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On the Existence of ϕ-Recurrent $(LCS)_n$-Manifolds

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Abstract: The object of the present paper is to provide the existence of ϕ-recurrent $(LCS)_n$-manifolds with several non-trivial examples.

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1. Introduction

Recently the first author ([1]) introduced the notion of Lorentzian concircular structure manifolds (briefly $(LCS)_n$-manifolds) with an example, which generalizes the notion of LP-Sasakian manifolds introduced by Matsumoto ([4]). The notion of local symmetry of a Riemannian manifold has been weakened by many authors in several ways to a different extent. As a weaker version of local symmetry, T. Takahashi ([5]) introduced the notion of local ϕ-symmetry on a Sasakian manifold.

In the context of Lorentzian geometry, the notion of local ϕ-symmetry is introduced and studied by Shaikh and Baishya ([2]) with several examples. Generalizing these notions, in the present paper we introduce the notion of *locally ϕ-recurrent $(LCS)_n$-manifolds*. Section 2 is concerned with some curvature properties of $(LCS)_n$-manifolds. Section 3 consists of locally ϕ-recurrent $(LCS)_n$-manifolds and obtained a necessary and sufficient condition for such a manifold to be of locally ϕ-recurrent. It is shown that in a locally ϕ-recurrent $(LCS)_n$-*manifold*, $\frac{r}{2}$ is an eigenvalue of the Ricci tensor corresponding to the eigenvector associated to the 1-form of the recurrence, r being the scalar curvature of the manifold. And also in a locally ϕ-recurrent $(LCS)_n$-manifold, the recurrent vector field is obtained as $\rho = \frac{1}{r}\,\mathrm{grad}\, r$. Finally, the existence of such a manifold is ensured by several non-trivial examples in both odd and even dimension.

2. $(LCS)_n$-MANIFOLDS

An n-dimensional Lorentzian manifold M is a smooth connected para-compact Hausdorff manifold with a Lorentzian metric g, that is, M admits a smooth symmetric tensor field g of type (0,2) such that for each point $p \in M$, the tensor $g_p : T_pM \times T_pM \to \mathbb{R}$ is a non-degenerate inner product of signature $(-, +, \dots, +)$, where T_pM denotes the tangent vector space of M at p and $\mathbb{R}$ is the real number space. A non-zero vector $v \in T_pM$ is said to be timelike (resp., non-spacelike, null, spacelike) if it satisfies $g_p(v, v) < 0$ (resp., ≤ 0, $= 0$, > 0) ([3]).

DEFINITION 2.1. In a Lorentzian manifold (M, g) a vector field P defined by

$$g(X, P) = A(X)$$

for any $X \in \chi(M)$ is said to be a concircular vector field if

$$(\nabla_X A)(Y) = \alpha\{g(X, Y) + \omega(X)A(Y)\}$$

where α is a non-zero scalar and ω is a closed 1-form.

Let M^n be a Lorentzian manifold admitting a unit timelike concircular vector field ξ, called the generator of the manifold. Then we have

$$g(\xi, \xi) = -1\,. \tag{2.1}$$

Since ξ is a unit concircular vector field, it follows that there exists a non-zero 1-form η such that for

$$g(X, \xi) = \eta(X)\,, \tag{2.2}$$

the equation of the following form holds

$$(\nabla_X \eta)(Y) = \alpha\{g(X, Y) + \eta(X)\eta(Y)\} \quad (\alpha \neq 0) \tag{2.3}$$

for all vector fields X, Y where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g and α is a non-zero scalar function satisfies

$$\nabla_X \alpha = (X\alpha) = d\alpha(X) = \rho\eta(X) \tag{2.4}$$

ρ being a certain scalar function given by $\rho = -(\xi\alpha)$. If we put

$$\phi X = \frac{1}{\alpha}\nabla_X \xi\,, \tag{2.5}$$

then from (2.3) and (2.5) we have

$$\phi X = X + \eta(X)\xi\,, \tag{2.6}$$

from which it follows that ϕ is a symmetric (1,1) tensor and called the structure tensor of the manifold. Thus the Lorentzian manifold M^n together with the unit timelike concircular vector field ξ, its associated 1-form η and (1,1) tensor field ϕ is said to be a Lorentzian concircular structure manifold (briefly $(LCS)_n$-*manifold*) ([1]). Especially, if we take $\alpha = 1$, then we can obtain the LP-Sasakian structure of Matsumoto ([4]). In a $(LCS)_n$-manifold, the following relations hold ([1]):

$$a)\quad \eta(\xi) = -1\,, \qquad b)\quad \phi\xi = 0\,, \tag{2.7}$$

$$c)\quad \eta(\phi X) = 0\,, \qquad d)\quad g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y)\,,$$

$$\eta(R(X,Y)Z) = (\alpha^2 - \rho)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]\,, \tag{2.8}$$

$$S(X,\xi) = (n-1)(\alpha^2 - \rho)\eta(X)\,, \tag{2.9}$$

$$R(X,Y)\xi = (\alpha^2 - \rho)[\eta(Y)X - \eta(X)Y]\,, \tag{2.10}$$

$$(\nabla_X \phi)(Y) = \alpha\{g(X,Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X\}\,, \tag{2.11}$$

$$(X\rho) = d\rho(X) = \beta\eta(X)\,. \tag{2.12}$$

We now state and prove some curvature properties of $(LCS)_n$-manifold which will be frequently used later on.

Lemma 2.1. *Let (M^n, g) be a $(LCS)_n$-manifold. Then for any X, Y, Z the following relation holds:*

$$R(X,Y)Z = \phi R(X,Y)Z + (\alpha^2 - \rho)\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}\xi \tag{2.13}$$

for any vector field X, Y, Z.

Proof. From (2.3), (2.4), (2.5), (2.6) and (2.10) we can easily get (2.13). ∎

Lemma 2.2. *Let (M^n, g) be a $(LCS)_n$-manifold. Then for any X, Y, Z the following relation holds:*

$$(\nabla_W R)(X,Y)\xi = (2\alpha\rho - \beta)\{\eta(Y)\eta(W)X - \eta(X)\eta(W)Y\} \tag{2.14}$$

$$+ \alpha(\alpha^2 - \rho)\{g(Y,W)X - g(X,W)Y\} - \alpha R(X,Y)W.$$

Proof. By virtue of (2.3), (2.6) and (2.10) we can easily get (2.14). ∎

LEMMA 2.3. *Let (M^n, g) be a $(LCS)_n$-manifold. Then for any X, Y, Z the following relation holds:*

$$g((\nabla_W R)(X,Y)Z,U) = -g((\nabla_W R)(X,Y)U,Z)\,. \tag{2.15}$$

Proof. By definition, we have

$$\begin{aligned} g((\nabla_W R)(X,Y)Z,U) = g(\nabla_W R(X,Y)Z,U) + \tilde{R}(X,Y,U,\nabla_W Z) \\ + \tilde{R}(\nabla_W X,Y,U,Z) + \tilde{R}(X,\nabla_W Y,U,Z)\,, \end{aligned} \tag{2.16}$$

where $\tilde{R}(X,Y,Z,U) = g(R(X,Y)Z,U)$ and the property of curvature tensor have been used. Since ∇ is a metric connection, it follows that

$$g(\nabla_W R(X,Y)Z,U) = g(R(X,Y)\nabla_W U,Z) - \nabla_W g(R(X,Y)U,Z) \tag{2.17}$$

and

$$\nabla_W g(R(X,Y)U,Z) = g(\nabla_W R(X,Y)U,Z) + g(R(X,Y)U,\nabla_W Z)\,. \tag{2.18}$$

From (2.17) and (2.18) we have

$$\begin{aligned} g(\nabla_W R(X,Y)Z,U) = &- g(\nabla_W R(X,Y)U,Z) \\ &- g(R(X,Y)U,\nabla_W Z) + g(R(X,Y)\nabla_W U,Z)\,. \end{aligned} \tag{2.19}$$

Using (2.19) in (2.16), we get the relation (2.15). ∎

3. LOCALLY ϕ-RECURRENT $(LCS)_n$-MANIFOLDS

DEFINITION 3.1. A $(LCS)_n$-manifold (M^n, g) is said to be locally ϕ-recurrent if and only if there exists a non-zero 1-form A such that

$$\phi^2((\nabla_W R)(X,Y)Z) = A(W)R(X,Y)Z \tag{3.1}$$

holds for any vector field X, Y, Z, W orthogonal to ξ, that is, for any horizontal vector field X, Y, Z, W.

If, in particular, the 1-form A vanishes identically, then the manifold is said to be a locally ϕ-symmetric manifold ([5]).

THEOREM 3.1. *A $(LCS)_n$-manifold (M^n, g) is locally ϕ-recurrent if and only if the relation*

$$(\nabla_W R)(X,Y)Z = \alpha(\alpha^2 - \rho)\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\}\xi \\ - \alpha g(R(X,Y)W,Z)\xi + A(W)R(X,Y)Z \tag{3.2}$$

holds for all horizontal vector fields X, Y, Z, W on M.

Proof. Let us consider a $(LCS)_n$-manifold (M^n, g) which is locally ϕ-recurrent. Then using (2.6) in (3.1) we have

$$(\nabla_W R)(X,Y)Z + \eta((\nabla_W R)(X,Y)Z)\xi = A(W)R(X,Y)Z \tag{3.3}$$

for any X, Y, Z, W orthogonal to ξ. In view of (2.15), it follows from (3.3) that

$$(\nabla_W R)(X,Y)Z = g((\nabla_W R)(X,Y)\xi, Z)\xi + A(W)R(X,Y)Z\,. \tag{3.4}$$

Using (2.14) in (3.4) we obtain the relation (3.2). Conversely, if in a $(LCS)_n$-manifold the relation (3.2) holds, then applying ϕ on both sides of (3.2) and keeping in mind that X, Y, Z and W are orthogonal to ξ, we obtain (3.1). This proves the theorem. ∎

THEOREM 3.2. *A $(LCS)_n$-manifold is of positive (resp. negative) constant curvature according as $\alpha^2 > \rho$ (resp. $\alpha^2 < \rho$) if and only if the relation*

$$\phi^2((\nabla_W R)(X,Y)\xi) = A(W)R(X,Y)\xi \tag{3.5}$$

holds for all horizontal vector fields X, Y, W.

Proof. Using (2.6) in (3.5) we have

$$(\nabla_W R)(X,Y)\xi + \eta((\nabla_W R)(X,Y)\xi)\xi = A(W)R(X,Y)\xi\,. \tag{3.6}$$

In view of (2.14) and (2.10), (3.6) yields

$$(\nabla_W R)(X,Y)\xi = 0 \tag{3.7}$$

for any horizontal vector field X, Y, W. Also for any X, Y, W orthogonal to ξ, the relation (2.14) reduces to

$$(\nabla_W R)(X,Y)\xi = (2\alpha\rho - \beta)\{\eta(Y)\eta(W)X - \eta(X)\eta(W)Y\} \tag{3.8}$$
$$+ \alpha(\alpha^2 - \rho)\{g(Y,W)X - g(X,W)Y\} - \alpha R(X,Y)W.$$

From (3.7) and (3.8), it follows that

$$R(X,Y)W = (\alpha^2 - \rho)\{g(Y,W)X - g(X,W)Y\} \tag{3.9}$$

for any horizontal vector field X, Y, W. We shall now show that $\alpha^2 - \rho$ is constant. Taking covariant derivative along any horizontal vector field X and then using (2.4) and (2.12) we obtain

$$\nabla_X(\alpha^2 - \rho) = 0$$

and hence $\alpha^2 - \rho$ =constant. Thus the manifold is of constant curvature.

Conversely, if a $(LCS)_n$-manifold is of constant curvature, then from (3.9) it follows that the relation (3.5) holds. This proves the theorem. ∎

THEOREM 3.3. *In a locally ϕ-recurrent $(LCS)_n$-manifold (M^n, g) $(n > 3)$, $\frac{r}{2}$ is an eigenvalue of the Ricci tensor corresponding to the eigenvector ρ, where ρ is the associated vector field of the 1-form A.*

Proof. In a locally ϕ-recurrent $(LCS)_n$-manifold the relation (3.1) holds. Changing W, X, Y cyclically in (3.1) and then adding the results we obtain

$$\begin{aligned}
&\big[(\nabla_W R)(X,Y)Z + (\nabla_X R)(Y,W)Z + (\nabla_Y R)(W,X)Z\big] \\
&+ \big[\eta((\nabla_W R)(X,Y)Z) + \eta((\nabla_X R)(Y,W)Z) + \eta((\nabla_Y R)(W,X)Z)\big]\xi \\
&= A(W)R(X,Y)Z + A(X)R(Y,W)Z + A(Y)R(W,X)Z\,,
\end{aligned}$$

which yields by virtue of Bianchi identity that

$$A(W)R(X,Y)Z + A(X)R(Y,W)Z + A(Y)R(W,X)Z = 0 \tag{3.10}$$

for all X, Y, Z, W orthogonal to ξ. Taking an inner product on both sides of (3.10) by any horizontal vector field U, we get

$$\begin{aligned}
A(W)g(R(X,Y)Z,U) &+ A(X)g(R(Y,W)Z,U) \\
&+ A(Y)g(R(W,X)Z,U) = 0\,.
\end{aligned} \tag{3.11}$$

Contraction over X and U in (3.11) yields

$$A(W)S(Y,Z) + A(R(Y,W)Z) - A(Y)S(W,Z) = 0\,. \tag{3.12}$$

Again, contracting X and U in (3.12) we obtain

$$S(W,\rho) = \frac{r}{2}\,A(W) = \frac{r}{2}\,g(W,\rho)\,.$$

This proves the theorem. ∎

THEOREM 3.4. *In a locally ϕ-recurrent $(LCS)_n$-manifold (M^n, g), the 1-form of recurrence A is given by*

$$A(W) = \frac{\mathrm{d}r(W)}{r} \tag{3.13}$$

for all W orthogonal to ξ, where r is the non-zero and non-constant scalar curvature of the manifold.

Proof. In a locally ϕ-recurrent $(LCS)_n$-manifold, the relation (3.2) holds good. Taking an inner product on both sides of (3.2) by an arbitrary horizontal vector field U tangent to M, we obtain

$$g((\nabla_W R)(X,Y)Z,U) = A(W)g(R(X,Y)Z,U)\,. \tag{3.14}$$

Contracting over X and U in (3.14), we get

$$(\nabla_W S)(Y,Z) = A(W)S(Y,Z) \tag{3.15}$$

which yields again by contraction over Y and Z, the relation (3.13). This proves the theorem. ∎

In particular, if r is a non-zero constant in the direction orthogonal to ξ, then the locally ϕ-recurrent $(LCS)_n$-manifold reduces to the locally ϕ-symmetric $(LCS)_n$-manifold. Thus we have the following corollary:

COROLLARY 3.1. *If in a locally ϕ-recurrent $(LCS)_n$-manifold (M^n, g) the scalar curvature is a non-zero constant along the orthogonal direction to ξ, then the manifold is locally ϕ-symmetric.*

We shall now construct several examples of locally ϕ-recurrent $(LCS)_n$-manifolds.

EXAMPLE 3.1. We consider a 3-dimensional manifold

$$M = \left\{(x,y,z) \in \mathbb{R}^3 \,:\, x \neq \pm\sqrt{2}z^2,\ x \neq 0,\ z \neq 0\right\},$$

where (x,y,z) are the standard coordinates in $\mathbb{R}^3$. Let $\{E_1, E_2, E_3\}$ be linearly independent global frame on M given by

$$E_1 = z\,\frac{\partial}{\partial x}\,, \qquad E_2 = zx\,\frac{\partial}{\partial y}\,, \qquad E_3 = \frac{\partial}{\partial z}\,.$$

Let g be the Lorentzian metric defined by

$$\begin{aligned} g(E_1, E_3) &= g(E_2, E_3) = g(E_1, E_2) = 0\,, \\ g(E_1, E_1) &= g(E_2, E_2) = 1\,, \\ g(E_3, E_3) &= -1\,. \end{aligned}$$

Let η be the 1-form defined by $\eta(U) = g(U, E_3)$ for any $U \in \chi(M)$. Let ϕ be the (1,1) tensor field defined by $\phi E_1 = E_1$, $\phi E_2 = E_2$, $\phi E_3 = 0$. Then using the linearity of ϕ and g we have

$$\begin{aligned} \eta(E_3) &= -1\,, \\ \phi U &= U + \eta(U)E_3\,, \\ g(\phi U, \phi W) &= g(U, W) + \eta(U)\eta(W) \end{aligned}$$

for any $U, W \in \chi(M)$.

Let ∇ be the Levi-Civita connection with respect to the Lorentzian metric g and R be the curvature tensor of g. Then we have

$$[E_1, E_2] = \frac{z}{x}E_2\,, \qquad [E_1, E_3] = -\frac{1}{z}E_1\,, \qquad [E_2, E_3] = -\frac{1}{z}E_2\,. \tag{3.16}$$

Taking $E_3 = \xi$ and using Koszul formula for the Lorentzian metric g, we can easily calculate

$$\begin{aligned} &\nabla_{E_1}E_3 = -\frac{1}{z}E_1\,, && \nabla_{E_1}E_1 = -\frac{1}{z}E_3\,, && \nabla_{E_1}E_2 = 0\,, \\ &\nabla_{E_2}E_3 = -\frac{1}{z}E_2\,, && \nabla_{E_3}E_2 = 0\,, && \nabla_{E_2}E_1 = -\frac{z}{x}E_2\,, \\ &\nabla_{E_3}E_3 = 0\,, && \nabla_{E_2}E_2 = \frac{z}{x}E_1 - \frac{1}{z}E_3\,, && \nabla_{E_3}E_1 = 0\,. \end{aligned}$$

From the above it can be easily seen that $E_3 = \xi$ is a unit timelike concircular vector field and hence (ϕ, ξ, η, g) is a $(LCS)_3$ structure on M. Consequently $M^3(\phi, \xi, \eta, g)$ is a $(LCS)_3$-manifold with $\alpha = -\frac{1}{z} \neq 0$ such that $(X\alpha) = \rho\eta(X)$ where $\rho = -\frac{1}{z^2}$. Using the above relations, we can easily calculate the non-vanishing components of the curvature tensor R as follows:

$$R(E_2,E_3)E_3 = -\frac{2}{z^2}\,E_2\,, \qquad R(E_1,E_3)E_3 = -\frac{2}{z^2}\,E_1\,,$$

$$R(E_1,E_2)E_2 = -\left[2\left(\frac{z}{x}\right)^2 - \frac{1}{z^2}\right]E_1\,, \qquad R(E_2,E_3)E_2 = -\frac{2}{z^2}\,E_3\,,$$

$$R(E_1,E_3)E_1 = -\frac{2}{z^2}\,E_3\,, \qquad R(E_1,E_2)E_1 = \left[2\left(\frac{z}{x}\right)^2 - \frac{1}{z^2}\right]E_2$$

and the components which can be obtained from these by the symmetry properties. From the above, we can easily calculate the non-vanishing components of the Ricci tensor S as follows:

$$S(E_1,E_1) = -\left[2\left(\frac{z}{x}\right)^2 - \frac{1}{z^2}\right],$$

$$S(E_2,E_2) = -\left[2\left(\frac{z}{x}\right)^2 - \frac{1}{z^2}\right],$$

$$S(E_3,E_3) = -\frac{4}{z^2}\,.$$

Hence the scalar curvature r is given by

$$r = \sum_{i=1}^{3}\epsilon_i S(E_i,E_i) = -\left[4\left(\frac{z}{x}\right)^2 - \frac{2}{z^2}\right] \neq 0\,,$$

where $\epsilon_i = g(E_i,E_i)$.

Consequently, $\mathrm{d}r(E_1) = -8(\frac{z}{x})^3 \neq 0$ but $\mathrm{d}r(E_2) = 0$. We shall show that the manifold (M^3,g) under consideration is locally ϕ-recurrent $(LCS)_3$-manifold. To verify this we calculate the covariant derivatives of the required non-zero components of the curvature tensor as follows:

$$(\nabla_{E_1}R)(E_1,E_2)E_1 = -4(\frac{z}{x})^3 E_2\,,$$

$$(\nabla_{E_1}R)(E_1,E_2)E_2 = 4(\frac{z}{x})^3 E_1\,,$$

$$(\nabla_{E_2}R)(E_1,E_2)E_1 = \left[\left(\frac{1}{z}\right)^3 - \frac{2z}{x^2}\right]E_3\,.$$

This implies that the manifold under consideration is not locally ϕ-symmetric. Let us now consider the components of the 1-form A as follows:

$$A(E_i) = \begin{cases} -\dfrac{4z^5}{x(2z^4 - x^2)} & \text{for } i = 1\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

at any point $x \in M$. In our M^3, (3.1) reduces with the 1-form to the following equations:

$$\phi^2((\nabla_{E_i} R)(E_j, E_k)E_l = A(E_i)R(E_j, E_k)E_l\,,$$

for $i, j, k, l = 1, 2$. Hence the manifold under consideration satisfies the following relation

$$\phi^2((\nabla_W R)(X, Y)Z) = A(W)R(X, Y)Z$$

for all vector fields X, Y, Z, W orthogonal to ξ, that is, for any horizontal vector field X, Y, Z, W. Thus the manifold (M^3, g) under consideration is neither locally symmetric nor locally ϕ-symmetric but locally ϕ-recurrent $(LCS)_3$-manifold. Hence we can state the following:

Theorem 3.5. *There exists a locally ϕ-recurrent $(LCS)_3$-manifold which is neither locally symmetric nor locally ϕ-symmetric.*

Example 3.2. We consider the 4-dimensional manifold

$$M = \left\{(x, y, z, u) \in \mathbb{R}^4 \,:\, u \neq 0,\ x \neq \pm 1, \pm\sqrt{2}\right\},$$

where (x, y, z, u) are the standard coordinates in $\mathbb{R}^4$. Let $\{E_1, E_2, E_3, E_4\}$ be linearly independent global frame on M given by

$$E_1 = -\frac{1}{u}\frac{\partial}{\partial x}\,, \quad E_2 = -\frac{x}{u}\left(\frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right), \quad E_3 = -\frac{1}{u}\frac{\partial}{\partial z}\,, \quad E_4 = \frac{\partial}{\partial u}\,.$$

Let g be the Lorentzian metric defined by

$$\begin{aligned} &g(E_1, E_1) = g(E_2, E_2) = g(E_3, E_3) = 1\,, \\ &g(E_4, E_4) = -1\,, \\ &g(E_i, E_j) = 0 \qquad \text{for } i \neq j\,. \end{aligned}$$

Let η be the 1-form defined by $\eta(U) = g(U, E_4)$ for any $U \in \chi(M)$. Let ϕ be the (1,1) tensor field defined by $\phi E_1 = E_1$, $\phi E_2 = E_2$, $\phi E_3 = E_3$ and $\phi E_4 = 0$. Then using the linearity of ϕ and g we have

$$\begin{aligned} \eta(E_4) &= -1\,, \\ \phi^2 U &= U + \eta(U)E_4\,, \\ g(\phi U, \phi W) &= g(U,W) + \eta(U)\eta(W) \end{aligned}$$

for any $U, W \in \chi(M)$. Let ∇ be the Levi-Civita connection with respect to the Lorentzian metric g and R be the curvature tensor of g. Then we have

$$\begin{aligned} [E_1, E_4] &= -\frac{1}{u}\,E_1\,, \qquad & [E_1, E_2] &= -\frac{1}{xu}\,E_2\,, \\ [E_2, E_4] &= -\frac{1}{u}\,E_2\,, \qquad & [E_3, E_4] &= -\frac{1}{u}\,E_3\,. \end{aligned}$$

Taking $E_4 = \xi$ and using Koszul formula for the Lorentzian metric g, we can easily calculate

$$\begin{aligned} \nabla_{E_1} E_4 &= -\frac{1}{u}\,E_1\,, \qquad & \nabla_{E_2} E_4 &= -\frac{1}{u}\,E_2\,, \qquad & \nabla_{E_3} E_4 &= -\frac{1}{u}\,E_3\,, \\ \nabla_{E_1} E_1 &= -\frac{1}{u}\,E_4\,, \qquad & \nabla_{E_2} E_1 &= \frac{1}{xu}\,E_2\,, \qquad & \nabla_{E_3} E_3 &= -\frac{1}{u}\,E_4\,, \\ \nabla_{E_2} E_2 &= -\frac{1}{u}\,E_4 - \frac{1}{ux}\,E_1\,. \end{aligned}$$

From the above it can be easily seen that $E_4 = \xi$ is a unit timelike concircular vector field and hence (ϕ, ξ, η, g) is a $(LCS)_4$ structure on M. Consequently $M^4(\phi, \xi, \eta, g)$ is a $(LCS)_4$-manifold with $\alpha = -\frac{1}{u} \neq 0$ and $\rho = -\frac{1}{u^2}$. Using the above relations, we can easily calculate the non-vanishing components of the curvature tensor as follows:

$$\begin{aligned} R(E_2, E_3)E_2 &= -\frac{1}{u^2}\,E_3\,, \qquad & R(E_2, E_3)E_3 &= \frac{1}{u^2}\,E_2\,, \\ R(E_1, E_3)E_1 &= -\frac{1}{u^2}\,E_3\,, \qquad & R(E_1, E_3)E_3 &= \frac{1}{u^2}\,E_1\,, \\ R(E_3, E_4)E_4 &= -\frac{2}{u^2}\,E_3\,, \qquad & R(E_3, E_4)E_3 &= -\frac{2}{u^2}\,E_4\,, \end{aligned}$$

$$R(E_1,E_4)E_1 = -\frac{2}{u^2}\,E_4\,, \qquad R(E_1,E_4)E_4 = -\frac{2}{u^2}\,E_1\,,$$
$$R(E_2,E_4)E_2 = -\frac{2}{u^2}\,E_4\,, \qquad R(E_2,E_4)E_4 = -\frac{2}{u^2}\,E_2\,,$$
$$R(E_1,E_2)E_2 = \frac{1}{u^2}\left(1-\frac{2}{x^2}\right)E_1\,, \qquad R(E_1,E_2)E_1 = -\frac{1}{u^2}\left(1-\frac{2}{x^2}\right)E_2\,,$$

and the components which can be obtained from these by the symmetry properties. From the above, we can easily calculate the scalar curvature r as follows:

$$r = \sum_{i=1}^{4} \epsilon_i S(E_i,E_i) = \frac{2}{u^2}\left(1-\frac{2}{x^2}\right) \neq 0\,,$$

where $\epsilon_i = g(E_i,E_i)$. Consequently, $\mathrm{d}r(E_1) = -\frac{8}{(ux)^3} \neq 0$, but $\mathrm{d}r(E_2) = 0$, $\mathrm{d}r(E_3) = 0$. We shall now show that the manifold (M^4,g) is locally ϕ-recurrent $(LCS)_4$-manifold. To verify this we calculate the covariant derivatives of the required non-zero components of the curvature tensor as follows:

$$(\nabla_{E_1}R)(E_1,E_2)E_1 = \frac{4}{(ux)^3}\,E_2\,,$$
$$(\nabla_{E_1}R)(E_1,E_2)E_2 = -\frac{4}{(ux)^3}\,E_1 + \left(\frac{1}{u^3}+\frac{1}{x^2u^3}\right)E_4\,,$$
$$(\nabla_{E_2}R)(E_1,E_2)E_1 = -\frac{2}{x^2u^3}\,E_4\,.$$

This implies that the manifold under consideration is not locally ϕ-symmetric. Let us now consider the components of the 1-form as follows:

$$A(E_i) = \begin{cases} -\dfrac{2}{ux(x^2-1)} & \text{for } i=1\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

at any point $x \in M$. In our M^3, (3.1) reduces with the 1-form to the following equations:

$$\phi^2((\nabla_{E_i}R)(E_j,E_k)E_l) = A(E_i)R(E_j,E_k)E_l\,,$$

for $i,j,k,l = 1,2$. Hence the manifold satisfies the following relation

$$\phi^2((\nabla_W R)(X,Y)Z) = A(W)R(X,Y)Z$$

for all vector fields X,Y,Z,W orthogonal to ξ. Thus the manifold (M^4,g) under consideration is neither locally symmetric nor locally ϕ-symmetric but locally ϕ-recurrent $(LCS)_4$-manifold. Hence we can state the following:

THEOREM 3.6. *There exists a locally ϕ-recurrent $(LCS)_4$-manifold which is neither locally symmetric nor locally ϕ-symmetric.*

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