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Perturbations of Operators Satisfying a Local Growth Condition

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Abstract: A Banach space operator $T \in B(\mathcal{X})$ satisfies a local growth condition of order m for some positive integer m , $T \in \operatorname{loc}(G_m)$, if for every closed subset F of the set of complex numbers and every x in the glocal spectral subspace $X_T(F)$ there exists an analytic function $f : \mathbb{C} \setminus F \to \mathcal{X}$ such that $(T - \lambda)f(\lambda) \equiv x$ and $||f(\lambda)|| \leq K[\operatorname{dist}(\lambda, F)]^{-m}||x||$ for some $K > 0$ (independent of F and x). Browder-Weyl type theorems are proved for perturbations by an algebraic operator of operators which are either $\operatorname{loc}(G_m)$ or polynomially $\operatorname{loc}(G_m)$.

Key words: Banach space, local growth condition, single valued extension property, Browder-Weyl theorems, Riesz operator, perturbation, polynomially locally (G_m) operator.

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1. Introduction

A Banach space operator T, $T \in B(\mathcal{X})$, has *the single-valued extension property* at $\lambda_0 \in \mathbb{C}$, SVEP at λ_0 for short, if for every open disc $\mathcal{D}_{\lambda_0}$ centered at λ_0 the only analytic function $f : \mathcal{D}_{\lambda_0} \to \mathcal{X}$ which satisfies

$$(T - \lambda)f(\lambda) = 0 \qquad \text{for all} \quad \lambda \in \mathcal{D}_{\lambda_0}$$

is the function $f \equiv 0$. *T has SVEP if it has SVEP at every $\lambda \in \mathbb{C}$.* The single valued extension property, introduced by Dunford [9, 10], plays an important role in local spectral theory and Fredholm theory (see [17] and [1]; also see [14]). Evidently, every T has SVEP at points in the resolvent $\rho(T) = \mathbb{C} \setminus \sigma(T)$ or the boundary $\partial\sigma(T)$ of the spectrum $\sigma(T)$. It is easily verified that SVEP is inherited by restrictions, and that if T has SVEP and $TX = XY$ for some injection X, then Y has SVEP.

An operator T is said to satisfy a *growth condition of order m*, or to be a (G_m)-operator, if there exists a constant $K > 0$ such that

$$||(T - \lambda)^{-1}|| \leq \frac{K}{[\operatorname{dist}(\lambda, \sigma(T))]^m}$$

for all $\lambda \notin \sigma(T)$. Hyponormal operators are (G_1)-operators [25] and spectral operators of type $m-1$ are (G_m)-operators [11, Theorem XV.6.7]. Not every $T \in (G_m)$ has SVEP. To see this, start by observing that $T \in (G_m) \Rightarrow T^* \in (G_m)$. Hence, if every $T \in (G_m)$ has SVEP, then both T and T^* have SVEP. But this is false, as follows from a consideration of the forward and backward unilateral shifts on a Hilbert space.

For an arbitrary closed subset F of the set $\mathbb{C}$ of complex numbers and $T \in B(\mathcal{X})$, let $X_T(F) = \{x \in \mathcal{X} : (T-\lambda)f_x(\lambda) \equiv x$ for some analytic function $f_x : \mathbb{C} \setminus F \to \mathcal{X}\}$. The *glocal spectral subspace* $X_T(F)$ is a hyperinvariant linear manifold of T [17, p. 220]. Let m be a positive integer. We say that $T \in \mathrm{loc}(G_m)$ (or, T satisfies a local growth condition of order m) if for every closed set $F \subset \mathbb{C}$ and every $x \in X_T(F)$ there exists an analytic function $f : \mathbb{C} \setminus F \to \mathcal{X}$ such that $(T-\lambda)f(\lambda) \equiv x$ and

$$||f(\lambda)|| \leq K[\mathrm{dist}(\lambda, F)]^{-m}||x|| \qquad \text{for some} \quad K > 0$$

(independent of F and x). Hyponormal operators are $\mathrm{loc}(G_1)$ [25, 15] and spectral operators of type $m-1$ are $\mathrm{loc}(G_m)$ [11, Proof of Theorem XV.6.7]. Evidently, $T \in \mathrm{loc}(G_m) \Rightarrow T \in (G_m)$. It is known, [15, Proposition 2], that if the Banach space $\mathcal{X}$ is reflexive (in particular, a Hilbert space), then operators $T \in \mathrm{loc}(G_m)$ satisfy Dunford's condition (C). Hence $\mathrm{loc}(G_m)$ operators $T \in B(\mathcal{X})$ such that $\mathcal{X}$ is reflexive have SVEP, which implies that both T and T^* satisfy a-Browder's theorem. This observation forms the starting point of this paper. Let $T \in \mathrm{loc}(G_m) \cap B(\mathcal{X})$, $\mathcal{X}$ is reflexive. We prove that $f(T)$ satisfies Weyl's theorem and $f(T^*)$ satisfies a-Weyl's theorem for every function f which is analytic on a neighbourhood of $\sigma(T)$. The spectrum of an operator and its various distinguished parts are not stable under perturbations by a compact operator, even a quasinilpotent operator. It is proved that a-Browder's theorem is inherited by perturbations of operators T by Riesz operators which commute with T. Specializing to algebraic operators A which commute with T, it is proved that $f(T+A)$ satisfies Weyl's theorem and $f(T+A)^*$ satisfies a-Weyl's theorem for every f which is analytic on a neighbourhood of $\sigma(T+A)$. A similar result holds for polynomially $\mathrm{loc}(G_m)$ operators T which commute with A. Problems of the type considered in this paper have been considered by other authors in the recent past, see [2, 4, 5, 6, 19], but for operator classes independent of the class $\mathrm{loc}(G_m)$. The results of this paper complement those from these papers.

2. Notation and terminology

An operator $T \in B(\mathcal{X})$ is semi-Fredholm if it is either upper Fredholm or lower Fredhlom, where T is *upper Fredholm*, $T \in \Phi_+(\mathcal{X})$, if $T(\mathcal{X})$ is closed and the *deficiency index* $\alpha(T) = \dim(T^{-1}(0))$ is finite, and T is *lower Fredholm*, $T \in \Phi_-(\mathcal{X})$, if the deficiency index $\beta(T) = \dim(\mathcal{X}/T(\mathcal{X}))$ is finite; T is Fredholm, $T \in \Phi(\mathcal{X})$, if $T \in \Phi_+(\mathcal{X}) \cap \Phi_-(\mathcal{X})$. The *semi-Fredholm index* of T, $\operatorname{ind}(T)$, is the (finite or infinite) integer $\operatorname{ind}(T) = \alpha(T) - \beta(T)$. The ascent of T, $\operatorname{asc}(T)$, is the least non-negative integer n such that $T^{-n}(0) = T^{-(n+1)}(0)$ and the descent of T, $\operatorname{dsc}(T)$, is the least non-negative integer n such that $T^n(\mathcal{X}) = T^{n+1}(\mathcal{X})$. We say that T is of *finite ascent* (resp., *finite descent*) if $\operatorname{asc}(T - \lambda I) < \infty$ (resp., $\operatorname{dsc}(T - \lambda I) < \infty$) for all complex numbers λ. We shall, henceforth, shorten $(T - \lambda I)$ to $(T - \lambda)$. The operator T is *Weyl* if it is Fredholm of zero index, and T is said to be *Browder* if it is Fredholm "of finite ascent $\operatorname{asc}(T)$ and descent $\operatorname{dsc}(T)$". Let $\mathbb{C}$ denote the set of complex numbers. The Browder spectrum $\sigma_b(T)$ and the Weyl spectrum $\sigma_w(T)$ of T are the sets $\sigma_b(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Browder}\}$ and $\sigma_w(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Weyl}\}$. Let $\pi(T)$, $\pi_0(T)$ and $\pi_{00}(T)$ denote, respectively, the set of poles of the resolvent of T, the set of *Riesz points* of T (i.e., the set of $\lambda \in \mathbb{C}$ such that $T - \lambda$ is Fredholm of finite ascent and descent [3]), and the set of isolated points λ of $\sigma(T)$, $\lambda \in \operatorname{iso}\sigma(T)$, which are eigenvalues of T of finite (geometric) multiplicity. In keeping with current usage [13, 1], we say that an operator $T \in B(\mathcal{X})$ satisfies *Browder's theorem* (resp., *Weyl's theorem*) if $\sigma(T) \setminus \sigma_w(T) = \pi_0(T)$ (resp., $\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T)$). Recall [13] that Weyl's theorem for T implies Browder's theorem for T, and Browder's theorem for T is equivalent to Browder's theorem for T^*.

The (Fredholm) essential spectrum $\sigma_e(T)$ of $T \in B(\mathcal{X})$ is the set $\sigma_e(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin \Phi(\mathcal{X})\}$. If we let $\operatorname{acc}\sigma(T)$ denote the set of accumulation points of $\sigma(T)$, then

$$\sigma_e(T) \subseteq \sigma_w(T) \subseteq \sigma_b(T) \subseteq \sigma_e(T) \cup \operatorname{acc}\sigma(T).$$

Let $\pi_{a0}(T)$ be the set of $\lambda \in \mathbb{C}$ such that λ is an isolated point of $\sigma_a(T)$, $\lambda \in \operatorname{iso}\sigma_a(T)$, and $0 < \alpha(T - \lambda) < \infty$, where $\sigma_a(T)$ denotes the approximate point spectrum of the operator T. Then $\pi_0(T) \subseteq \pi_{00}(T) \subseteq \pi_{a0}(T)$. T is said to satisfy *a-Weyl's theorem* if

$$\sigma_{wa}(T) = \sigma_a(T) \setminus \pi_{a0}(T),$$

where $\sigma_{wa}(T)$ denotes *the essential approximate point spectrum* of T (i.e., $\sigma_{wa}(T) = \cap\{\sigma_a(T+K) : K \in K(\mathcal{X})\}$ with $K(\mathcal{X})$ denoting the ideal of compact operators on $\mathcal{X}$). Let $\Phi_+^-(\mathcal{X}) = \{T \in \Phi_+(\mathcal{X}) : \operatorname{ind}(T) \leq 0\}$. Then $\sigma_{wa}(T)$ is the complement in $\mathbb{C}$ of all those λ for which $(T-\lambda) \in \Phi_+^-(\mathcal{X})$ [20]. The concept of a-Weyl's theorem was introduced by Rakočvić: a-Weyl's theorem for $T \Rightarrow$ Weyl's theorem for T, but the converse is generally false [21]. If we let $\sigma_{ba}(T)$ denote the *Browder essential approximate point spectrum of* T,

$$\begin{aligned}\sigma_{ba}(T) &= \bigcap \{\sigma_a(T+K) : TK = KT \text{ and } K \in K(\mathcal{X})\} \\ &= \{\lambda \in \mathbb{C} : T-\lambda \notin \Phi_+^-(\mathcal{X}) \text{ or } \operatorname{asc}(T-\lambda) = \infty\},\end{aligned}$$

then $\sigma_{wa}(T) \subseteq \sigma_{ba}(T)$. We say that T satisfies *a-Browder's theorem* if $\sigma_{ba}(T) = \sigma_{wa}(T)$ [20]. It is known [5, Lemma 2.18] that *a Banach space operator T with SVEP satisfies a-Browder's theorem.* Let $\sigma_s(T)$ denote the *surjectivity spectrum* of T. The *essential surjectivity spectrum* $\sigma_{ws}(T)$ of T is the set $\cap\{\sigma_s(T+K) : K \in K(\mathcal{X})\}$. If we let $\Phi_-^+(\mathcal{X}) = \{T \in \Phi_-(\mathcal{X}) : \operatorname{ind}(T) \geq 0\}$, then $\sigma_{ws}(T)$ is the complement in $\mathbb{C}$ of all those λ for which $(T-\lambda) \in \Phi_-^+(\mathcal{X})$.

The *quasinilpotent part* $H_0(T-\lambda)$ and the *analytic core* $K(T-\lambda)$ of $(T-\lambda)$ are defined by

$$H_0(T-\lambda) = \left\{x \in \mathcal{X} : \lim_{n\to\infty} ||(T-\lambda)^n x||^{\frac{1}{n}} = 0\right\}$$

and

$$K(T-\lambda) = \left\{ x \in \mathcal{X} \ : \ \begin{array}{c} \text{there exists a sequence } \{x_n\} \subset \mathcal{X} \text{ and } \delta > 0 \\ \text{for which } x = x_0,\ (T-\lambda)(x_{n+1}) = x_n \\ \text{and } \|x_n\| \leq \delta^n \|x\| \text{ for all } n = 1, 2, \ldots \end{array} \right\}.$$

We note that $H_0(T-\lambda)$ and $K(T-\lambda)$ are (generally) non-closed hyperinvariant subspaces of $(T-\lambda)$ such that $(T-\lambda)^{-q}(0) \subseteq H_0(T-\lambda)$ for all $q = 0, 1, 2, \ldots$ and $(T-\lambda)K(T-\lambda) = K(T-\lambda)$ [18]. The operator $T \in B(\mathcal{X})$ is said to be *semi-regular* if $T(\mathcal{X})$ is closed and $T^{-1}(0) \subset T^\infty(\mathcal{X}) = \cap_{n\in\mathbb{N}} T^n(\mathcal{X})$; T admits a *generalized Kato decomposition*, GKD for short, if there exists a pair of T-invariant closed subspaces (M, N) such that $\mathcal{X} = M \oplus N$, the restriction $T|_M$ is quasinilpotent and $T|_N$ is semi-regular. An operator $T \in B(\mathcal{X})$ has a GKD at every $\lambda \in \operatorname{iso}\sigma(T)$, namely $\mathcal{X} = H_0(T-\lambda) \oplus K(T-\lambda)$. We say that

T is of *Kato type* at a point λ if $(T-\lambda)|_M$ is nilpotent in the GKD for $(T-\lambda)$. If $T-\lambda$ is Kato type, then $K(T-\lambda)=(T-\lambda)^\infty(\mathcal{X})$. Fredholm (also, semi-Fredholm) operators are Kato type [16, Theorem 4]. (For more information on semi-Fredholm operators, semi-regular operators and Kato type operators, see [1] and [17].)

In the following, $H(\sigma(T))$ shall denote the class of functions f which are analytic on a neighbourhood of $\sigma(T)$. *We assume in the following that our Banach space $\mathcal{X}$ is reflexive: this ensures that operators $T \in \mathrm{loc}(G_m)$ have SVEP.*

3. $\mathrm{loc}(G_m)$ OPERATORS AND SVEP

The following (essentially known) lemma proves that operators $T \in \mathrm{loc}(G_m)$ are *isoloid*, i.e., points $\lambda \in \mathrm{iso}\,\sigma(T)$ are eigenvalues of T. Recall [8] that T is *polaroid* if every $\lambda \in \mathrm{iso}\,\sigma(T)$ is a pole (no restriction on rank) of the resolvent of T. Polaroid operators are isoloid.

LEMMA 3.1. *Operators $T \in \mathrm{loc}(G_m)$ are polaroid.*

Proof. Evidently, $\mathrm{loc}(G_m) \subseteq (G_m)$; hence it would suffice to prove $\lambda \in \mathrm{iso}\,\sigma(T) \Rightarrow \lambda \in \pi(T)$ for operators $T \in (G_m)$. Let $\lambda_0 \in \mathrm{iso}\,\sigma(T)$, and let $\Gamma=\{\lambda : |\lambda-\lambda_0|=\epsilon\} \subset \rho(T)$ for some $\epsilon \leq \mathrm{dist}(\lambda_0, \sigma(T)\setminus\{\lambda_0\})$. Then

$$(\lambda_0-T)^m = \frac{1}{2\pi\mathrm{i}}\int_\Gamma (\lambda_0-\lambda)^m(\lambda-T)^{-1}\mathrm{d}\lambda\,,$$

and

$$||(\lambda_0-T)^m|| \leq \frac{1}{2\pi}\int_\Gamma |\lambda_0-\lambda|^m||(\lambda-T)^{-1}||\,|\mathrm{d}\lambda| \leq \frac{1}{2\pi}\epsilon^m\frac{K}{\epsilon^m}2\pi\epsilon\,,$$

which tends to zero with ϵ. Hence $H_0(T-\lambda_0) \subseteq (T-\lambda_0)^{-m}(0)$. Since $(T-\lambda_0)^{-n}(0) \subseteq H_0(T-\lambda_0)$ for every positive integer n, $H_0(T-\lambda_0) = (T-\lambda_0)^{-m}(0)$. The point λ_0 being isolated in $\sigma(T)$,

$$\mathcal{X} = H_0(T-\lambda_0)\oplus K(T-\lambda_0) = (T-\lambda_0)^{-m}(0)\oplus K(T-\lambda_0)\,.$$

Hence

$$(T-\lambda_0)^m\mathcal{X} = 0\oplus(T-\lambda_0)^mK(T-\lambda_0) = K(T-\lambda_0)\,,$$

which implies that

$$\mathcal{X} = (T - \lambda_0)^{-m}(0) \oplus (T - \lambda_0)^m \mathcal{X},$$

i.e., λ_0 is a pole (of order $\leq m$) of the resolvent of T. ∎

The fact that operators $T \in \mathrm{loc}(G_m)$ have SVEP (recall that our Banach space $\mathcal{X}$ is reflexive) implies that T and T^* satisfy a-Browder's [5, Lemma 2.18], hence also Browder's theorem. More is true.

THEOREM 3.2. *If $T \in \mathrm{loc}(G_m)$, then $f(T)$ satisfies Weyl's theorem and $f(T^*)$ satisfies a-Weyl's theorem for every $f \in H(\sigma(T))$.*

Proof. T satisfies Browder's theorem implies that $\sigma(T) \setminus \sigma_w(T) = \pi_0(T) \subseteq \pi_{00}(T)$. Since $\lambda \in \mathrm{iso}\,\sigma(T) \Rightarrow \lambda \in \pi(T)$ (Lemma 3.1), $\pi_{00}(T) \subseteq \pi_0(T)$. Hence $\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T)$. Furthermore, since $\sigma(T) = \sigma(T^*)$, $\sigma_w(T) = \sigma_w(T^*)$ and $\pi_{00}(T) = \pi_{00}(T^*)$ (observe that $\lambda \in \pi_{00}(T) \Rightarrow \lambda \in \pi_0(T) = \pi_0(T^*) \subseteq \pi_{00}(T^*)$, and conversely), T^* also satisfies Weyl's theorem. Recall from [24, Theorem 1] that $f(B)$ satisfies Weyl's theorem for every $f \in H(\sigma(B))$ for an isoloid operator B satisfying Weyl's theorem such that $\mathrm{ind}(B - \lambda) \leq 0$ for $\lambda \in \Phi_{\pm}(B)$; recall also that if B has SVEP, then $B - \lambda \in \Phi_{\pm}(\mathcal{X}) \Rightarrow \mathrm{ind}(B - \lambda) \leq 0$ [1, Theorem 3.16 and Theorem 3.4]. Hence $f(T)^* = f(T^*)$ satisfies Weyl's theorem. Evidently, $f(T)$ has SVEP. Since, for an operator B with SVEP, B^* satisfies Weyl's theorem if and only if B^* satisfies a-Weyl's theorem, see [1, Theorem 3.108], $f(T^*)$ satisfies a-Weyl's theorem. ∎

4. Perturbation by Riesz operators

An operator $R \in B(\mathcal{X})$ is a *Riesz operator* if $R - \lambda \in \Phi(\mathcal{X})$ for all $\lambda \in \mathbb{C} \setminus \{0\}$. Equivalently, R is a Riesz operator if and only if *the essential spectral radius* $r_e(R)$ of R equals 0 [3, Theorem 3.3.1]. Compact operators, also quasinilpotent operators, are Riesz operators. It is well known that if $Q \in B(\mathcal{X})$ is a quasinilpotent operator which commutes with an operator $T \in B(\mathcal{X})$, $[Q, T] = QT - TQ = 0$, then $\sigma(T + Q) = \sigma(T)$. Moreover, if R is a Riesz operator such that $[R, T] = 0$, then $\sigma_w(T + R) = \sigma_w(T)$ [19, Lemma 2.2]. The following theorem extends this result to $\sigma_{wa}(T+R)$ and $\sigma_{ws}(T+R)$. We remark that parts (i) and (ii) of the following theorem are independent of our standing hypothesis that the Banach space $\mathcal{X}$ is reflexive.

THEOREM 4.1. *If $R \in B(\mathcal{X})$ is a Riesz operator such that $[R,T] = 0$ for some operator $T \in B(\mathcal{X})$, then we have the following.*

(i) $T \in \Phi_+(\mathcal{X}) \Rightarrow T + R \in \Phi_+(\mathcal{X})$ *and* $T \in \Phi_-(\mathcal{X}) \Rightarrow T + R \in \Phi_-(\mathcal{X})$.

(ii) $\sigma_{wa}(T + R) = \sigma_{wa}(T)$, $\sigma_{ws}(T + R) = \sigma_{ws}(T)$ *and* $\sigma_{wa}(T) = \sigma_{ba}(T) \Rightarrow \sigma_{wa}(T + R) = \sigma_{ba}(T + R)$.

(iii) *Furthermore, if $T \in \text{loc}(G_m)$, then $\sigma_{wa}(T + R) = \sigma_{ba}(T + R)$.*

Proof. (i) A proof of (i) appears in [23]: we include it here for completeness. Let $T \in \Phi_+(\mathcal{X})$, and let

$$\begin{aligned} r_+(T) &= \sup\{\epsilon \geq 0 : T - \lambda \in \Phi_+(\mathcal{X}) for|\lambda| < \epsilon\} \\ &= \lim\{\text{dist}(T^n, B(\mathcal{X}) \setminus \Phi_+(\mathcal{X}))\}^{\frac{1}{n}} = \lim\{d_+(T^n)\}^{\frac{1}{n}} \end{aligned}$$

denote the upper semi-Fredholm radius of T [26]. Since $0 = r_e(R) < r_+(T)$, there exist a compact operator K and a positive integer n such that $||R^n - K|| < d_+(T^n)$. But then

$$\begin{aligned} & 0 < d_+(T^n) - ||R^n - K|| \leq d_+(T^n - R^n + K) \\ \Rightarrow \quad & T^n - R^n + K \in \Phi_+(\mathcal{X}) \quad \Rightarrow \quad T^n - R^n \in \Phi_+(\mathcal{X}) . \end{aligned}$$

Since $[R,T] = 0$ and $T^n - R^n = (T - R) \sum_{j=1}^{n-1} T^{n-1-j} R^j$, it follows [3, Corollary 1.3.4] that $T - R \in \Phi_+(\mathcal{X})$.

If $T \in \Phi_-(\mathcal{X})$, then $T^* \in \Phi_+(\mathcal{X}^*)$, $[R^*, T^*] = 0$ and R^* is Riesz. Arguing as above, it follows that $T^* + R^* \in \Phi_+(\mathcal{X}^*) \Rightarrow T + R \in \Phi_-(\mathcal{X})$.

(ii) The argument of part (i) shows, indeed, that $T \in \Phi_\pm(\mathcal{X}) \Rightarrow T + tR \in \Phi_\pm(\mathcal{X})$ for all $0 \leq t \leq 1$. Since the semi-Fredholm index is a continuous function, $\text{ind}(T + R) = \text{ind}(T)$. Hence,

$$\begin{aligned} T - \lambda \in \Phi_+^-(\mathcal{X}) \quad &\Longleftrightarrow \quad T + R - \lambda \in \Phi_+^-(\mathcal{X}) , \\ T - \lambda \in \Phi_-^+(\mathcal{X}) \quad &\Longleftrightarrow \quad T + R - \lambda \in \Phi_-^+(\mathcal{X}) ; \end{aligned}$$

this proves $\sigma_{wa}(T+R) = \sigma_{wa}(T)$ and $\sigma_{ws}(T+R) = \sigma_{ws}(T)$. Recall that $\sigma_{ba}(T)$ is the largest subset of $\sigma_a(T)$ which remains invariant under perturbations by Riesz operators which commute with T [22, Theorem 5]; hence $\sigma_{wa}(T) = \sigma_{ba}(T) \Rightarrow \sigma_{wa}(T + R) = \sigma_{ba}(T + R)$.

(iii) Since T has SVEP, T satisfies a-Browder's theorem (i.e., $\sigma_{wa}(T) = \sigma_{ba}(T)$) [5, Lemma 2.18]. Thus (ii) implies that $\sigma_{wa}(T + R) = \sigma_{ba}(T + R)$. ∎

The conclusion $\sigma_{wa}(T+R) = \sigma_{ba}(T+R)$ of Theorem 4.1(iii) implies that $\sigma(T+R) \setminus \sigma_w(T+R) = \pi_0(T+R) \subseteq \pi_{00}(T+R)$ for Riesz operators R commuting with operators $T \in \text{loc}(G_m)$. The conclusion does not, however, extend to $\sigma(T+R) \setminus \sigma_w(T+R) = \pi_{00}(T+R)$, as the following example shows. Evidently, the trivial operator $T = 0 \in \text{loc}(G_m)$, and the compact operator $K(x_1, x_2, x_3, \dots) = (\frac{x_2}{2}, \frac{x_3}{3}, \dots)$ is a Riesz operator which commutes with T. Trivially, $\sigma(T+K) = \sigma_w(T+K)$ $(= \sigma_{wa}(T+K) = \sigma_{ba}(T+K))$, but $\sigma(T+K) \setminus \sigma_w(T+K) = \emptyset \neq \pi_{00}(T+K) = \{0\}$. Something more is required. One such condition, which has been considered by Han and Lee [12], Oudghiri [19], and Aiena and Guillen [2], is the following. An operator $A \in B(\mathcal{X})$ is *finitely isoloid* if the points $\lambda \in \text{iso}\,\sigma(A) \in \pi_{00}(A)$.

THEOREM 4.2. *Let T be a finitely isoloid $\text{loc}(G_m)$ operator. If R is a Riesz operator which commutes with T, then $T+R$ and $T^* + R^*$ satisfy Weyl's theorem. Furthermore, if R is quasinilpotent, then $T^* + R^*$ satisfies a-Weyl's theorem.*

Proof. Both T and T^* satisfy Weyl's theorem (by Theorem 3.2). Evidently, $[R, T] = 0 = [R^*, T^*]$, and both R and R^* are Riesz operators. Since $\lambda \in \text{iso}\,\sigma(T^*) \Rightarrow \lambda \in \text{iso}\,\sigma(T) \Rightarrow \lambda \in \pi(T) = \pi(T^*)$, the finitely isoloid hypothesis on T implies that T^* is (also) finitely isoloid. Hence [19, Theorem 2.7] applies, and we conclude that $T+R$ and $(T+R)^*$ satisfy Weyl's theorem. Now let R be a quasinilpotent operator. Then $T+R$ has SVEP [1, Corollary 2.12]. Arguing as in the proof of Theorem 3.2, it follows that $T^* + R^*$ satisfies a-Weyl's theorem. ∎

PERTURBATION BY ALGEBRAIC OPERATORS. $A \in B(\mathcal{X})$ is *algebraic* if $p(A) = 0$ for some non-trivial complex polynomial $p(\cdot)$. It is well known that an operator $F \in B(\mathcal{X})$ such that F^n is finite dimensional for some positive integer n is algebraic. Let A be an algebraic operator which commutes with a $\text{loc}(G_m)$ operator T. We prove in the following that $T + A$ satisfies Weyl's theorem and $T^* + A^*$ satisfies a-Weyl's theorem. Similar results for operators S which satisfy $H_0(S-\lambda) = (S-\lambda)^{-p_\lambda}(0)$ for some positive integer p_λ and all $\lambda \in \mathbb{C}$, or which are paranormal Hilbert space operators, or which are completely hereditarily normaloid have been proved, respectively, by Oudghiri [19], Aiena and Guillen [2] and the author [6].

Fix an algebraic operator A (such that $p(A) = 0$) and operator $T \in \text{loc}(G_m)$ such that $[A, T] = 0$. Evidently, $\sigma(A) = \{\mu_1, \mu_2, \dots, \mu_n\}$ for some integer $n \geq 1$. Let $A_i = A|_{H_0(A-\mu_i)}$ and $T_i = T|_{H_0(A-\mu_i)}$. Then,

LEMMA 4.3. *$\sigma_b(T_i + A_i) = \sigma_w(T_i + A_i)$ for all $1 \leq i \leq n$.*

Proof. Since $\sigma(A_i) = \{\mu_i\}$ and $p(A_i) = 0$, $p(\mu_i) = p(\sigma(A_i)) = \sigma(p(A_i)) = \{0\}$. Hence

$$0 = p(A_i) = p(A_i) - p(\mu_i) = (A_i - \mu_i)^{t_i} g(A_i)$$

for some positive integer t_i and invertible $g(A_i)$. Consequently, $A_i - \mu_i$ $\big(= A|_{H_0(A-\mu_i)} - \mu_i I|_{H_0(A-\mu_i)}\big)$ is nilpotent. The commutativity of A and T implies that $H_0(A - \mu_i)$ is invariant for T; hence $T_i \in \mathrm{loc}(G_m)$ has SVEP. Applying Theorem 4.1(iii), we conclude that $T_i + A_i$ satisfies a-Browder's theorem $\Rightarrow T_i + A_i$ satisfies Browder's theorem. ∎

The proofs of the following two lemmas differ but only slightly from the proofs of [6, Lemma 3.4 and Lemma 3.6]; the proofs are included here for the reader's convenience. (We remark here that the proof of Lemma 4.4 below, as also of [6, Lemma 3.4], is inspired by that of [19, Lemma 3.3].)

LEMMA 4.4. *If N is a nilpotent operator which commutes with T, then $H_0(T + N - \lambda) = (T + N - \lambda)^{-r}(0)$, for some integer $r \geq 1$, at points $\lambda \in \mathrm{iso}\,\sigma(T)$.*

Proof. We may assume that $N^t = 0$ for some positive integer t. Choose an integer $s > t$. Then, for every $x \in \mathcal{X}$ and $\lambda \in \mathbb{C}$,

$$\begin{aligned}
||(T-\lambda)^s x||^{\frac{1}{s}} &= ||((T+N-\lambda)-N)^s x||^{\frac{1}{s}} \\
&= \left\| \sum_{j=0}^{s-1} (-1)^j \;\, {}^sC_j N^j (T+N-\lambda)^{s-j} x \right\|^{\frac{1}{s}} \\
&\leq \sum_{j=0}^{s-1} [{}^sC_j ||N||^j]^{\frac{1}{s}} ||(T+N-\lambda)^{s-j} x||^{\frac{1}{s}},
\end{aligned}$$

which implies that

$$H_0(T-\lambda) \subseteq H_0(T+N-\lambda)\,.$$

By symmetry,

$$H_0(T+N-\lambda) \subseteq H_0(T+N-\lambda-N) = H_0(T-\lambda)\,.$$

Hence

$$H_0(T-\lambda) = H_0(T+N-\lambda)\,.$$

Choose $\lambda \in \operatorname{iso}\sigma(T)$. Then $H_0(T-\lambda) = (T-\lambda)^{-m}(0)$. Set $r = mt$. Since $x \in (T-\lambda)^{-m}(0)$ implies that

$$(T+N-\lambda)^{mt}x = \sum_{r-m+1}^{r} \left\{ {}^{r}C_j (T-\lambda)^{r-j} N^{j-t} \right\} N^t x = 0\,,$$

it follows that

$$H_0(T+N-\lambda) = (T-\lambda)^{-mt}(0) \subseteq (T+N-\lambda)^{-mt}(0)\,.$$

Since $(T+N-\lambda)^{-q}(0) \subseteq H_0(T+N-\lambda)$ for all integers $q \geq 1$, $H_0(T+N-\lambda) = (T+N-\lambda)^{-r}(0)$. ∎

The following lemma relates $H_0(T+A-\lambda)$ to $(T+A-\lambda)^{-r}(0)$ at points $\lambda \in \operatorname{iso}\sigma(T+A)$.

Lemma 4.5. *If $\lambda \in \operatorname{iso}\sigma(T+A)$, then $H_0(T+A-\lambda) = (T+A-\lambda)^{-r}(0)$ for some positive integer r.*

Proof. Since the subspace $H_0(A-\mu_i)$ coincides with the range of the spectral projection of A associated with μ_i [1, Theorem 3.74], the hypothesis $[T,A]=0$ implies that $[T_i, A_i] = 0$. Apparently, see the proof of Lemma 4.3, $(A_i-\mu_i)^{t_i} = 0$ for some integer $t_i \geq 1$. Let $\lambda \in \operatorname{iso}\sigma(T+A)$. Then either $\lambda - \mu_i \notin \sigma(T_i)$ or $\lambda - \mu_i \in \operatorname{iso}\sigma(T_i)$, $1 \leq i \leq n$. If $\lambda - \mu_i \notin \sigma(T_i)$, then the invertibility of $T_i - (\lambda - \mu_i)$ implies that $\{T_i - (\lambda-\mu_i)\} + \{A_i - \mu_i\}$ is invertible, and hence that

$$\begin{aligned} H_0(T_i + A_i - \lambda) &= H_0((T_i + A_i - \mu_i) - (\lambda - \mu_i)) = \{0\} \\ &= (T_i + A_i - \lambda)^{-r_i}(0) \end{aligned}$$

for every positive integer r_i and all $1 \leq i \leq n$. Now let $\lambda - \mu_i \in \operatorname{iso}\sigma(T_i)$; then, by Lemma 4.4,

$$\begin{aligned} H_0(T_i + A_i - \lambda) &= H_0((T_i + A_i - \mu_i) - (\lambda - \mu_i)) \\ &= ((T_i + A_i - \mu_i) - (\lambda - \mu_i))^{-r_i}(0) = (T_i + A_i - \lambda)^{-r_i}(0) \end{aligned}$$

for some positive integer r_i and all $1 \leq i \leq n$. Let $m = \max\{r_1, r_2, \ldots, r_n\}$.

Then

$$\begin{aligned} H_0(T+A-\lambda) &= \bigoplus_{i=1}^{n} H_0(T_i+A_i-\lambda) \\ &= \bigoplus_{i=1}^{n}(T_i+A_i-\lambda)^{-r_i}(0) = (T+A-\lambda)^{-r}(0)\,. \end{aligned}$$

This completes the proof. ∎

THEOREM 4.6. *$f(T+A)$ satisfies Weyl's theorem and $f(T^*+A^*)$ satisfies a-Weyl's theorem for every $f \in H(\sigma(T+A))$.*

Proof. Each T_i, being the restriction of an operator with SVEP to an invariant subspace, has SVEP; hence $T_i+\mu_i$ has SVEP for all $1 \le i \le n$. Since $A_i-\mu_i$ is nilpotent, and commutes with $T_i+\mu_i$, T_i+A_i has SVEP for all $1 \le i \le n$. Thus, the upper triangular operator matrix $T+A = \bigoplus_{i=1}^{n}(T_i+A_i)$ has SVEP (and so satisfies Browder's theorem). We claim that $T+A$ is Kato type at points $\lambda \in \operatorname{iso}\sigma(T+A)$. Indeed, if $\lambda \in \operatorname{iso}\sigma(T+A)$, then Lemma 4.5 implies the existence of a positive integer r such that $H_0(T+A-\lambda) = (T+A-\lambda)^{-r}(0)$. Hence, since $(T+A-\lambda)^r K(T+A-\lambda) = K(T+A-\lambda)$, we have

$$\begin{aligned} \mathcal{X} &= H_0(T+A-\lambda) \oplus K(T+A-\lambda) \\ &= (T+A-\lambda)^{-r}(0) \oplus K(T+A-\lambda) \\ \Rightarrow \quad \mathcal{X} &= (T+A-\lambda)^{-r}(0) \oplus (T+A-\lambda)^r \mathcal{X}. \end{aligned}$$

Evidently, λ is a pole of the resolvent of $T+A$; in particular, $T+A$ is Kato type at λ, and our claim is proved. Recall from [7, Theorem 3.3 and Theorem 3.6(ii)] that a sufficient condition for a Banach space operator with SVEP to satisfy Weyl's theorem, and its conjugate to satisfy a-Weyl's theorem, is that it is Kato type at the isolated points of its spectrum. Hence $T+A$ satisfies Weyl's theorem, and T^*+A^* satisfies a-Weyl's theorem. Evidently, $T+A$ is isoloid and $\operatorname{ind}(T+A-\lambda) \le 0$ for $\lambda \in \Phi_{\pm}(T+A)$, [24, Theorem 1] applies and we conclude that $f(T+A)$ satisfies Weyl's theorem for every $f \in H(\sigma(T+A))$. Since $f(T+A)$ has SVEP, an argument similar to that in the proof of Corollary 3.2 shows that $f(T^*+A^*)$ satisfies a-Weyl's theorem for every $f \in H(\sigma(T+A))$. ∎

Polynomially $\text{loc}(G_m)$ operators. An operator $T \in B(\mathcal{X})$ is (*algebraically*) *polynomially* $\text{loc}(G_m)$, $T \in \text{pl}(G_m)$, if there exists a non-constant polynomial $q(\cdot)$ such that $q(T) \in \text{loc}(G_m)$. Given a *class* $\mathcal{C}$ *of operators in* $B(\mathcal{X})$ (thus, the elements of $\mathcal{C}$ are characterized by a property or a set of properties), it is known that if T is polynomially $\mathcal{C}$, then T inherits Browder-Weyl theorem type properties for a number of classes $\mathcal{C}$ (see [5] for some references). We prove in the following that an analogue of Theorem 4.6 holds for operator $T \in \text{pl}(G_m)$.

Throughout the following A shall denote an algebraic operator which commutes with $T \in \text{pl}(G_m)$ (such that $q(T) \in \text{loc}(G_m)$). The operators A_i and T_i, $1 \le i \le n$, shall be defined as above. We start by proving some complementary lemmas.

Lemma 4.7. *$T + A$ and $T_i + A_i$, $1 \le i \le n$, have SVEP.*

Proof. Since $q(T)$ has SVEP, [17, Proposition 3.3.9] implies that T has SVEP. This, as in the proof of Theorem 4.6, implies that $T_i + A_i$ has SVEP for all $1 \le i \le n$. ∎

Lemma 4.7 implies, in particular, that $T + A$ satisfies Browder's theorem.

Lemma 4.8. *Points $\lambda \in \text{iso}\,\sigma(T)$ are poles of the resolvent of T. In particular, T is isoloid.*

Proof. The hypothesis $\lambda \in \text{iso}\,\sigma(T)$ implies that $\mathcal{X} = H_0(T-\lambda) \oplus K(T-\lambda)$. Let $T_0 = T|_{H_0(T-\lambda)}$; then $\sigma(T_0) = \{\lambda\}$ and $\sigma(q(T_0)) = q(\sigma(T_0)) = \{q(\lambda)\}$. Since $q(T_0) \in \text{loc}(G_m)$, and the isolated points of a $\text{loc}(G_m)$ operator are poles (of order m) of the resolvent of the operator, $\{q(T_0) - q(\lambda)\}^m = 0$. Let

$$\{q(T_0) - q(\lambda)\}^m = c(T_0 - \lambda)^{mt} \prod_{i=1}^{s} (T_0 - \lambda_i)$$

for some numbers $c, \lambda_1, \ldots, \lambda_s \in \mathbb{C}$. Since each $T_0 - \lambda_i$ is invertible, we conclude that $T_0 - \lambda$ is nilpotent (of some order $r \le mt$). Hence

$$\begin{aligned} &\mathcal{X} = H_0(T-\lambda) \oplus K(T-\lambda) = (T-\lambda)^{-r}(0) \oplus K(T-\lambda) \\ \Rightarrow \quad &\mathcal{X} = (T-\lambda)^{-r}(0) \oplus (T-\lambda)^r \mathcal{X}, \end{aligned}$$

i.e., λ is a pole (of order $\le r$) of the resolvent of T. ∎

Lemma 4.8 says that T is Kato type at points $\lambda \in \mathrm{iso}\,\sigma(T)$ for operators $T \in \mathrm{pl}(G_m)$. More is true, as the following lemma shows.

LEMMA 4.9. *$T + A$ is Kato type at points $\lambda \in \mathrm{iso}\,\sigma(T + A)$.*

Proof. Argue as in the proof of Lemma 4.4, Lemma 4.5 and Theorem 4.6. Remark that the part of the argument of the proof of Theorem 4.6 (also, the proof of Lemma 4.4 and Lemma 4.5) leading to the conclusion $T + A$ is Kato type at $\lambda \in \mathrm{iso}\,\sigma(T + A)$ depends only upon the property that points $\lambda \in \mathrm{iso}\,\sigma(T)$ of $T \in \mathrm{loc}(G_m)$ are poles, a property guaranteed by Lemma 4.8. ∎

THEOREM 4.10. *If A is an algebraic operator which commutes with an operator $T \in \mathrm{pl}(G_m)$, then $f(T + A)$ satisfies Weyl's theorem and $f(T^* + A^*)$ satisfies a-Weyl's theorem for every $f \in H(\sigma(T + A))$.*

Proof. Since $T + A$ has SVEP (Lemma 4.7) and is Kato type at points $\lambda \in \mathrm{iso}\,\sigma(T + A)$, $T + A$ satisfies Weyl's theorem and $(T + A)^*$ satisfies a-Weyl's theorem [7, Theorem 3.3 and Theorem 3.6(ii)]. Furthermore, since $T + A$ is isoloid and $\mathrm{ind}(T + A - \lambda) \leq 0$ at $\lambda \in \Phi_\pm(T + A)$, [24, Theorem 1] and [1, Theorem 3.108] apply. ∎

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REFERENCES

[1] P. AIENA, "Fredholm and Local Spectral Theory, with Applications to Multipliers", Kluwer Academic Publishers, Dordrecht, 2004.

[2] P. AIENA, J.R. GUILLEN, Weyl's theorem for perturbations of paranormal operators, *Proc. Amer. Math. Soc.* **135** (2007), 2443–2451.

[3] S.R. CARADUS, W.E. PFAFFENBERGER, Y. BERTRAM, "Calkin Algebras and Algebras of operators on Banach Spaces", Marcel Dekker, Inc., New York, 1974.

[4] R.E. CURTO, Y.M. HAN, Weyl's theorem for algebraically paranormal operators, *Integral Equations Operator Theory* **47** (2003), 307–314.

[5] B.P. DUGGAL, Hereditarily normaloid operators, *Extracta Math.* **20** (2005), 203–217.

[6] B.P. DUGGAL, Perturbations and Weyl's theorem, *Proc. Amer. Math. Soc.* **135** (2007), 2899–2905.

[7] B.P. Duggal, S. Djordjević, C. Kubrusly, Kato type operators and Weyl's theorems, *J. Math. Anal. Appl.* **309** (2005), 433 – 441.

[8] B.P. Duggal, R. Harte, I.H. Jeon, Polaroid operators and Weyl's theorem, *Proc. Amer. Math. Soc.* **132** (2004), 1345 – 1349.

[9] N. Dunford, Spectral theory I. Resolution of identity, *Pacific J. Math.* **2** (1952), 559 – 614.

[10] N. Dunford, Spectral operators, *Pacific J. Math.* **4** (1954), 321 – 354.

[11] N. Dunford, J.T. Schwartz, "Linear Operators, Parts I and III", Interscience, New York, 1964, 1971.

[12] Y.M. Han, W.Y. Lee, Weyl spectra and Weyl's theorem, *Studia Math.* **148** (2001), 193 – 206.

[13] R.E. Harte, W.Y. Lee, Another note on Weyl's theorem, *Trans. Amer. Math. Soc.* **349** (1997), 2115 – 2124.

[14] H.G. Heuser, "Functional Analysis", John Wiley & Sons, Ltd., Chichester, 1982.

[15] A.A. Jafarian, M. Radjabalipour, Transitive algebra problem and local resolvent techniques, *J. Operator Theory* **1** (1979), 273 – 285.

[16] T. Kato, Perturbation theory for nullity, deficiency and other quantities of linear operators, *J. Math. Anal.* **6** (1958), 261 – 322.

[17] K.B. Laursen, M.N. Neumann, "An Introduction to Local Spectral Theory", The Clarendon Press, Oxford University Press, New York, 2000.

[18] M. Mbekhta, Généralisation de la décomposition de Kato aux opérateurs paranormaux et spectraux, *Glasg. Math. J.* **29** (1987), 159 – 175.

[19] M. Oudghiri, Weyl's theorem and perturbations, *Integral Equations Operator Theory* **53** (2005), 535 – 545.

[20] V. Rakočević, On the essential approximate point spectrum II, *Mat. Vesnik* **36** (1981), 89 – 97.

[21] V. Rakočević, Operators obeying a-Weyl's theorem, *Rev. Roumaine Math. Pures Appl.* **34** (1989), 915 – 919.

[22] V. Rakočević, Semi-Browder operators and perturbations, *Studia Math.* **122** (1997), 131 – 137.

[23] M. Schechter, R. Whitley, Best Fredholm perturbation theorems, *Studia Math.* **90** (1988), 175 – 190.

[24] C. Schmoeger, On operators T such that Weyl's theorem holds for $f(T)$, *Extracta Math.* **13** (1998), 27 – 33.

[25] J.G. Stampfli, A local spectral theory for operators: Spectral subspaces for hyponormal operators, *Trans. Amer. Math. Soc.* **217** (1976), 359 – 365.

[26] J. Zemánek, The semi-Fredholm radius of a linear operator, *Bull. Pol. Acad. Sci. Math.* **32** (1984), 67 – 76.