $\sum_{n\in\mathbb{N}} x_1^n, \ldots, \sum_{n\in\mathbb{N}} x_k^n$, the sequence
$$(T(s_i^m, \ldots, s_k^m))_{m=1}^{\infty}$$
is norm-convergent, where $s_i^m := \sum_{n=1}^{m} x_i^n$.

There are other (non equivalent) definitions of completely continuous and unconditionally convergent multilinear operators in the literature (see [19]), but the above ones have shown to be the more convenient to extend most of the results of the linear theory (see, for instance, [6] where the above defined unconditionally converging multilinear operators are introduced and studied, comparing it with other definitions appeared in the literature.)

Coming back again to the $C(K)$ spaces, we have

Theorem 3.4. ([8], [35], [36]) *Let $T \in \mathcal{L}^k(C(K_1), \ldots, C(K_k); X)$, $T^{**} \in \mathcal{L}^k(C(K_1)^{**}, \ldots, C(K_k)^{**}; X^{**})$ its Aron-Berner extension, $\tilde{T} = T^{**}|_{\mathcal{B}(K_1)\times\ldots\times\mathcal{B}(K_k)}$ and Γ its representing polymeasure. Then, the following statements are equivalent:*

1) *$\tilde{T}$ takes values in X.*

2) Γ *takes values in* X.
3) Γ *is countably additive.*
4) Γ *is regular.*
5) T^{**} *takes values in* X.
6) T *is completely continuous.*
7) $\tilde{T}$ *is completely continuous.*
8) T^{**} *is completely continuous.*
9) T *is unconditionally converging.*
10) $\tilde{T}$ *is unconditionally converging.*
11) T^{**} *is unconditionally converging.*

If we compare the above theorem with its linear analogous Theorem 3.1 we see that the main difference is that in the multilinear case does not appear among the equivalent assertions the condition of T being weakly compact. And Example 3.2 shows that, indeed, this condition is not equivalent to the others (although, of course, it implies all of them). It seems that the weak compactness of an operator in the multilinear setting is a too restrictive condition, and its place is taken by the property *"some Aron-Berner extension of* T *takes its values in* X*"*. See [20] for more related results that support this fact.

4. When a polymeasure can be extended to a measure?

It is easy to see that every polymeasure can be extended to a finitely additive measure γ_m on the algebra $a(\Sigma_1 \times \cdots \times \Sigma_k)$ generated by the measurable rectangles but, in general, this set function can not be extended to a measure on the generated σ-algebra $\sigma(\Sigma_1 \times \cdots \times \Sigma_k)$. However, if there is a countably additive measure μ of bounded variation on $\sigma(\Sigma_1 \times \cdots \times \Sigma_k)$ that extends γ_m, then we have

$$v(\gamma_m)(A_1 \times \cdots \times A_k) = v_1(\gamma)(A_1, \ldots, A_k) = v(\mu)(A_1 \times \cdots \times A_k) \qquad (*)$$

$(A_i \in \Sigma_i)$, where the *p-variation* $(1 \leq p < \infty)$ of γ is the set function

$$v_p(\gamma) : \Sigma_1 \times \cdots \times \Sigma_n \longrightarrow [0, +\infty]$$

given by

$$v_p(\gamma)(A_1,\ldots,A_n)=\sup\left\{\left(\sum_{j_1=1}^{n_1}\cdots\sum_{j_k=1}^{n_k}\left\|\gamma(A_1^{j_1},\ldots A_k^{j_k})\right\|^p\right)^{\frac{1}{p}}\right\}$$

where the supremum is taken over all the finite Σ_i-partitions $(A_i^{j_i})_{j_i=1}^{n_i}$ of A_i $(1\le i\le k)$. We write simply $v(\gamma)$ instead of $v_1(\gamma)$.

Suppose now that $T\in\mathcal{L}^k(C(K_1),\ldots,C(K_k))$ with representing regular polymeasure γ on $Bo(K_1)\times\cdots\times Bo(K_k)$. If there exists a regular finite measure μ on the Borel σ-algebra of $K_1\times\cdots\times K_k$ (hence of finite variation) that extends γ_m, then, by the Riesz representation theorem, μ is the representing measure of some continuous linear form $\hat{T}$ on $C(K_1\times\cdots\times K_k)\approx C(K_1)\check{\otimes}\cdots\check{\otimes}C(K_k)$ and clearly

$$\begin{aligned}\hat{T}(f_1\otimes\cdots\otimes f_k)=&\int_{K_1\times\cdots\times K_k}f_1\cdots f_k d\mu=\\ =&(T(f_1,\ldots,f_k)=)\int_{K_1\times\cdots\times K_k}(f_1,\ldots,f_k)d\gamma. \qquad (\dagger)\end{aligned}$$

Note also that $\|\hat{T}\|=v(\mu)=v(\gamma)$.

Conversely, if T is such that its linearization $\hat{T}$ on $C(K_1)\otimes\cdots\otimes C(K_k)$ is continuous for the ϵ-topology (we shall call a map of this type *integral*), again the Riesz representation theorem yields a regular finite measure μ on $Bo(K_1\times\cdots\times K_k)$ such that ($\dagger$) holds. By the uniqueness of the representation theorem for k-linear maps, we have

$$\mu(A_1\times\cdots\times A_k)=\gamma(A_1,\ldots,A_k)\quad(\text{for every } A_i\in\mathcal{B}(K_i)$$

and so μ extends γ. This is the trivial part of the following

THEOREM 4.1. [10, Theorem 3.3, Corollary 3.4] *Let $T\in\mathcal{L}^k(C(K_1),\ldots,C(K_k))$ with representing polymeasure γ. Then the following are equivalent:*

a) $v(\gamma)<\infty$.

b) *T is integral.*

c) *γ can be extended to a finite regular measure μ on $Bo(K_1\times\cdots\times K_k)$.*

d) *γ can be extended to a countably additive (not necessarily regular) measure μ_2 on $\sigma(Bo(K_1)\times\cdots\times Bo(K_k))$.*

e) *γ can be decomposed in the way $\gamma=\gamma^+-\gamma^-$, where γ^+,γ^- are positive polymeasures.*

Hence, the representing polymeasures of multilinear forms on $C(K)$ spaces that cannot be extended to a countably additive measure are precisely those with infinite variation.

The first example of a bimeasure (in fact, a representing bimeasure of a bilinear form on c_0) with finite semivariation but with infinite variation was given by Littlewood in the seminal paper [24]. An easier one (essentially [37, Example 1.4]) is the following:

EXAMPLE 4.2. Let $T : c_0 \times C([0,1]) \longrightarrow \mathbb{R}$ be defined by the formula

$$T((a_n), f) := \sum_{n=1}^{\infty} \frac{a_n}{n} \int_0^1 f r_n \, d\lambda,$$

where (r_n) are the Rademacher functions and λ denotes the Lebesgue measure. T is continuous, with representing bimeasure

$$\gamma(A, B) = \sum_{n \in A} \frac{1}{n} \int_B r_n \, d\lambda, \quad A \in \mathcal{P}(\mathbb{N}), B \in Bo([0,1]).$$

If we choose $\{\{1\}, \{2\}, \ldots, \{m\}\}$ as a partition of $A_m := \{1, 2, \ldots, m\}$ and, calling $B_j := [\frac{j-1}{2^m}, \frac{j}{2^m}),\ 1 \leq j < 2^m; B_{2^m} = [1 - 2^{-m}, 1]$, we choose $(B_j)_{j=1}^{2^m}$ as a partition of $[0,1]$, we have

$$v(\gamma) \geq v(\gamma)(A_m, [0,1]) \geq \sum_{i=1}^{m} \sum_{j=1}^{2^m} |\gamma(\{i\}, B_j)| = \sum_{i=1}^{m} \sum_{j=1}^{2^m} \frac{1}{i} \frac{1}{2^m} = \sum_{i=1}^{m} \frac{1}{i}.$$

Remark 4.3. The equivalence (a) $\Leftrightarrow$ (d) in the above theorem was proved for bimeasures in [23, Corollary 9], with techniques which do not seem to extend easily to the case $k \geq 3$.

There is a similar characterization of extendible vector-valued representing polymeasures, although in this case they do not need to have finite variation, but a milder condition suffices (see [10, Theorem 4.1]).

5. Vector polymeasures with finite variation

Even when $k = 1$ not all the representing *measures* of operators $T : C(K) \to X$ (even if they are countably additive) have finite variation. In fact, this happens if and only if the operator is *absolutely summing* ([15, Chapter VI]). The study of the corresponding result for multilinear operators requires some new concepts:

Recall that an operator $S \in \mathcal{L}(E;X)$ is called G-integral (the "G" comes from "Grothendieck") if the associated bilinear form

$$\begin{array}{rccc} B_S: & E \times X^* & \longrightarrow & \mathbb{K} \\ & (x,y) & \mapsto & y(S(x)) \end{array}$$

is integral in the sense previously defined.

Next is a multilinear version of the well known notion of (p,q)-summing operator between Banach spaces. We refer the reader to [14] for non explained notation and any result concerning this subject.

Definition 5.1. ([11, Definition 2.1]; see also [25]) Let $1 \le q_1, \dots, q_k \le p < +\infty$. A multilinear operator $T : E_1 \times \cdots \times E_k \longrightarrow X$ is *multiple* $(p;q_1,\dots,q_k)$*-summing* if there exists a constant $K > 0$ such that, for every choice of positive integers m_j $(1 \le j \le k)$ and for every choice of elements $x^j_{i_j} \in E_j$ $(1 \le i_j \le m_j)$, the following relation holds

$$\left(\sum_{i_1,\dots,i_k=1}^{m_1,\dots,m_k} \|T(x^1_{i_1},\dots,x^k_{i_k})\|^p \right)^{\frac{1}{p}} \le K \prod_{j=1}^{k} \|(x^j_{i_j})_{i_j=1}^{m_j}\|^{\omega}_{q_j}, \tag{1}$$

where for any finite sequence $(x_i)_{i=1}^m$ in a Banach space E we write

$$\|(x_i)_{i=1}^m\|^{\omega}_p := \sup\left\{ \left(\sum_{i=1}^{m} |x^*(x_i)|^p \right)^{\frac{1}{p}} : x^* \in B(E^*) \right\}.$$

In that case, we define the *multiple* $(p;q_1,\dots,q_k)$*-summing norm* of T by

$$\pi_{(p;q_1,\dots,q_k)}(T) = \min\{K : K \text{ verifies } (1)\}.$$

A multiple $(p;q,\dots,q)$-summing operator will be called *multiple* (p,q)*-summing* and we write $\pi_{(p,q)}$ for the associated norm. Moreover, a multiple (p,p)-summing operator will be called *multiple* p*-summing* and we write π_p for the associated norm. We have that the class $\Pi^k_{(p;q_1,\dots,q_k)}(E_1,\dots,E_k;X)$ of multiple $(p;q_1,\dots,q_k)$-summing k-linear operators is a Banach space with its associated norm $\pi_{(p;q_1\dots,q_k)}$.

This definition is more restrictive than the usual notion of multilinear $(p; q_1, \dots, q_k)$ summing operator, introduced by Pietsch in [32], in which only the "diagonal" sum

$$\left(\sum_{i=1}^{m} \|T(x_i^1, \dots, x_i^k)\|^p \right)^{\frac{1}{p}}$$

has to be dominated by the right hand side of (1). This class is endowed again with a natural Banach norm. When $\frac{1}{p} = \frac{1}{q_1} + \cdots \frac{1}{q_k}$ the $(p; q_1, \dots, q_k)$ -summing operators are called *dominated operators*, and they are denoted by $\mathcal{D}^n_{(q_1,\dots,q_k)}$. In the case $q_1 = \cdots = q_k = q$ they will be called simply q-dominated operators, and denoted by $\mathcal{D}^k_q$.

The dominated operators share many properties of the p-summing linear operators. For instance, they are exactly those multilinear operators for which a domination theorem and a factorization theorem of Pietsch type hold (see [32] for the case of multilinear *forms*. A general proof appears in [28].) From the domination theorem for dominated operators follows immediately that when $1 \leq r \leq p < \infty$,

$$\mathcal{D}^k_r(E_1, \dots, E_k; X) \subset \Pi^k_p(E_1, \dots, E_k; X),$$

also with norm-one inclusion (see [31, Theorem 3.10] for a more precise result.) And, of course, the inclusions are in general strict.

Through the domination theorem a large part of the theory and applications of linear p-summing operators can be extended to the multilinear setting. However, no multilinear version of Grothendieck's famous theorems concerning operators on $C(K)$ or $L_1(\mu)$ spaces were known (except for some very special cases, as in [3].)

The class of multiple summing operators allows to obtain such a multilinear version of p-summing formulations of Grothendieck's theorems:

THEOREM 5.2. ([11, Theorem 3.1]) *Let E_j be a $\mathcal{L}_{\infty,\lambda_j}$-space for $1 \leq j \leq k$ and let X be a space with cotype 2. Then, every multilinear operator $T : E_1 \times \cdots \times E_k \longrightarrow X$ is multiple 2-summing and*

$$\pi_2(T) \leq K_k \prod_{j=1}^{k} \lambda_j \|T\|,$$

where $K_k = (3^{\frac{1}{4}} C_2(Y))^{2k}$, with $C_2(Y)$ the cotype 2 constant of Y.

THEOREM 5.3. ([11, Theorem 5.2]) *Let E_j be a $\mathcal{L}_{1,\lambda_j}$-space $(1 \leq j \leq k)$ and let H be a Hilbert space. Then, every multilinear operator $T : E_1 \times \cdots \times E_k \longrightarrow H$ is multiple 1-summing and*

$$\pi_1(T) \leq C^k \prod_{j=1}^{k} \lambda_j \|T\|,$$

where C is an absolute constant.

As in the linear case, Theorem 5.3 gives rise (and, in fact, it is equivalent) to a matrix inequality, which can be considered as a multilinear generalization of Grothendieck's inequality:

THEOREM 5.4. *Let $k \geq 2$ and for every $l \in \{1, \ldots, k\}$, let $(a_{i_l,j_l})_{i_l,j_l=1}^{p_l,q_l} \subset \mathbb{K}$ be a matrix such that*

$$\sup \left\{ \left| \sum_{i_l,j_l=1}^{p_l,q_l} a_{i_l,j_l} s_{i_l} t_{j_l} \right| : |s_{i_l}| \leq 1, |t_{j_l}| \leq 1 \right\} \leq 1. \tag{\dagger}$$

Let H be a Hilbert space and consider elements $x_{i_1,\ldots,i_k}, y_{j_1,\ldots,j_k}$ $(1 \leq i_l \leq p_l, 1 \leq j_l \leq q_l, l = 1, \ldots, k)$ in the unit ball of H. Then

$$\left| \sum_{i_1,\ldots,i_k,j_1,\ldots,j_k=1}^{p_1,\ldots,p_n,q_1\ldots,q_k} a_{i_1,j_1} \cdots a_{i_k,j_k} \langle x_{i_1,\ldots,i_k}, y_{j_1,\ldots,j_k} \rangle \right| \leq C^k. \tag{\ddagger}$$

(*Here C is the absolute constant that appears in Theorem 5.3.*)

Proof. Consider the multilinear map

$$T : \ell_1^{p_1} \times \cdots \times \ell_1^{p_k} \to H$$

defined by $T(e_{i_1}, \ldots, e_{i_k}) := x_{i_1,\ldots,i_k}$. It is easy to see that T is continuous and $\|T\| \leq 1$. Hence, by Theorem 5.3, T is multiple 1-summing and $\pi_1(T) \leq C^k$.

Now, for each $1 \leq l \leq k$ consider the map $u_l : \ell_\infty^{q_l} \to \ell_1^{p_l}$ given by $u_l(e_{j_l}) = \sum_{i_l=1}^{p_l} a_{i_l,j_l} e_{i_l}$. Then $\|u_l\|$ is just the left hand side of (†) and hence

$$\|(u_l(e_{j_l}))_{j_l=1}^{q_l}\|_1^w = \|u_l\| \leq 1 \ (1 \leq l \leq k)$$

Consequently,

$$\sum_{j_1,\ldots,j_k=1}^{q_1,\ldots,q_k} \|T(u_1(e_{j_1}), \ldots, u_k(e_{j_k}))\| \leq \pi_1(T) \leq C^k$$

The inequality (‡) follows, taking into account that for every choice of $j_1, \dots, j_k$,

$$\left| \sum_{i_1,\dots,i_k=1}^{p_1,\dots,p_k} a_{i_1,j_1} \cdots a_{i_k,j_k} \langle x_{i_1,\dots,i_k}, y_{j_1,\dots,j_n k} \rangle \right| \leq \|T(u_1(e_{j_1}), \dots, u_k(e_{j_k}))\|.$$

In general, the class of multiple summing operators behaves better in many ways than other previously defined. Besides the mentioned multilinear versions of Grothendieck's theorems, also several relations with nuclear and Hilbert-Schmidt multilinear operators that extend and generalize classical linear results, are true for this class ([29], [30], [31]). It is easy to see that these results are not true for the general class of Pietsch's multilinear summing operators.

The most basic example of a $(p;q)$-multiple summing operator is given by a multilinear integral operator $T : E_1 \times \cdots \times E_k \longrightarrow X$ such that its linearization $\hat{T} : E_1 \check{\otimes} \cdots \check{\otimes} E_k \longrightarrow X$ is (p,q)-summing. Also, it should be mentioned that not every continuous multilinear *form* is necessarily multiple summing.

With this definition at hand, we can obtain the announced characterization of the vector representing polymeasures with finite variation:

Theorem 5.5. ([31, Proposition 3.1]) *Let* $T \in \mathcal{L}^k(C(K_1), \dots, C(K_k); X)$ *with representing polymeasure* γ*. Then the following are equivalent:*

a) T *is 1-multiple summing.*

b) T *is integral and its linearization* $\hat{T} : C(K_1) \check{\otimes} \cdots \check{\otimes} C(K_k) \longrightarrow X$ *is G-integral.*

c) $v(\gamma) < \infty$.

Proof. (b) and (c) are equivalent by Theorem 4.1. As we have commented before, (b) always implies (a). Finally (a)⇒ (b) follows from the fact that the Aron-Berner extension of a $(p; q_1, \dots, q_k)$ multiple summing operator is also $(p; q_1, \dots, q_k)$-summing ([31, Theorem 2.3] and the (easily checked) fact that if $(A_i)_{i=1}^m$ is a Borel partition of K, then $\|(\chi_{A_i})_{i=1}^m\|_1^\omega \leq 1$.

Remark 5.6. a) By a theorem of A. Defant and J.Voigt ([1, Theorem 3.10]), every continuous multilinear *form* is $(1; 1, \dots, 1)$ summing (in Pietsch's sense). Hence, example 4.2 shows that condition (a) in the theorem above is not implied by the condition of T being $(1; 1, \dots, 1)$-summing.

b) By using some of the results contained in [3] it can be proved that any continuous multilinear *form* on the product of $\mathcal{L}_\infty$ spaces is in fact $(1; 2, \dots, 2)$

summing and, by [11, Theorem 3.1], also multiple 2-summing. Again, Example 4.2 shows that neither of these two conditions imply condition (a) of Theorem 5.5.

As in the linear case, the coincidence of multiple 1-summing operators with those integral operators such that its linearization is G-integral characterizes, in fact, the $\mathcal{L}_\infty$-spaces ([31, Proposition 3.6]).

Next theorem, which extends the equivalence (a) $\Leftrightarrow$ (c) in Theorem 5.5, is one of the main results of [31]:

THEOREM 5.7. ([31, Theorem 3.8]) *Let* (Ω_j, Σ_j), $(1 \leq j \leq k)$ *be measurable spaces, let* $1 \leq p < \infty$ *and let* X *be a Banach space. Let* $\gamma : \Sigma_1 \times \cdots \times \Sigma_k \longrightarrow X$ *be a polymeasure with bounded semivariation and let us consider the associated multilinear operator* $T : \mathcal{B}(\Sigma_1) \times \cdots \times \mathcal{B}(\Sigma_k) \longrightarrow X$. *Then* $v_p(\gamma) < \infty$ *if and only if* T *is multiple* $(p; 1)$*-summing.*

Moreover, in that case,

$$v_p(\gamma) \leq \pi_{(p;1)}(T) \leq 2^{k\left(1-\frac{1}{p}\right)} v_p(\gamma) \qquad \text{(real case)}$$

$$v_p(\gamma) \leq \pi_{(p;1)}(T) \leq 2^{k\left(2-\frac{1}{p}\right)} v_p(\gamma) \qquad \text{(complex case)}$$

Of course, a corresponding result for representing polymeasures of multilinear operators on $C(K)$ spaces also holds.

If we take into account that every multilinear operator T with values in a space X of cotype q is multiple $(q; 1)$-summing ([11, Theorem 3.2]), we obtain the surprising result that *every polymeasure of finite semivariation, with values in a cotype* q *space, has finite* q*-variation.* In particular, every scalar bounded polymeasure has finite 2-variation (although, as we know, it might have *infinite* variation).

This last result connects with the already mentioned Littlewood's paper [24] where a bimeasure with bounded semivariation and infinite variation appears for the first time. But the most important part of the paper is the assertion that *every* scalar bimeasure on $\mathcal{P}(\mathbb{N})$ of bounded semivariation, has finite p-variation for $p \geq \frac{4}{3}$, and the number $\frac{4}{3}$ is sharp, in the sense that for every $1 \leq q < \frac{4}{3}$ there is a bimeasure with finite semivariation and infinite q-variation. In fact, Littlewood proved ([24, Theorem 1(1)]) that for any continuous bilinear form $T : c_0 \times c_0 \longrightarrow \mathbb{K}$,

$$\left(\sum_{n,m=1}^{\infty} |T(e_n, e_m)|^{\frac{4}{3}} \right)^{\frac{3}{4}} \leq C\|T\|,$$

where (e_n) is the usual base of c_0 and $C > 0$ is an absolute constant.

This inequality has had far reaching consequences in Harmonic analysis (specifically, in the theory of *Sidon sets*; see [4, Chapter VII]) The k-linear extension of Littlewood's inequality was stated without proof by Davie ([13]) and independently by Johnson and Woodward in [22], where a proof was given of the following fact: For any continuous k-linear form $T : c_0 \times \cdots \times c_0 \longrightarrow \mathbb{K}$,

$$\left(\sum_{i_1,\ldots,i_k=1}^{\infty} \|T(e^1_{i_1},\ldots,e^k_{i_k})\|^{\frac{2k}{k+1}} \right)^{\frac{k+1}{2k}} \leq 2^{\frac{k-1}{2}} \|T\|.$$

With this result at hand and Theorem 5.7, it is easy to prove the following

Proposition 5.8. ([31, Corollary 3.21]) *Every scalar k-measure γ of bounded semivariation has finite $\frac{2k}{k+1}$-variation and*

$$v_{\frac{2k}{k+1}}(\gamma) \leq 2^{\frac{k-1}{2}} \|\gamma\|.$$

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