EXTRACTA MATHEMATICAE Vol. **22**, Núm. 3, 297 – 314 (2007)

On the Existence of Prolongations of Connections by Bundle Functors

W. M. Mikulski

Institute of Mathematics, Jagiellonian University, Reymonta 4, Kraków, Poland,
mikulski@im.uj.edu.pl

Presented by Marcelo Epstein Received January 1, 2007

Abstract: We construct canonically a general connection $A^F(\Gamma, \nabla)$ on $Fp : FY \to FM$ from a general connection Γ on a fibred manifold $p : Y \to M$ by means of a projectable classical linear connection ∇ on Y, where $F : \mathcal{M}f \to \mathcal{VB}$ is a vector bundle functor. In the case of a not necessarily vector bundle functor $F : \mathcal{M}f \to \mathcal{FM}$ we find some simple equivalent condition on the existence of a general connection $A(\Gamma, \nabla)$ on $Fp : FY \to FM$ from a general connection Γ on $Y \to M$ by means of a projectable classical linear connection ∇ on Y. We present a construction of a classical linear connection $A^F(\nabla)$ on FY from a projectable classical linear connection ∇ on Y for any fiber product preserving bundle functor $F : \mathcal{FM}_m \to \mathcal{FM}$. We characterize bundle functors $F : \mathcal{FM}_{m,n} \to \mathcal{FM}$ which admit a construction of a classical linear connection $A(\nabla)$ on FY from a projectable classical linear connection ∇ on Y. We characterize gauge bundle functors $F : \mathcal{VB}_{m,n} \to \mathcal{FM}$ which admit a construction of a classical linear connection $A(D, \nabla)$ on FE from a linear general connection D on $E \to M$ by means of a classical linear connection ∇ on M.

Key words: General connection, classical linear connection, (vector) (gauge) bundle functor, fiber product preserving bundle functor, Weil algebra, natural isomorphism, natural (gauge) operator.

AMS *Subject Class.* (2000): 58A05, 58A20, 58A32.

0. Introduction

From now on, let $\mathcal{M}f$ be the category of all manifolds and all maps, $\mathcal{M}f_m$ be the category of m-dimensional manifolds and their embeddings, $\mathcal{FM}$ be the category of all fibred manifolds (i.e. surjective submersions between manifolds) and their fibred maps, $\mathcal{FM}_m$ be the category of all fibred manifolds with m-dimensional bases and their fibred maps covering embeddings, $\mathcal{FM}_{m,n}$ be the category of fibred manifolds with m-dimensional bases and n-dimensional fibres and their fiber embeddings, $\mathcal{VB}$ be the category of all vector bundles and their vector bundle maps, and $\mathcal{VB}_{m,n}$ be the category of vector bundles with m-dimensional bases and n-dimensional fibres and their vector bundle embeddings.

A general connection on a fibred manifold $p : Y \to M$ is a section $\Gamma : Y \to J^1Y$ of the first jet prolongation $J^1Y \to Y$ of $p : Y \to M$. Equivalently, Γ can be treated as the corresponding lifting map

$$\Gamma : TM \times_M Y \to TY,$$

see [10]. If $E \to M$ is a vector bundle, then a general connection $\Gamma : E \to J^1E$ is called linear if it is a vector bundle map. In particular if $E = TM$ is the tangent bundle of M, a linear connection $\Gamma : TM \to J^1TM$ is a classical linear connection on M (it can be equivalently defined by its covariant derivative $\nabla_X Y$ on vector fields, or equivalently defined as the corresponding section of the affine bundle of connections $QM = \pi^{-1}(id_{TM}) \subset T^*M \otimes J^1TM$). Given a fibered manifold $p : Y \to M$, a classical linear connection ∇ on Y is called projectable if there exists a (unique) p-related with ∇ classical linear connection $\underline{\nabla}$ on M

The theory of canonical constructions on connections has its origin in the works of C. Ehresmann, [3], [4]. Some canonical constructions on connections have motivations in quantum mechanics, higher order dynamics, field theories and gauge theories of mathematical physics, [6], [19], [21]. That is why, canonical constructions on connections have been studied in many papers, [1], [2], [5], [7]–[10], [12]–[20], [22]. Roughly speaking, a canonical construction on connections is a rule A transforming given connections $\Gamma_1, \dots, \Gamma_k$ on a manifold Y or fibred manifold $Y \to M$ into a connection $A(\Gamma_1, \dots, \Gamma_k)$ on a functor bundle FY of Y, which is well defined (i.e., the definition of $A(\Gamma_1, \dots, \Gamma_k)$ is independent of the choice of local coordinates on Y). Such constructions have reflection in the corresponding natural operators in the sense of Kolář-Michor-Slovák [10]. The theory and precise definitions of bundle functors and natural operators (canonical constructions) can be found in the fundamental monograph [10].

In the first part of the paper, we study the following two problems.

Problem 1. Let $F : \mathcal{M}f \to \mathcal{VB}$ be a vector bundle functor. To construct a general connection $A^F(\Gamma, \nabla)$ on $Fp : FY \to FM$ from a general connection Γ on $p : Y \to M$ by means of a projectable classical linear connection ∇ on Y.

Problem 2. To characterize (not necessarily vector) bundle functors $F : \mathcal{M}f \to \mathcal{FM}$ such that there exists a general connection $A(\Gamma, \nabla)$ on $Fp : FY \to FM$ induced from a general connection Γ on $p : Y \to M$ by means of a projectable classical linear connection ∇ on Y.

We remark that in [14], we studied the problem whether for a given general connection $\Gamma : Y \to J^1Y$ on a fibred manifold $p : Y \to M$ one can construct canonically a general connection $A(\Gamma) : FY \to J^1(FY \to FM)$ on $Fp : FY \to FM$, where $F : \mathcal{M}f \to \mathcal{VB}$ is a vector bundle functor with the point property $F(\{point\}) = \{0\}$. We proved that a construction $A(\Gamma)$ in question exists if and only if F is product preserving.

In the second part of the paper we study the following three problems.

Problem 3. Let $F : \mathcal{FM}_m \to \mathcal{FM}$ be a fiber product preserving bundle functor. To construct a classical linear connection $A^F(\nabla)$ on FY from a projectable classical linear connection ∇ on $Y \to M$.

Problem 4. To characterize bundle functors $F : \mathcal{FM}_{m,n} \to \mathcal{FM}$, which admits a canonical construction of a classical linear connection $A(\nabla)$ on FY from a projectable classical linear connection ∇ on $Y \to M$.

Problem 5. To give an example of a bundle functor $F : \mathcal{FM}_{m,n} \to \mathcal{FM}$ which does not admit any construction of a classical linear connection $A(\nabla)$ on FY from a projectable classical linear connection ∇ on $Y \to M$.

We inform that the most important example of a fiber product preserving bundle functor is the r-jet prolongation functor $J^r : \mathcal{FM}_m \to \mathcal{FM}$. All fiber product preserving bundle functors $F : \mathcal{FM}_m \to \mathcal{FM}$ have been classified in [11].

Fiber product preserving bundle functors on $\mathcal{FM}_m$ play a similar role as product preserving bundle functors (Weil bundles) on manifolds. On the Weil bundle T^AM we have the classical linear connection ∇^A from a given classical linear connection ∇ on M, the complete lift of ∇ in the sense of A. Morimoto [17]. To construct ∇^A from ∇, A. Morimoto defined a lot of canonical lifts of functions, vector fields and forms. Unfortunately, in the case of $J^r : \mathcal{FM}_m \to \mathcal{FM}$ any natural operator lifting projectable vector fields X on $Y \to M$ to J^rY is the constant multiple of the flow operator, [10]. Also (one can show) that any natural lifting of functions $f : Y \to \mathbf{R}$ to $\pi^r_0 : J^rY \to Y$ is the vertical lift $f^V = f \circ \pi^r_0 : J^rY \to \mathbf{R}$ composed with a function $\mathbf{R} \to \mathbf{R}$. In other words, J^r is a very rigid functor. Thus it is very unexpected the positive answer to Problem 3 for $F = J^r$. It must use quite different method than the one by A. Morimoto [17].

In the special case $m = 0$, we have $\mathcal{FM}_{0,n} = \mathcal{M}f_n$ under the identification $Y \to \{point\}$ with Y. Any classical linear connection on Y is projectable on

$Y \to \{point\}$. Thus the solution of Problem 4 gives a characterization of bundle functors (natural bundles) $F : \mathcal{M}f_n \to \mathcal{FM}$ which admits a construction of a classical linear connection $A(\nabla)$ on N from a classical linear connection ∇ on N. This (in particular) shows the reason why a prolongation of connections ∇ on N to $T^A N$ exists.

In the third part we solve the following problems.

PROBLEM 6. To characterize all gauge bundle functors $F : \mathcal{VB}_{m,n} \to \mathcal{FM}$, which admit a canonical construction of a classical linear connection $A(D, \nabla)$ on FE from a linear general connection D on an $\mathcal{VB}_{m,n}$-object $E \to M$ by means of a classical linear connection ∇ on M.

PROBLEM 7. To give an example of a gauge bundle functor $F : \mathcal{VB}_{m,n} \to \mathcal{FM}$ which does not admit any canonical construction of a classical linear connection $A(D, \nabla)$ on FE from a linear general connection D on an $\mathcal{VB}_{m,n}$-object $E \to M$ by means of a classical linear connection ∇ on M.

We inform that in [15], we proved that there is no canonical construction of a classical linear connection $A(D)$ on FE from a linear general connection D on a $\mathcal{VB}_{m,n}$-object $E \to M$. So, the using of an auxiliary classical linear connection ∇ on M is unavoidable in Problem 6.

All manifolds and maps are assumed to be of class $\mathbf{C}^\infty$.

PART I. SOME CONSTRUCTIONS ON GENERAL CONNECTIONS

1. SOME DEFINITIONS

Let $B : \mathcal{FM} \to \mathcal{M}f$ be the base functor, $B(Y \to M) = M$, $B(f, \underline{f}) = \underline{f}$.

DEFINITION 1. A *bundle functor over manifolds* is a covariant functor $F : \mathcal{M}f \to \mathcal{FM}$ satisfying $B \circ F = id$ and the localization condition: for every inclusion of an open subset $i_U : U \to M$, FU is the restriction $p_M^{-1}(U)$ of $p_M : FM \to M$ over U and $Fi_U : FU \to FM$ is the inclusion $p_M^{-1}(U) \to FM$, [10]. If a bundle functor F has values in the category $\mathcal{VB}$, we say that $F : \mathcal{M}f \to \mathcal{VB}$ is a *vector bundle functor*.

A simple example of a vector bundle functor is the tangent functor $T : \mathcal{M}f \to \mathcal{VB}$ sending a manifold M into its tangent bundle TM over M and any map $f : M \to M_1$ into the tangent map $Tf : TM \to TM_1$ over f. An example of a bundle functor F which is not vector is the tangent functor $T^r : \mathcal{M}f \to \mathcal{FM}$ for $r \geq 2$ sending any manifold M into the r-tangent bundle $T^rM = J^r_0(\mathbf{R}, M)$ and any map $f : M \to M_1$ into the induced fibred map $T^rf : T^rM \to T^rM_1$ covering f, $T^rf(j^r_0\gamma) = j^r_0(f \circ \gamma)$, $j^r_0\gamma \in T^rM$. More examples of bundle functors over manifolds can be found in [10].

Let $F : \mathcal{M}f \to \mathcal{FM}$ be a bundle functor.

Definition 2. An $\mathcal{FM}_{m,n}$-*natural operator* (a *canonical construction*) transforming connections Γ on $\mathcal{FM}_{m,n}$-objects $Y \to M$ and a projectable classical linear connection ∇ on $Y \to M$ into general connections $A(\Gamma, \nabla)$ on fibred manifold $Fp : FY \to FM$ is a family of $\mathcal{FM}_{m,n}$-invariant regular operators (functions)

$$A : Con(p : Y \to M) \times Con_{proj-clas-lin}(p : Y \to M) \to Con(Fp : FY \to FM)$$

for any $\mathcal{FM}_{m,n}$-object $p : Y \to M$, where $Con(p : Y \to M)$ is the set of all general connections on $p : Y \to M$ and $Con_{proj-clas-lin}(p : Y \to M)$ is the set of all projectable classical linear connections on $p : Y \to M$. The invariance means that for any general connections Γ and Γ_1 on $\mathcal{FM}_{m,n}$-objects $p : Y \to M$ and $p_1 : Y_1 \to M_1$ (respectively) and projectable classical linear connections ∇ and ∇_1 on $p : Y \to M$ and $p_1 : Y_1 \to M_1$ (respectively), if Γ and Γ_1 are f-related and ∇ and ∇_1 are f-related for some $\mathcal{FM}_{m,n}$-map $f : Y \to Y_1$ covering $\underline{f} : M \to M_1$, then $A(\Gamma, \nabla)$ and $A(\Gamma_1, \nabla_1)$ are $(Ff, F\underline{f})$-related. The regularity means that A transforms smoothly parametrized families of pairs of connections into smoothly parametrized families of connections.

2. Solution of Problem 1

Let $F : \mathcal{M}f \to \mathcal{VB}$ be a vector bundle functor. We have

$$F_0\mathbf{R}^m \tilde{=} F(i_{m,m+n})(F_0(\mathbf{R}^m)) \subset F_{(0,0)}(\mathbf{R}^m \times \mathbf{R}^n)\,,$$

where $i_{m,m+n} : \mathbf{R}^m \to \mathbf{R}^m \times \mathbf{R}^n$, $x \to (x, 0)$. Define

$$C^F : (\mathbf{R}^m \times TF_0\mathbf{R}^m) \times_{F_0\mathbf{R}^m} F_{(0,0)}(\mathbf{R}^m \times \mathbf{R}^n) \to (\mathbf{R}^m \times \mathbf{R}^n) \times TF_{(0,0)}(\mathbf{R}^m \times \mathbf{R}^n)$$

by

$$C^F\Big(\big(a, \frac{d}{d\tau}_{|0}(Fp(f) + \tau u)\big), f\Big) = \big((a, 0), \frac{d}{d\tau}_{|0}(f + \tau u)\big)\,,$$

$a \in \mathbf{R}^m$, $u \in F_0\mathbf{R}^m \subset F_{(0,0)}(\mathbf{R}^m \times \mathbf{R}^n)$, $f \in F_{(0,0)}(\mathbf{R}^m \times \mathbf{R}^n)$. From the translation identification $F\mathbf{R}^m = \mathbf{R}^m \times F_0\mathbf{R}^m$ we have the identification $TF\mathbf{R}^m = T\mathbf{R}^m \times TF_0\mathbf{R}^m$. Thus

$$\mathbf{R}^m \times TF_0\mathbf{R}^m = (TF\mathbf{R}^m)_0 \,.$$

Similarly,

$$(\mathbf{R}^m \times \mathbf{R}^n) \times TF_{(0,0)}(\mathbf{R}^m \times \mathbf{R}^n) = (TF(\mathbf{R}^m \times \mathbf{R}^n))_{(0,0)} \,.$$

Thus

$$C^F : (TF\mathbf{R}^m)_0 \times_{F_0\mathbf{R}^m} F_{(0,0)}(\mathbf{R}^m \times \mathbf{R}^n) \to (TF(\mathbf{R}^m \times \mathbf{R}^n))_{(0,0)} \,.$$

One can easily observe that

LEMMA 1. (a) *The mapping C^F is fiber linear in the first factor.*

(b) *We have the lifting property*

$$TFp(C^F(w,f)) = w$$

for any $(w,f) \in (TF\mathbf{R}^m)_0 \times_{F_0\mathbf{R}^m} F_{(0,0)}(\mathbf{R}^m \times \mathbf{R}^n)$.

(c) *We have the invariant condition*

$$C^F\big(TF\varphi(w), F(\varphi \times \psi)(f)\big) = TF(\varphi \times \psi)(C^F(w,f))$$

for any $(w,f) \in (TF\mathbf{R}^m)_0 \times_{F_0\mathbf{R}^m} F_{(0,0)}(\mathbf{R}^m \times \mathbf{R}^n)$ and any linear isomorphisms $\varphi : \mathbf{R}^m \to \mathbf{R}^m$ and $\psi : \mathbf{R}^n \to \mathbf{R}^n$.

Let Γ be a general connection on a fibered manifold $p : Y \to M$ with $\dim(M) = m$ and $\dim(Y) = m + n$. Let ∇ be a projectable classical linear connection on Y with the underlying classical linear connection $\underline{\nabla}$ on M. Let $y \in Y$, $p(y) = x$. The following lemma is almost clear.

LEMMA 2. (a) *There is a normal fiber coordinate system $\Psi : (U, y) \to (\mathbf{R}^m \times \mathbf{R}^n, (0,0))$ on Y of ∇ with center y covering a normal coordinate system $\underline{\Psi} : (\underline{U}, x) \to (\mathbf{R}^m, 0)$ on M of $\underline{\nabla}$ with center x and sending $\Gamma(y)$ into $j^1_0(\theta)$, where θ is the zero section of $\mathbf{R}^m \times \mathbf{R}^n \to \mathbf{R}^m$.*

(b) *If $\overline{\Psi}$ is another such system then there are linear isomorphisms $\varphi : \mathbf{R}^m \to \mathbf{R}^m$ and $\psi : \mathbf{R}^n \to \mathbf{R}^n$ such that $\overline{\Psi} = (\varphi \times \psi) \circ \Psi$ near y.*

EXAMPLE 1. Let $F : \mathcal{M}f \to \mathcal{VB}$ be a vector bundle functor. Let Γ be a general connection on a fibred manifold $p : Y \to M$ and ∇ be a projectable classical linear connection on $Y \to M$, $\dim(Y) = m + n$, $\dim(M) = m$. We are going to construct a general connection $A^F(\Gamma, \nabla)$ on $Fp : FY \to FM$. Let $z \in F_yY$, $y \in Y$. Define $A^F(\Gamma, \nabla)(z) \in (J^1_{Fp(z)}FY)_z$ as follows. Choose a system Ψ as in Lemma 2(a) and put

$$A^F(\Gamma, \nabla)(z) = J^1F\Psi^{-1}\big(j^1_{Fp\,(F\Psi(z))}(\sigma^F_{F\Psi(z)})\big)\,,$$

where $j^1_{Fp(f)}(\sigma^F_f) \in \big(J^1_{Fp(f)}(F(\mathbf{R}^m \times \mathbf{R}^n))\big)_f$ is the unique element such that $C^F(u, f) = d_{Fp(f)}\sigma^F_f(u)$, $f \in F_{(0,0)}(\mathbf{R}^m \times \mathbf{R}^n)$, $u \in T_{Fp(f)}F\mathbf{R}^m$ (the existence of such $j^1_{Fp(f)}(\sigma^F_f)$ follows from Lemma 1(a)(b)). If $\overline{\Psi}$ is another such system, then by Lemma 2(b) and the invariant condition from Lemma 1(c) we obtain the same value $A^F(\Gamma, \nabla)(z)$. Thus the definition of $A^F(\Gamma, \nabla)(z)$ is correct. The resulting map $A^F(\Gamma, \nabla) : FY \to J^1(FY \to FM)$ is a general connection on $Fp : FY \to FM$.

Because of the canonical character of the construction $A^F(\Gamma, \nabla)$ we immediately have

PROPOSITION 1. (1) *Given non-negative integers m and n with $n \geq 1$, the family of operators $A^F : (\Gamma, \nabla) \to A^F(\Gamma, \nabla)$ (described in Example 1) is an $\mathcal{FM}_{m,n}$-natural operator.*

(2) *Let $a = \{a_M\} : F_1 \to F_2$ be an $\mathcal{M}f$-natural isomorphism of vector bundle functors (i.e. $a_M : F_1M \to F_2M$ is a base preserving vector bundle isomorphism for any manifold M such that $a_{M_2} \circ F_1 f = F_2 f \circ a_{M_1}$ for any map $f : M_1 \to M_2$). Then for any $\mathcal{FM}_{m,n}$-object $p : Y \to M$ any projectable classical linear connection ∇ on $Y \to M$ and any general connection Γ on $Y \to M$, general connections $A^{F_1}(\Gamma, \nabla)$ and $A^{F_2}(\Gamma, \nabla)$ are (a_Y, a_M)-related.*

3. SOLUTION OF PROBLEM 2

Let $F : \mathcal{M}f \to \mathcal{FM}$ be a (not necessarily vector) bundle functor. Suppose that there exists a $\mathcal{FM}_{m,n}$-canonical construction of a general connection $A(\Gamma, \nabla)$ on $Fp : FY \to FM$ from a general connection Γ on $p : Y \to M$ by means of a projectable classical linear connection ∇ on Y. Then by the composition of the restrictions of

$$A(\Gamma^0, \nabla^0) : TF\mathbf{R}^m \times_{F\mathbf{R}^m} F(\mathbf{R}^m \times \mathbf{R}^n) \to TF(\mathbf{R}^m \times \mathbf{R}^n)\,,$$

where Γ^0 is the trivial general connection on $p : \mathbf{R}^m \times \mathbf{R}^n \to \mathbf{R}^m$ and ∇^0 is the projectable classical linear connection on $\mathbf{R}^m \times \mathbf{R}^n$ with vanishing Christoffel symbols, with the projection

$$(TF(\mathbf{R}^m \times \mathbf{R}^n))_{(0,0)} \tilde{=} (\mathbf{R}^m \times \mathbf{R}^n) \times T(F_{(0,0)}(\mathbf{R}^m \times \mathbf{R}^n)) \to T(F_{(0,0)}(\mathbf{R}^m \times \mathbf{R}^n))\,,$$

we have the general connection

$$\Gamma^o : T(F_0\mathbf{R}^m) \times_{F_0\mathbf{R}^m} F_{(0,0)}(\mathbf{R}^m \times \mathbf{R}^n) \to T(F_{(0,0)}(\mathbf{R}^m \times \mathbf{R}^n))$$

on $F_{(0,0)}p : F_{(0,0)}(\mathbf{R}^m \times \mathbf{R}^n) \to F_0\mathbf{R}^m$. This connection is $Gl(m) \times Gl(n)$-invariant because of the $Gl(m) \times Gl(n)$-invariance of A^F, Γ^0 and ∇^0.

Conversely suppose that there exists a $Gl(m) \times Gl(n)$-invariant general connection

$$\Gamma^o : T(F_0\mathbf{R}^m) \times_{F_0\mathbf{R}^m} F_{(0,0)}(\mathbf{R}^m \times \mathbf{R}^n) \to T(F_{(0,0)}(\mathbf{R}^m \times \mathbf{R}^n))$$

on $F_{(0,0)}p : F_{(0,0)}(\mathbf{R}^m \times \mathbf{R}^n) \to F_0\mathbf{R}^m$. Then we have the map

$$C^o : (TF\mathbf{R}^m)_0 \times_{F_0\mathbf{R}^m} F_{(0,0)}(\mathbf{R}^m \times \mathbf{R}^n) \to (TF(\mathbf{R}^m \times \mathbf{R}^n))_{(0,0)}$$

given by

$$C^o((a,u),f) = ((a,0),\Gamma^o(u,f)),$$

$(a,u) \in (TF\mathbf{R}^m)_0 \tilde{=} \mathbf{R}^m \times TF_0\mathbf{R}^m$, $f \in F_{(0,0)}(\mathbf{R}^m \times \mathbf{R}^n)$, $Fp(f) = \pi(u)$, where $\pi : TF\mathbf{R}^m \to F\mathbf{R}^m$ is the tangent projection, $(\mathbf{R}^m \times \mathbf{R}^n) \times TF_{(0,0)}(\mathbf{R}^m \times \mathbf{R}^n) \tilde{=} (TF(\mathbf{R}^m \times \mathbf{R}^n))_{(0,0)}$. This map C^o has the properties as C^F in Lemma 1. Then similarly as in the Example 1 we can construct a general connection $A(\Gamma,\nabla)$ on $Fp : FY \to FM$. Thus we have proved

THEOREM 1. *Let $F : \mathcal{M}f \to \mathcal{FM}$ be a bundle functor. The following conditions are equivalent:*

(i) *There exists a canonical construction (an $\mathcal{FM}_{m,n}$-natural operator) of a general connection $A(\Gamma,\nabla)$ on $Fp : FY \to FM$ from a general connection Γ on a fibred manifold $p : Y \to M$ with $\dim(Y) = m+n$ and $\dim(M) = m$ by means of a projectable classical linear connection ∇ on Y.*

(ii) *There exists an $Gl(m) \times Gl(n)$-invariant general connection Γ^o on $F_{(0,0)}p : F_{(0,0)}(\mathbf{R}^m \times \mathbf{R}^n) \to F_0\mathbf{R}^m$.*

Remark 1. (i) Let $F : \mathcal{M}f \to \mathcal{FM}$ be a bundle functor. Assume that: (a) $F_{(0,0)}(\mathbf{R}^m \times \mathbf{R}^n)$ and $F_0\mathbf{R}^m$ are vector spaces; (b) the map $F_0(i_{m,m+n}) : F_0\mathbf{R}^m \to F_{(0,0)}(\mathbf{R}^m \times \mathbf{R}^n)$ is linear; (c) the map $F_{(0,0)}p : F_{(0,0)}(\mathbf{R}^m \times \mathbf{R}^n) \to F_0\mathbf{R}^m$ is linear (then linear epimorphism); (d) the actions of $Gl(m) \times Gl(n)$ on $F_{(0,0)}(\mathbf{R}^m \times \mathbf{R}^n)$ and of $Gl(m)$ on $F_0\mathbf{R}^m$ are by linear isomorphism. Then we have a $Gl(m) \times Gl(n)$-invariant general connection Γ^o on $F_{(0,0)}p : F_{(0,0)}(\mathbf{R}^m \times \mathbf{R}^n) \to F_0\mathbf{R}^m$ by $\Gamma^o\big(\frac{d}{d\tau}_{|0}(u + \tau w), f\big) = \frac{d}{d\tau}_{|0}\big(f + \tau F(i_{m,m+n})(w)\big)$, $u, w \in F_0\mathbf{R}^m$, $f \in F_{(0,0)}(\mathbf{R}^m \times \mathbf{R}^n)$, $Fp(f) = u$.

(ii) An example of a (not necessarily vector) bundle functor F satisfying conditions (a)–(d) is a Weil functor T^A corresponding to a Weil algebra A. So, we can construct a general connection $A(\Gamma, \nabla)$ on $T^A p : FY \to FM$ from a general connection Γ on $p : Y \to M$ by means of a projectable classical linear connection on Y. In [19], J. Slovák gave an example of a general connection $A(\Gamma)$ on $T^A p : FY \to FM$ from a general connection Γ on $p : Y \to M$ without using a projectable classical linear connection ∇ on Y.

Open problem. If exist a $Gl(m) \times Gl(n)$-invariant general connection on $F_{(0,0)}p : F_{(0,0)}(\mathbf{R}^m \times \mathbf{R}^n) \to F_0\mathbf{R}^m$ for any bundle functor $F : \mathcal{M}f \to \mathcal{FM}$?

PART II. Some constructions on projectable linear connections

4. Some definitions

Let $B : \mathcal{FM} \to \mathcal{M}f$ be the base functor and $\tau : \mathcal{FM}_m \to \mathcal{M}f$ be the total space functor, $\tau(Y \to M) = Y$, $\tau(f, \underline{f}) = f$.

Definition 3. A *bundle functor on* $\mathcal{FM}_m$ is a covariant functor $F : \mathcal{FM}_m \to \mathcal{FM}$ such that $B \circ F = \tau$ satisfying the the localization condition: for any inclusion of an open subset $i_U : U \to Y$, FU is the restriction $p^{-1}_{Y \to M}(U)$ of $p_{Y \to M} : FY \to Y$ and Fi_U is the inclusion $p^{-1}_{Y \to M}(U) \to FY$. A bundle functor $F : \mathcal{FM}_m \to \mathcal{FM}$ is fiber-product preserving if $F(Y_1 \times_M Y_2) = FY_1 \times_M FY_2$ (modulo a fibred diffeomorphism) for any $\mathcal{FM}_m$-objects $Y_1 \to M$ and $Y_2 \to M$ over the same base.

A more important example of a fiber product preserving bundle functor on $\mathcal{FM}_m$ is the r-jet prolongation functor $J^r : \mathcal{FM}_m \to \mathcal{FM}$ sending any $\mathcal{FM}_m$-object $Y \to M$ into its r-jet prolongation bundle $J^rY = \{j^r_x\sigma \mid \sigma : M \to Y$ is a section of $Y \to M, x \in M\}$ over Y, and any $\mathcal{FM}_m$-map $f : Y \to Y_1$ covering (embedding) $\underline{f} : M \to M_1$ into its induced map $J^rf : J^rY \to$

J^rY_1, $J^r(j^r_s\sigma) = j^r_{\underline{f}(x)}(f \circ \sigma \circ \underline{f}^{-1})$, $j^r_x\sigma \in J^rY$. Fiber product preserving bundle functors $F : \mathcal{FM}_m \to \mathcal{FM}_m$ have been completely described in [11] in terms of triples (A, H, t), where A is a Weil algebra of order r, H is a group homomorphism from the rth jet group G^r_m into group $Aut(A)$ of all automorphisms of A and t is a G^r_m-invariant algebra homomorphism from the algebra $\mathcal{D}^r_m = J^r_0(\mathbf{R}^m, \mathbf{R})$ into A. An example of not fiber product preserving bundle functor on $\mathcal{FM}_m$ is the tangent functor $T : \mathcal{FM}_m \to \mathcal{FM}$ sending any $\mathcal{FM}_m$-object $Y \to M$ into $TY \to Y$ and any $\mathcal{FM}_m$-map $f : Y \to Y_1$ into $Tf : TY \to TY_1$. More examples of bundle functors $F : \mathcal{FM}_m \to \mathcal{FM}$ can be found in [11].

Let $F : \mathcal{F}_{m,n} \to \mathcal{FM}$ be a bundle functor (defined quite similarly as bundle functors on $\mathcal{FM}_m$).

Definition 4. An $\mathcal{FM}_{m,n}$-*natural operator (a canonical construction)* transforming projectable classical linear connections ∇ on an $\mathcal{FM}_{m,n}$-object $Y \to M$ into classical linear connections $A(\nabla)$ on FY is a family of $\mathcal{FM}_{m,n}$-invariant regular operators (functions)

$$A : Con_{proj-clas-lin}(Y \to M) \to Con_{clas-lin}(FY)$$

for any $\mathcal{FM}_{m,n}$-object $Y \to M$, where $Con_{clas-lin}(FY)$ is the set of all classical linear connections on FY. (The invariance and regularity is defined quite similarly as in Definition 2.)

5. Solution of Problem 3

Let $p : Y \to M$ be a fibred manifold with m-dimensional base and n-dimensional fibers. Let ∇ be a projectable classical linear connection on Y with the underlying classical linear connection $\underline{\nabla}$ on M. The following lemma is almost clear.

Lemma 3. (a) *There exists a fibred normal coordinate system* $\Psi^y : (U, y) \to (\mathbf{R}^m \times \mathbf{R}^n, (0, 0))$ *on* Y *of* ∇ *with center* y *covering a normal coordinate system* $\underline{\Psi}^x : (\underline{U}, x) \to (\mathbf{R}^m, 0)$ *on* M *of* $\underline{\nabla}$ *with center* x.

(b) *If* Ψ^y_1 *is another such fibered normal coordinate system then there exists a linear isomorphism* $\Phi : \mathbf{R}^m \times \mathbf{R}^n \to \mathbf{R}^m \times \mathbf{R}^n$ *the form* $\Phi(x, y) = \big(\varphi(x), \psi_1(x) + \psi_2(y)\big)$, $(x, y) \in \mathbf{R}^m \times \mathbf{R}^n$ *such that* $\Psi^y_1 = \Phi \circ \Psi^y$ *near* y.

Let $F:\mathcal{FM}_m\to\mathcal{FM}$ be a fiber product preserving bundle functor. Let (A,H,t) be the triple of F. Then $F(\mathbf{R}^m\times\mathbf{R}^n)=(\mathbf{R}^m\times\mathbf{R}^n)\times\times\underline{A}^n$, where $\underline{A}$ is the maximal ideal of A. The following lemma is clear under the classification result of [11].

Lemma 4. *Let $Gl(m,n)$ be the linear group of fibred linear isomorphisms $\Phi:\mathbf{R}^m\times\mathbf{R}^n\to\mathbf{R}^m\times\mathbf{R}^n$ which are (of course) of the form $\Phi(x,y)=\big(\varphi(x),\psi_1(x)+\psi_2(y)\big)$. Given $\Phi\in Gl(m,n)$ (of the above form) we have*

$$F\Phi((x,y),\underline{a})=\Big(\big(\varphi(x),\psi_1(x)+\psi_2(y)\big),H(j^r_0\varphi)\big(\psi_2\otimes id_A(\underline{a})+t(j^r_1\psi_1)\big)\Big)\,,$$

$(x,y)\in\mathbf{R}^m\times\mathbf{R}^n$, $\underline{a}\in\underline{A}^n=\mathbf{R}^n\otimes\underline{A}$. In particular, the action of $Gl(m,n)$ on the standard fiber $F_{(0,0)}(\mathbf{R}^m\times\mathbf{R}^n)$ is by affine isomorphisms.

Example 2. Let $F:\mathcal{FM}_m\to\mathcal{FM}$ be a fiber product preserving bundle functor. Let ∇ be a projectable classical linear connection on a fibred manifold $p:Y\to M$, $\dim(Y)=m+n$, $\dim(M)=m$. We are going to construct a classical linear connection $A^F(\nabla)$ on FY. Let $v\in F_yY$, $y\in Y$. Let ∇^F be the classical linear connection on the affine space $F_{(0,0)}(\mathbf{R}^m\times\mathbf{R}^n)$ with vanishing Christoffel symbols (in any affine coordinates). Because of Lemma 4, ∇^F is $Gl(m,n)$-invariant. Let Ψ^y be a normal fibred coordinate system from Lemma 3. Let $\Psi^y_*\nabla$ be the image of ∇ by Ψ^y. Thus on $F(\mathbf{R}^m\times\mathbf{R}^n)=(\mathbf{R}^m\times\mathbf{R}^n)\times F_{(0,0)}(\mathbf{R}^m\times\mathbf{R}^n)$ we have the classical linear connection $(\Psi^y_*\nabla)\times\nabla^F$. (We recall that given classical linear connections ∇_1 on M and ∇_2 on N we have the product $\nabla_1\times\nabla_2$ of ∇_1 and ∇_2. This is a classical linear connection on $M\times N$ defined as follows. Let $\lambda_1:TM\to J^1TM$ and $\lambda_2:TN\to J^1TN$ be the corresponding to ∇_1 and ∇_2 fiber linear sections. Then the fiber linear section $\lambda:TM\times TN\to J^1(TM\times TN)$ corresponding to $\nabla_1\times\nabla_2$ is given by $\lambda(u,v)=j^1_{(p,q)}(X_1\times X_2)$, $u\in T_pM$, $v\in T_qN$, $p\in M$, $q\in N$, $\lambda_1(u)=j^1_p(X_1)$, $\lambda_2(v)=j^1_q(X_2)$.) We define

$$A^F(\nabla)_v=QF(\Psi^y)^{-1}\big(((\Psi^y_*\nabla)\times\nabla^F)_{F\Psi^y(v)}\big)\,,$$

where we treat classical linear connections on Y as sections of the connection natural bundle QY. If Ψ^y_1 is another fibred normal coordinate system in question, then because of Lemma 3(b) and the $Gl(m,n)$-invariance of ∇^F we obtain the same $A^F(\nabla)_v$. Thus the definition of $A^F(\nabla)_v$ is correct. Thus we have the resulting classical linear connection $A^F(\nabla)$ on FY.

Because of the canonical character of the construction $A^F(\nabla)$ we have

PROPOSITION 2. (1) *The family of operators $A^F : \nabla \to A^F(\nabla)$ (described in Example 2) is an $\mathcal{FM}_{m,n}$-natural operator.*

(2) *Let $a = \{a_p\} : F_1 \to F_2$ be an isomorphism of fiber product preserving functors (i.e. $a_p : F_1Y \to F_2Y$ is a fibred diffeomorphism covering id_Y for any $\mathcal{FM}_m$-object $p : Y \to M$ such that $F_2 f \circ a_{p_1} = a_{p_2} \circ F_1 f$ for any $\mathcal{FM}_m$-map $f : Y_1 \to Y_2$ covering $\underline{f} : M_1 \to M_2$). Then for any projectable classical linear connection ∇ on an $\mathcal{FM}_{m,n}$-object $p : Y \to M$, connections $A^{F_1}(\nabla)$ and $A^{F_2}(\nabla)$ are a_p-related.*

6. SOLUTION OF PROBLEM 4

Let $F : \mathcal{FM}_{m,n} \to \mathcal{FM}$ be a (not necessarily fiber product preserving) bundle functor. Suppose that we have a construction A of a classical linear connection $A(\nabla)$ on FY from a projectable classical linear connection ∇ on Y. Then we have the connection $A(\nabla^0)$ on $F(\mathbf{R}^m \times \mathbf{R}^n) = (\mathbf{R}^m \times \mathbf{R}^n) \times F_{(0,0)}(\mathbf{R}^m \times \mathbf{R}^n)$, the trivialization by translations, where ∇^0 is the flat projectable classical linear connection on $\mathbf{R}^m \times \mathbf{R}^n$ with vanishing Christoffel symbols. Then by the Gauss formula we have the classical linear connection ∇' on $F_{(0,0)}(\mathbf{R}^m \times \mathbf{R}^n) = \{(0,0)\} \times F_{(0,0)}(\mathbf{R}^m \times \mathbf{R}^n)$ induced by $A(\nabla^0)$ via the product $(\mathbf{R}^m \times \mathbf{R}^n) \times F_{(0,0)}(\mathbf{R}^m \times \mathbf{R}^n)$. The connection ∇' is $Gl(m,n)$-invariant because of ∇^0 is.

Conversely, suppose we have a $Gl(m,n)$-invariant classical linear connection ∇' on $F_{(0,0)}(\mathbf{R}^m \times \mathbf{R}^n)$. Then similarly as in Example 2 (starting from ∇' instead of ∇^F) we can construct a classical linear connection $A(\nabla)$ on FY from a projectable classical linear connection ∇ on Y. Thus we have proved

THEOREM 2. *Let $F : \mathcal{FM}_{m,n} \to \mathcal{FM}$ be a bundle functor. The following conditions are equivalent:*

(a) *There exists a canonical construction ($\mathcal{FM}_{m,n}$-natural operator) of a classical linear connection $A(\nabla)$ on FY from a projectable classical linear connection ∇ on Y.*

(b) *There exist a $Gl(m,n)$-invariant classical linear connection ∇' on the fiber $F_{(0,0)}(\mathbf{R}^m \times \mathbf{R}^n)$.*

Remark 2. We can apply Theorem 2 to vector bundle functors $F : \mathcal{FM}_{m,n} \to \mathcal{VB}$. In this situation $Gl(m,n)$ acts on the vector space $F_{(0,0)}(\mathbf{R}^m \times \mathbf{R}^n)$ by linear isomorphisms. Then the flat classical linear connection ∇^F on

$F_{(0,0)}(\mathbf{R}^m \times \mathbf{R}^n)$ is $Gl(m,n)$-invariant. So, we have the classical linear connection $A^F(\nabla)$ on FY from a projectable classical linear connection ∇ on Y.

In the case $m = 0$ we have $\mathcal{FM}_{0,n} = \mathcal{M}f_n$. Thus we have the following corollary of Theorem 2.

Corollary 1. ([16]) *Let $F : \mathcal{M}f_n \to \mathcal{FM}$ be a bundle functor (natural bundle). Then the following conditions are equivalent:*

(a) *There exists a canonical construction (a $\mathcal{M}f_n$-natural operator) of a classical linear connection $A(\nabla)$ on FN from a classical linear connection ∇ on a n-dimensional manifold N;*

(b) *There is a $GL(n)$-invariant classical linear connection $\tilde{\nabla}$ on $F_0\mathbf{R}^n$.*

Remark 3. In particular, if $F = T^A : \mathcal{M}f \to \mathcal{FM}$ is the Weil bundle functor for some Weil algebra $A = \mathbf{R} \times \underline{A}$ then $T_0^A\mathbf{R}^n = (\underline{A})^n$ is the vector bundle. More, the action $GL(n)$ on $T_0^A\mathbf{R}^n$ is linear. Then the standard flat classical linear connection on the vector space $T_0^A\mathbf{R}^n$ is $GL(n)$-invariant. Then by Corollary 1, there exists a canonical construction of a classical linear connection $A(\nabla)$ on T^AN from a classical linear connection ∇ on N. That is why the Morimoto prolongation [17] of classical linear connections to Weil bundles is possible.

7. Solution of Problem 5

Let $\mathbf{P}(\mathbf{R}^m \times \mathbf{R}^n)$ be the projective space. The group $Gl(m,n)$ acts on $\mathbf{P}(\mathbf{R}^m \times \mathbf{R}^n)$ by the projectivization.

Lemma 5. *Let $m \geq 2$. There is no $Gl(m,n)$-invariant classical linear connection ∇' on $\mathbf{P}(\mathbf{R}^m \times \mathbf{R}^n)$.*

Proof. Suppose that ∇' is in question. $\mathbf{P}^{fin} = \{[1, x^2, \dots, x^m, y^1, \dots, y^n] \in \mathbf{P}(\mathbf{R}^m \times \mathbf{R}^n) \mid (x^2, \dots, x^m, y^1, \dots, y^n) \in \mathbf{R}^{m-1} \times \mathbf{R}^n\}$ is the affine space of "finite points". $P^{fin} = \mathbf{R}^{m-1} \times \mathbf{R}^n$ by the identification $[1, x^2, \dots, x^m, y^1, \dots, y^n] = (x^2, \dots, x^m, y^1, \dots, y^n)$. By the suppose, $\nabla' | \mathbf{P}^{fin}$ is invariant by the translations $\tau_{(a,b)} = [x^1, x^2 + a_2x^1, \dots, x^m + a_mx^1, y^1 + b_1x^1, \dots, y^m + b_nx^1] | \mathbf{P}^{fin}$ and and by the homotheties $tid = [x^1, tx^2, \dots, tx^m, ty^1, \dots, ty^n] | \mathbf{P}^{fin}$. Then $\nabla' | \mathbf{P}^{fin}$ has constant Christoffel symbols wchich are vanishing in the origin. Then $\nabla' | \mathbf{P}^{fin}$ is the usual flat connection with vanishing Christoffel symbols.

On the other hand, if $\Phi \in Gl(m,n)$ sends $(0,1,0,\dots,0)$ into $(1,0,\dots,0)$ then the projectivization $[\Phi]$ is not locally affine on $\mathbf{P}^{fin}$. Then ∇' is not $[\Phi]$-invariant. Contradiction. ∎

EXAMPLE 3. Let $\mathbf{P}(T) : \mathcal{FM}_{m,n} \to \mathcal{FM}$ be the projectivization of the tangent functor,

$$\mathbf{P}(T)(Y) = \bigcup_{y\in Y} \mathbf{P}(T_yY)\,, \quad \mathbf{P}(T)(\Phi) = \bigcup_{y\in Y_1} [T_y\Phi] : \mathbf{P}(T)(Y_1) \to \mathbf{P}(T)(Y_2)\,.$$

Then $\mathbf{P}(T)_{(0,0)}(\mathbf{R}^m \times \mathbf{R}^n) = \mathbf{P}(\mathbf{R}^m \times \mathbf{R}^n)$ with the usual action of $Gl(m,n)$. Because of Lemma 5 there is no $Gl(m,n)$-invariant classical linear connection ∇' on $\mathbf{P}(T)_{(0,0)}(\mathbf{R}^m \times \mathbf{R}^n)$. So, by Theorem 2, there is no canonical construction of a classical linear connection $A(\nabla)$ on $\mathbf{P}(T)(Y)$ from a projectable classical linear connection ∇ on Y.

PART III. SOME GAUGE CONSTRUCTIONS ON LINEAR CONNECTIONS

8. SOME DEFINITIONS

Let $B' : \mathcal{VB}_{m,n} \to \mathcal{M}f$ and $B : \mathcal{FM} \to \mathcal{M}f$ be the base functors.

DEFINITION 5. A *gauge bundle functor on* $\mathcal{VB}_{m,n}$ is a covariant functor $F : \mathcal{VB}_{m,n} \to \mathcal{FM}$ satisfying $B \circ F = B'$ and the localization property: for every $\mathcal{VB}_{m,n}$-object $p : E \to M$ and every inclusion of an open sub-bundle $i_U : E|U \to E$, $F(E|U)$ is the restriction $p_E^{-1}(U)$ of $p_E : FE \to M$ over U and Fi_U is the inclusion $p_E^{-1}(U) \to FE$.

A simple example of a gauge bundle functor on $\mathcal{VB}_{m,n}$ is the r-jet prolongation functor $J^r : \mathcal{VB}_{m,n} \to \mathcal{FM}$ sending any $\mathcal{VB}_{m,n}$-object $E \to M$ into the $r$jet prolongation bundle $J^rE = \{j^r_x\sigma \mid \sigma : M \to E \text{ is a section}, x \in M\}$ over M and any $\mathcal{VB}_{m,n}$-map $f : E \to E_1$ covering $\underline{f} : M \to M_1$ into the induced map $J^rf : J^rE \to J^rE_1$. In fact, $J^r : \mathcal{VB}_{m,n} \to \mathcal{VB}$ is a gauge vector bundle functor. More examples of such functors can be found in [10].

Let $F : \mathcal{VB}_{m,n} \to \mathcal{FM}$ be a gauge bundle functor.

DEFINITION 6. A $\mathcal{VB}_{m,n}$-natural gauge operator transforming linear connections D on $\mathcal{VB}_{m,n}$-object $E \to M$ and classical linear connections ∇ on M

into classical linear connections $A(D,\nabla)$ on FE is a family of $\mathcal{VB}_{m,n}$-invariant regular operators

$$A : Con_{lin}(E \to M) \times Con_{clas-lin}(M) \to Con_{clas-lin}(FE)$$

for any $\mathcal{VB}_{m,n}$-object $p : E \to M$, where $Con_{lin}(E \to M)$ is the set of linear general connections on $E \to M$ and $Con_{clas-lin}(M)$ is the set of all classical linear connections on M. (The invariance and regularity we mean quite similarly as in Definition 2.)

9. Solution of Problems 6 and 7

Let $F : \mathcal{VB}_{m,n} \to \mathcal{FM}$ be a gauge bundle functor. On the standard fiber $F_0(\mathbf{R}^m \times \mathbf{R}^n)$, $0 \in \mathbf{R}^m$, we have the left action of $GL(m) \times GL(n)$ by $(B, C).f = F(B \times C)(f)$, $f \in F_0(\mathbf{R}^m \times \mathbf{R}^n)$.

(I) Suppose we have a $GL(m)\times GL(n)$-invariant classical linear connection $\tilde{\nabla}$ on $F_0(\mathbf{R}^m \times \mathbf{R}^n)$. Let D be a linear general connection on an $\mathcal{VB}_{m,n}$-object $p : E \to M$ and let ∇ be a classical linear connection on M. We are going to construct a classical linear connection $A(D, \nabla)$ on FE. Let $f \in F_xE$, $x \in M$.

Firstly, for any basis $b = (b_1, \dots, b_n)$ of E_x and any basis $l = (l_1, \dots, l_m)$ of T_xM, we construct a vector bundle trivialization $\Phi^{b,l} : E|U \to \mathbf{R}^m \times \mathbf{R}^n$ over some neighborhood of x as follows. Let $\Gamma(D, \nabla)$ be the classical linear connection on E induced by D and ∇, see [5], [10; Sect. 54.2]. Given $v \in E_x$, we define a (smooth) section $\tilde{v} : U \to E$ of $E \to M$ (on some neighborhood $U \subset M$ of x) by

$$\tilde{v}(y) = exp_v\big(D(exp_x^{-1}(y), v)\big)\,,\ y \in U\,,$$

where $exp_x : T_xM \to M$ is the (defined locally) exponent of ∇, and $exp_v : T_vE \to E$ is the (defined locally) exponent of $\Gamma(D, \nabla)$. Analyzing the definition of $\Gamma(D, \nabla)$ one can see that $\Gamma(D, \nabla)$ is projectable on ∇. Then exp_v is fibred over exp_x. So, $\tilde{v}$ is really a section. One can also observe that $\Gamma(D, \nabla)$ is invariant with respect to fiber homotheties of E. Then $\tilde{v}$ depends linearly on v. Then the $\tilde{b}_i$ form a basis of sections of $E|U \to U$. We choose the (unique) normal coordinate system $\varphi^l : U \to \mathbf{R}^m$ of ∇ with center x which maps l into the usual basis of $T_0\mathbf{R}^m = \mathbf{R}^m$. We define $\Phi^{b,l} : E|U \to \mathbf{R}^m \times \mathbf{R}^n$ to be the unique vector bundle isomorphism covering φ^l and sending basis $\tilde{b}_i$ of local sections of $E \to M$ into the usual basis of sections of $\mathbf{R}^m \times \mathbf{R}^n \to \mathbf{R}^m$. One can easily observe that if b' and l' are another bases in E_x and T_xM then

$$(*)\qquad \Phi^{b',l'} = (B^{-1} \times C^{-1}) \circ \Phi^{b,l}\,,$$

where B is the matrix between bases l to l' (i.e., $l' = l.B$) and C is the matrix between bases b to b' (i.e., $b' = b.C$).

Let $f \in F_x E$. We choose b and l and $\Phi^{b,l}$ over φ^l as above. We have classical linear connection $\varphi^l_* \nabla \times \tilde{\nabla}$ on some neighborhood of the fibre over zero of $\mathbf{R}^m \times F_0(\mathbf{R}^m \times \mathbf{R}^n) = F(\mathbf{R}^m \times \mathbf{R}^n)$. We put

$$A(D, \nabla)_f = (QF\Phi^{b,l})^{-1}\big((\varphi^l)_* \nabla \times \tilde{\nabla})_{F\Phi^{b,l}(f)}\big),$$

where Q is the bundle functor of classical linear connections. Because of $(*)$ and the $GL(m) \times GL(n)$-invariance of $\tilde{\nabla}$, the definition of $A(D, \nabla)_f$ is correct (it is independent of the choice of (b, l)).

(II) Conversely, suppose we have a canonical construction ($\mathcal{VB}_{m,n}$-natural gauge operator) A transforming linear general connections D on $E \to M$ and classical linear connections ∇ in M into classical linear connections $A(D, \nabla)$ on FE. Let ∇^o be the flat classical linear connection on $\mathbf{R}^m$ and D^o be the trivial linear general connection on $\mathbf{R}^m \times \mathbf{R}^n \to \mathbf{R}^m$. Then we have the classical linear connection $A(D^o, \nabla^o)$ on $F(\mathbf{R}^m \times \mathbf{R}^n) = \mathbf{R}^m \times F_0(\mathbf{R}^m \times \mathbf{R}^n)$. Thus (by the Gauss formula) we have the classical linear connection $\tilde{\nabla}$ on $F_0(\mathbf{R}^m \times \mathbf{R}^n)$. Since D^o is $GL(m) \times GL(n)$-invariant and ∇^o is $GL(m)$-invariant and A is invariant, then $\tilde{\nabla}$ is $GL(m) \times GL(n)$-invariant. Thus we have proved

THEOREM 3. *Let $F : \mathcal{VB}_{m,n} \to \mathcal{FM}$ be a gauge bundle functor. The following conditions are equivalent:*

(a) *There exists a canonical construction (a $\mathcal{VB}_{m,n}$-natural gauge operator) of a classical linear connection $A(D, \nabla)$ from a linear general connection D on $E \to M$ by means of a classical linear connection ∇ on M;*

(a) *There exists a $GL(m) \times GL(n)$-invariant classical linear connection $\tilde{\nabla}$ on the standard fibre $F_0(\mathbf{R}^m \times \mathbf{R}^n)$ of F.*

In the case of a vector gauge bundle functor $F : \mathcal{VB}_{m,n} \to \mathcal{VB}$ we have the action of $GL(m) \times GL(n)$ on the vector space $F_0(\mathbf{R}^m \times \mathbf{R}^n)$. Let $\tilde{\nabla} = \nabla^F$ be the usual flat connection on $F_0(\mathbf{R}^m \times \mathbf{R}^n)$. It is $GL(m) \times GL(n)$-invariant. Therefore (because of Theorem 3) we have a $\mathcal{VB}_{m,n}$-natural gauge operator A^F transforming linear general connections D on $\mathcal{VB}_{m,n}$-objects $E \to M$ and classical linear connections ∇ on M into classical linear connections $A^F(D, \nabla)$ on FE. Because of the canonical construction we have

Proposition 3. *Let $a : F_1 \to F_2$ be a natural isomorphism of two gauge vector bundle functors $F_1, F_2 : \mathcal{VB}_{m,n} \to \mathcal{VB}$. Then for any linear general connection D on $E \to M$ and any classical linear connection ∇ on M, connections $A^{F_1}(D,\nabla)$ and $A^{F_2}(D,\nabla)$ are $a_{E\to M}$-related.*

Example 4. We modify Example 3 as follows. Let $\tilde{\mathbf{P}}(T) = \mathbf{P}(T) \circ B' : \mathcal{VB}_{m,n} \to \mathcal{FM}$ be the gauge bundle functor

$$\tilde{\mathbf{P}}(T)(E) = \bigcup_{x\in M} \mathbf{P}(T_xM)\,, \quad \tilde{\mathbf{P}}(T)(f) = \bigcup_{x\in M} \mathbf{P}(T_x\underline{f})\,.$$

By Lemma 5 for $n = 0$ we have that there is no $GL(m)$-invariant classical linear connection on $\mathbf{P}(\mathbf{R}^m)$ for $m \geq 2$. That is why, there is no $GL(m) \times GL(n)$-invariant classical linear connection on $\tilde{\mathbf{P}}(T)_0(\mathbf{R}^m \times \mathbf{R}^n) \tilde{=} \mathbf{P}(\mathbf{R}^m)$. By Theorem 3, there is no canonical construction of a classical linear connection $A(D,\nabla)$ on $\tilde{\mathbf{P}}(T)(E)$ from a linear general connection D on $E \to M$ by means of a classical linear connection ∇ on M.

References

[1] Dębecki, J., Affine liftings of linear connections to Weil bundled, in review.

[2] Doupovec, M., Mikulski, W.M., On the existence of prolongation of connections, *Czechoslovak Math. J.*, **56** (131) (2006), 1323 – 1334.

[3] Ehresmann, C., Les prolongements d'un espace fibre différentiable, *C. R. Acad. Sci. Paris*, **240** (1955), 1755 – 1757.

[4] Ehresmann, C., Sur les connections d'ordre supérieur, Atti del C. Cang. del'Unione Mat. Italiana 1955, Roma Cremonese (1956), 344 – 346.

[5] Gancarzewicz, J., Kolář, I., Some gauge-natural operators on linear connections, *Monatsh. Math.*, **111** (1) (1991), 23 – 33.

[6] Fatibene, L., Francaviglia, M., "Natural and Gauge Formalism for Classical Field Theories, A Geometric Perspective Including Spinors and Gauge Theories", Kluwer Academic Publishers, Dordrecht, 2003.

[7] Janyska, J., Modugno, M., Relations between linear connections on the tangent bundle and connections n the jet bundle of a fibered manifolds, *Arch. Math. (Brno)*, **32** (4) (1996), 281 – 288.

[8] Kolář, I., Prolongation of generalized connections, in Differential geometry (Budapest, 1979), 317 – 325, Colloq. Math. Soc. János Bolyai, 31, North-Holland, Amsterdam, 1982.

[9] Kolář, I., Torsion-free connections on higher order frame bundles, in "New Developments in Differential Geometry (Debrecen, 1994)", L. Tamássy and J. Szenthe (eds), Kluver Academic Publishers, Dordrecht, 1996, 233 – 241.

[10] Kolář, I., Michor, P.W., Slovák, J., "Natural Operations in Differential Geometry", Springer-Verlag, Berlin, 1993.

[11] Kolář, I., Mikulski, W.M., On the fiber product preserving bundle functors, *Differential Geom. Appl.*, **11** (1999), 105 – 115.

[12] Kureš, M., Natural lifts of classical linear connections to the cotangent bundle, in "The Proceedings of the 15th Winter School Geometry and Physics" (Srní, 1995). *Rend. Circ. Mat. Palermo (2)* Suppl. **43** (1996), 181 – 187.

[13] Mikulski, W.M., A construction of a connection on $GY \to Y$ from a connection on $Y \to M$ by means of a classical linear connections on M and Y, *Comment. Math. Univ. Carolinae*, **46** (4)(2005), 759 – 770.

[14] Mikulski, W.M., Non-existence of some canonical constructions on connections, *Comment. Math. Univ Carolinae*, **44** (4) (2003), 691 – 695.

[15] Mikulski, W.M., Negative answers to some questions about constructions on connections, *Demonstratio Math.*, **39** (3)(2006), 685 – 689.

[16] Mikulski, W.M., The natural bundles admitting natural lifting of linear connections, *Demonstratio Math.*, **39** (1) (2006), 223 – 232.

[17] Morimoto, A., Prolongations of connections to bundles of infinitely near points, *J. Differential Geom.*, **11** (4) (1976), 476 – 498.

[18] Paluszny, M., Zajtz, A., "Foundation of the Geometry of Natural Bundles", Lect. Notes Univ. Caracas, 1984.

[19] Pohl, F.W., Connexions in differential geometry of higher order, *Trans. Amer. Math. Soc.*, **125** (1966), 310 – 325.

[20] Slovák, J., Prolongations of connections and sprays with respect to Weil functors, in "Proceedings of the 14th Winter School on Abstract Analysis (Srní, 1986)", *Rend. Circ. Mat. Palermo (2)*, Suppl. **14** (1987), 143 – 155.

[21] Vondra, A., Higher order differential equations represented by connections on prolongations of fibered manifolds, *Extracta Math.*, **15** (3) (2000), 421 – 512.

[22] Yano, K., Patterson, E.M., Vertical and complete lifts from a manifold to its cotangent bundle, *J. Math. Soc. Japan*, **19** (1967), 91 – 113.