

The $SL(2, \mathbb{C})$ Character Variety of a Class of Torus Knots

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Abstract: Let $K_{\frac{m}{2}}$ be the torus knot of type $(m, 2)$. It is well-known that the fundamental group of $S^3 \setminus K_{\frac{m}{2}}$ is $G = \langle A, B \mid A^m = B^2 \rangle$. In this paper we obtain a defining polynomial of the character variety $X(G)$ which allows us to give an easy geometrical description of it.

Key words: Torus knot, character variety

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1. THE CHARACTER VARIETY OF A FINITELY PRESENTED GROUP

Let G be a group, a representation $\rho : G \rightarrow SL(2, \mathbb{C})$ is just a group homomorphism. We say that two representations ρ and ρ' are equivalent if there exists $P \in SL(2, \mathbb{C})$ such that $\rho'(g) = P^{-1}\rho(g)P$ for every $g \in G$. A representation ρ is *reducible* if the elements of $\rho(G)$ all share a common eigenvector, otherwise we say ρ is *irreducible*.

Now, let us consider a finitely presented group $G = \langle x_1, \dots, x_k \mid r_1, \dots, r_s \rangle$ and let $\rho : G \rightarrow SL(2, \mathbb{C})$ be a representation. It is clear that ρ is completely determined by the k -tuple $(\rho(x_1), \dots, \rho(x_k))$ and thus we can identify

$$R(G) = \{(\rho(x_1), \dots, \rho(x_k)) \mid \rho \text{ is a representation of } G\} \subseteq \mathbb{C}^{4k}$$

with the set of all representations of G into $SL(2, \mathbb{C})$, which is therefore (see [1]) a well-defined affine algebraic set, up to canonical isomorphism.

Recall that given a representation $\rho : G \rightarrow SL(2, \mathbb{C})$ its character $\chi_\rho : G \rightarrow \mathbb{C}$ is defined by $\chi_\rho(g) = \text{tr } \rho(g)$. Note that two equivalent representations ρ and ρ' have the same character, and the converse is also true if ρ or ρ' is

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irreducible [1, Prop. 1.5.2]. Now choose any $g \in G$ and define $t_g : R(G) \rightarrow \mathbb{C}$ by $t_g(\rho) = \chi_\rho(g)$. Let T denote the ring generated by $\{t_g \mid g \in G\}$, then ([1, Prop. 1.4.1]) T is finitely generated ring and, moreover, it can be shown using the well-known identities

$$\begin{aligned} \operatorname{tr} A &= \operatorname{tr} A^{-1} \\ \operatorname{tr} AB &= \operatorname{tr} BA \\ \operatorname{tr} AB &= \operatorname{tr} A \operatorname{tr} B - \operatorname{tr} AB^{-1} \end{aligned}$$

which hold in $SL(2, \mathbb{C})$ (see [2, Cor. 4.1.2]) that T is generated by the set:

$$\{t_{x_i}, t_{x_i x_j}, t_{x_i x_j x_h} \mid 1 \leq i < j < h \leq k\}.$$

Now choose $\gamma_1, \dots, \gamma_\nu \in G$ such that $T = \langle t_{\gamma_i} \mid 1 \leq i \leq \nu \rangle$ and define the map $t : R(G) \rightarrow \mathbb{C}^\nu$ by $t(\rho) = (t_{\gamma_1}(\rho), \dots, t_{\gamma_\nu}(\rho))$. Observe that $\nu \leq \frac{k(k^2+5)}{6}$. Put $X(G) = t(R(G))$, then $X(G)$ is an algebraic variety which is well defined up to canonical isomorphism [1, Cor. 1.4.5] and is called the *character variety* of the group G in $SL(2, \mathbb{C})$. Note that $X(G)$ can be identified with the set of all characters χ_ρ of representations $\rho \in R(G)$.

For every $1 \leq j \leq k$ and for every $1 \leq i \leq s$ we have that $p_{ij} = t_{r_i x_j} - t_{x_j}$ is a polynomial with rational coefficients in the variables $\{t_{x_{i_1} \dots x_{i_m}} \mid m \leq 3\}$, (see [2, Cor. 4.1.2]). Then, we have the following explicit description of $X(G)$.

THEOREM 1.1. ([2, Theor. 3.2]) $X(G) = \{\bar{x} \in X(F_k) \mid p_{ij}(\bar{x}) = 0, \forall i, j\}$, where F_k is the free group in k generators.

2. TORUS KNOTS

Recall that \mathbb{R}^2 is the universal covering of the torus T^2 . We define the action $\Phi : (\mathbb{Z} \times \mathbb{Z}) \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by $\Phi((m, n), (x, y)) = (x + m, y + n)$, this action induces an isomorphism $\mathbb{R}^2 / (\mathbb{Z} \times \mathbb{Z}) \cong T^2$ that we shall denote by ϕ . If we now consider the family $\{r_p : y = px \mid p \in \mathbb{R}\}$ of straight lines passing through the origin, it is easily seen that if p is irrational then $\phi(r_p)$ is dense in T^2 and if $p = m/n$ with $\operatorname{g.c.d.}(m, n) = 1$, then $\phi(r_p) \subseteq T^2 \subseteq \mathbb{R}^3$ is a knot. We denote this knot by $K_{\frac{m}{n}}$ and call it the torus knot of type (m, n) (see [5, Chapter 3] for further considerations).

If we denote, as usual, by $G(K)$ the fundamental group of the exterior of any knot K it is well-known that

$$G(K_{\frac{m}{n}}) \cong \langle A, B \mid A^m = B^n \rangle.$$

Now, if $m \geq 1$ is an odd integer, let us define the following group:

$$H_m = \langle x, y \mid \underbrace{xyxy \dots yx}_{\text{length } m} = \overbrace{yxyx \dots xy}^{\text{length } m} \rangle.$$

Where length m means that there are m letters counting y 's and x 's together. Note that since m is odd, the word starts and ends with the same letter. Now, the following isomorphism will be useful in the sequel.

LEMMA 2.1. *Let $m \geq 1$ be an odd integer. Then $G(K_{\frac{m}{2}}) \cong H_m$.*

Proof. We define $\varphi : H_m \rightarrow G(K_{\frac{m}{2}})$ given by $\varphi(x) = B^{-1}A^{\frac{m+1}{2}}$, $\varphi(y) = A^{-\frac{m-1}{2}}B$ and $\psi : G(K_{\frac{m}{2}}) \rightarrow H_m$ given by $\psi(A) = yx$, $\psi(B) = \overbrace{yxyx \dots y}^{\text{length } m}$. The result follows from some easy computations. ■

3. SOME FAMILIES OF POLYNOMIALS

We will start this section by defining recursively the following family of polynomials:

$$\begin{aligned} q_1(T) &= T - 2, \\ q_2(T) &= T + 2, \\ \prod_{1 \neq d|n} q_d \left(X + \frac{1}{X} \right) &= \frac{X^{n-1} + X^{n-2} + \dots + X + 1}{X^{\frac{n-1}{2}}} \quad \text{if } n \text{ is odd,} \\ \prod_{1, 2 \neq d|n} q_d \left(X + \frac{1}{X} \right) &= \frac{X^{n-2} + X^{n-4} + \dots + X^2 + 1}{X^{\frac{n-2}{2}}} \quad \text{if } n \text{ is even.} \end{aligned}$$

Remark 1. If we recall the recursive definition of the cyclotomic polynomials (see [3, Chapter 5]) by

$$\prod_{d|n} g_d(T) = T^n - 1,$$

then it is easily seen that for $n > 1$

$$g_n(T) = T^{\frac{\varphi(n)}{2}} q_n \left(T + \frac{1}{T} \right)$$

where φ is the Euler function.

Now we introduce another family of polynomials:

$$\begin{aligned} p_1(X) &= X, \\ p_2(X) &= X^2 - 2, \\ p_n(X) &= Xp_{n-1}(X) - p_{n-2}(X), \quad \forall n \geq 3. \end{aligned}$$

Remark 2. Let G be a group and $\rho : G \longrightarrow SL(2, \mathbb{C})$ a representation. Then $p_n(\text{tr}\rho(x)) = \text{tr}\rho(x^n)$ for every $n \geq 1$. For the sake of completeness we will set, where necessary, $p_0(X) = 1$.

We have the following relationship between the families we have just defined:

PROPOSITION 3.1. $p_n(X) - 2 = q_1(X) \prod_{1 \neq d|n} q_d^2(X)$ if n is odd, and $p_n(X) - 2 = q_1(X)q_2(X) \prod_{1,2 \neq d|n} q_d^2(X)$ if n is even.

Proof. We will just show the odd case, the even case being completely analogous.

Consider the cyclic group $G = \langle x \rangle$ and a representation $\rho : G \longrightarrow SL(2, \mathbb{C})$. We can suppose, conjugating if necessary, that

$$\rho(x) = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}.$$

In such case it must be

$$\rho(x^n) = \rho(x)^n = \begin{pmatrix} a^n & c \\ 0 & a^{-n} \end{pmatrix}.$$

Set $X = \text{tr}(\rho(x)) = a + a^{-1}$, then

$$\begin{aligned} p_n(X) - 2 &= \text{tr}(\rho(x^n)) - 2 = a^n + a^{-n} - 2 = \frac{(a^n - 1)^2}{a^n} \\ &= \frac{1}{a^n} \left(\prod_{d|n} g_d(a) \right)^2 = \frac{(a-1)^2}{a^n} \left(\prod_{1 \neq d|n} g_d(a) \right)^2 \\ &= \frac{(a + a^{-1} - 2)a}{a^n} \left(\prod_{1 \neq d|n} a^{\frac{\varphi(d)}{2}} q_d(a + a^{-1}) \right)^2 \\ &= q_1(X) \prod_{1 \neq d|n} q_d^2(X). \end{aligned}$$

where the identity $\sum_{d|n} \varphi(d) = n$ was used. ■

Remark 3. The roots of $p_n(X) - 2$ are precisely the possible values of $\text{tr}(\rho(x))$ if $\rho : G \rightarrow SL(2, \mathbb{C})$ is a representation and $x^n = 1$.

Let R be any ring and consider a polynomial $g(T) = \sum_{i=0}^n a_i T^i \in R[T]$. We define

$$* : R[T] \rightarrow R[T] \quad \text{by} \quad g^*(T) = \sum_{i=0}^n (-1)^{n-i} a_i T^i.$$

In the next lemma we show some useful properties of this application.

LEMMA 3.2. *Given $g, h \in R[T]$ we have:*

- a) $g^{**} = g$.
- b) $(gh)^* = g^*h^*$.
- c) *If $g(T) = \sum_{i=0}^n a_i T^i$, then $g^* = g$ if and only if $a_i = 0$ for every i such that $(n - i) \equiv 1 \pmod{2}$.*

Proof. a) and b) follow from the identity $g^*(T) = (-1)^{\deg(g)} g(-T)$. c) is straightforward. ■

We can use the involution just defined to show another relation between our two families of polynomials.

PROPOSITION 3.3. *If $s \geq 1$ is an integer, then*

$$\sum_{i=0}^s (-1)^i p_{s-i}(Z) = \prod_{1 \neq d|2s+1} q_d^*(Z).$$

Proof. We observe that the degree of every term in $p_s(Z)$ has the same parity as $s = \deg p_s(Z)$. This fact together with the definition of $*$ shows that

$$\left(\sum_{i=0}^s (-1)^i p_{s-i}(Z) \right)^* = \sum_{i=0}^s p_i(Z).$$

Now, we claim that

$$\sum_{i=0}^s p_i(Z) = \prod_{1 \neq d|2s+1} q_d(Z).$$

We will prove this by induction on s , the case $s = 1$ being trivial since $p_0(Z) + p_1(Z) = 1 + Z = q_3(Z)$. Now let $s > 1$ be an odd integer (the even case is similar), by hypothesis we have

$$\sum_{i=0}^s p_i(Z) = \sum_{i=0}^{s-1} p_i(Z) + p_s(Z) = \prod_{1 \neq d|2s-1} q_d(Z) + p_s(Z)$$

and thus, setting $Z = X + \frac{1}{X}$ one obtains:

$$\begin{aligned} \sum_{i=0}^s p_i\left(X + \frac{1}{X}\right) &= \prod_{1 \neq d|2s-1} q_d\left(X + \frac{1}{X}\right) + p_s\left(X + \frac{1}{X}\right) \\ &= \frac{\sum_{i=0}^{2s-2} X^i}{X^{s-1}} + q_1\left(X + \frac{1}{X}\right) \prod_{1 \neq d|s} q_s^2\left(X + \frac{1}{X}\right) + 2 \\ &= \frac{\sum_{i=0}^{2s-2} X^i}{X^{s-1}} + \frac{(X-1)^2 \left(\sum_{i=0}^{s-1} X^i\right)^2}{X X^{s-1}} + 2 \\ &= \frac{\sum_{i=0}^{2s-2} X^i}{X^{s-1}} + \frac{X^{2s} + 1}{X^s} = \frac{\sum_{i=0}^{2s} X^i}{X^s} = \prod_{1 \neq d|2s+1} q_d\left(X + \frac{1}{X}\right). \end{aligned}$$

The proof is now completed by applying 3.2 a), b). ■

4. THE $SL(2, \mathbb{C})$ CHARACTER VARIETY OF THE KNOTS $K_{\frac{m}{2}}$

The aim of this section is to give a generating family of polynomials for $X(G)$ with $G = G(K_{\frac{m}{2}})$ ($m > 1$, odd) as well as a geometric description of this variety. Since we know that $G(K_{\frac{m}{2}}) \cong H_m$ we will work with $X(H_m)$ instead.

Before going into our main result we have to introduce another polynomial. We set $h(X, Z) = X^2 - Z$ and $k(X) = X^2 - 2$. Now we define

$$\alpha_l(X, Z) = \begin{cases} h(X, Z) & \text{if } l \text{ is even.} \\ k(X) & \text{if } l \text{ is odd.} \end{cases}$$

and finally we write for $s \geq 1$

$$f_s(X, Z) = p_s(Z)(h(X, Z) - 1) + \sum_{i=1}^s (-1)^i p_{s-i}(Z) \alpha_i(X, Z).$$

With these definitions we can prove the following result.

PROPOSITION 4.1. *If $m > 1$ is an odd integer, then*

$$X(H_m) = \{(X, Z) \in \mathbb{C}^2 \mid f_{\frac{m-1}{2}}(X, Z) = 0\}.$$

Proof. We set $w = \underbrace{xyxy \dots yx}_{\text{length } m} \underbrace{y^{-1}x^{-1}y^{-1}x^{-1} \dots y^{-1}}_{\text{length } m}$. Then, using Theorem 3.2 in [2], we have

$$X(H_m) = \{(X, Y, Z) \in \mathbb{C}^3 \mid p_0(X, Y, Z) = p_1(X, Y, Z) = p_2(X, Y, Z) = 0\}$$

where

$$\begin{aligned} X &= \tau_x, & p_0(X, Y, Z) &= \tau_w - \tau_1 \\ Y &= \tau_y, & p_1(X, Y, Z) &= \tau_{wx} - \tau_x \\ Z &= \tau_{xy}, & p_2(X, Y, Z) &= \tau_{wy} - \tau_y \end{aligned}$$

Now, $wy = \underbrace{xyx \dots y}_{\text{length } m-1} \underbrace{x(xy \dots y)^{-1}}_{\text{length } m-1}$ so we have $\tau_{wy} = \tau_x$ obtaining that $p_2(X, Y, Z) = X - Y$.

On the other hand $\tau_{wx} = \tau_w \tau_x - \tau_{wx^{-1}}$ and $wx^{-1} = \underbrace{xy \dots x}_{\text{length } m} y^{-1} \underbrace{(xy \dots x)^{-1}}_{\text{length } m}$

so we get $\tau_{wx^{-1}} = \tau_{y^{-1}} = \tau_y$ and thus

$$\begin{aligned} p_1(X, Y, Z) &= \tau_{wx} - \tau_x = \tau_w \tau_x - \tau_y - \tau_x = \tau_x(\tau_w - 1) - \tau_y \\ &= X p_0(X, Y, Z) + X - Y. \end{aligned}$$

Set now $w_1 = (xy)^{\frac{m-1}{2}}$ and $w_2 = (yx)^{\frac{m-1}{2}} yx^{-1}$. Since $w = 1$ if and only if $w_1 = w_2$, then it is easy to see that $p_0(X, Y, Z) = \tau_w - \tau_1$ vanishes if and only if $f(X, Y, Z) = \tau_{w_2} - \tau_{w_1}$ does. As a consequence

$$\begin{aligned} X(H_m) &= \{(X, Y, Z) \in \mathbb{C}^3 \mid f(X, Y, Z) = 0 = X - Y\} \\ &\cong \{(X, Z) \in \mathbb{C}^2 \mid f(X, X, Z) = 0\} \end{aligned}$$

Let us compute now the polynomial $f(X, Y, Z)$.

Firstly it is obvious by definition that $\tau_{w_1} = p_{\frac{m-1}{2}}(Z)$. In addition we have

$$\tau_{w_2} = \tau_{(yx)\frac{m-1}{2}}\tau_{yx^{-1}} - \tau_{(xy)\frac{m-3}{2}xx} = p_{\frac{m-1}{2}}(Z)(XY - Z) - \tau_{(xy)\frac{m-3}{2}xx}.$$

Thus, we have that

$$f(X, Y, Z) = \tau_{w_2} - \tau_{w_1} = p_{\frac{m-1}{2}}(Z)(XY - Z - 1) - \tau_{(xy)\frac{m-3}{2}xx}.$$

Now, we claim that if $X = Y$ then

$$\tau_{(xy)\frac{m-3}{2}xx} = - \sum_{i=1}^{\frac{m-1}{2}} (-1)^i p_{\frac{m-1}{2}-i}(Z) \alpha_i(X, Z).$$

We will proceed by induction on m , the cases $m = 3, 5$ being an easy verification. If $n \geq 7$, then some straightforward computations and the use of the recursive definition of the family $\{p_n\}$ gives

$$\begin{aligned} \tau_{(xy)\frac{m-3}{2}xx} &= \tau_{xy}\tau_{(xy)\frac{m-5}{2}xx} - \tau_{(xy)\frac{m-7}{2}xx} \\ &= -Z \sum_{i=1}^{\frac{m-3}{2}} (-1)^i p_{\frac{m-3}{2}-i}(Z) \alpha_i(X, Z) + \sum_{i=1}^{\frac{m-5}{2}} (-1)^i p_{\frac{m-5}{2}-i}(Z) \alpha_i(X, Z) \\ &= - \sum_{i=1}^{\frac{m-7}{2}} (-1)^i [Z p_{\frac{m-3}{2}-i}(Z) - p_{\frac{m-3}{2}-i-1}(Z)] \alpha_i(X, Z) \\ &\quad - (-1)^{\frac{m-5}{2}} Z p_1(Z) \alpha_{\frac{m-5}{2}}(X, Z) - (-1)^{\frac{m-3}{2}} Z p_0(Z) \alpha_{\frac{m-3}{2}}(X, Z) \\ &\quad + (-1)^{\frac{m-5}{2}} p_0(Z) \alpha_{\frac{m-5}{2}}(X, Z) \\ &= - \sum_{i=1}^{\frac{m-1}{2}} (-1)^i p_{\frac{m-1}{2}-i}(Z) \alpha_i(X, Z). \end{aligned}$$

Consequently, and recalling the definition of $f_s(X, Z)$ we get that $f(X, X, Z) = f_{\frac{m-1}{2}}(X, Z)$ and the proof is complete. ■

In order to obtain a geometrical description of $X(H_m)$ we are interested in factorizing the polynomial $f_s(X, Z)$. We will start by rewriting it in a different way:

$$f_s(X, Z) = (X^2 - Z - 2) \left(\sum_{i=0}^s (-1)^i p_{s-i}(Z) \right) + p_s(Z) + \sum_{i=1}^s (-1)^i \beta_i(Z) p_{s-i}(Z),$$

where $\beta_k(Z) = \begin{cases} Z & \text{if } k \text{ is odd.} \\ 2 & \text{if } k \text{ is even.} \end{cases}$

LEMMA 4.2. $p_s(Z) + \sum_{i=1}^s (-1)^i \beta_i(Z) p_{s-i}(Z) = 0$

Proof. It is enough to use the fact that $p_s(Z) - Zp_{s-1}(Z) = -p_{s-2}(Z)$. ■

COROLLARY 4.3. *If $m > 1$ is an odd integer, then*

$$X(H_m) \cong \{(X, Z) \in \mathbb{C}^2 \mid (X^2 - Z - 2) \prod_{1 \neq d \mid m} q_d^*(Z) = 0\}.$$

Proof. Just apply Proposition 3.3 and Lemma 4.2 to Proposition 4.1. ■

Now, we will find the roots of $q_d(Z)$. This is done in the following lemma.

LEMMA 4.4. *Let $\{a_1, \bar{a}_1, \dots, a_{\frac{\varphi(r)}{2}}, \bar{a}_{\frac{\varphi(r)}{2}}\}$ be set of the $\varphi(r)$ primitive r th roots of unity. Then*

$$q_r(Z) = \prod_{i=1}^{\frac{\varphi(r)}{2}} (Z - 2\operatorname{Re}(a_i))$$

Proof. Recall that, for $r > 2$ we have $g_r(X) = X^{\frac{\varphi(r)}{2}} q_r(X + 1/X)$ with $g_r(X)$ being the r th cyclotomic polynomial. As for all $1 \leq j \leq \frac{\varphi(r)}{2}$ it holds that $\frac{1}{a_j} = \bar{a}_j$ we obtain that $q_r(Z)$ has exactly $\frac{\varphi(r)}{2}$ different roots, namely $\{2\operatorname{Re}(a_1), \dots, 2\operatorname{Re}(a_{\frac{\varphi(r)}{2}})\}$. This together with the fact that the degree of $q_r(Z)$ is $\frac{\varphi(r)}{2}$ completes the proof. ■

This lemma allows us to go one step further in our description of the curve $X(H_m)$.

COROLLARY 4.5. *Let $m > 1$ be an odd integer. In the complex plane (X, Z) the curve $X(H_m)$ consists of the parabola $Z = X^2 - 2$ and the union of $\frac{m-1}{2}$ horizontal lines of the form $Z = -2\operatorname{Re}(w)$, being $1 \neq w$ an m th root of unity.*

Proof. It is enough to apply the previous lemma together with the fact that given a polynomial g , then a number a is a root of g if and only if $-a$ is a root of g^* . ■

Remark 4. Recall that the genus of the torus knot $K_{\frac{m}{n}}$ is $\frac{(m-1)(n-1)}{2}$. In our case, where $n = 2$ the genus of $K_{\frac{m}{2}}$ is $\frac{m-1}{2}$, which precisely coincides with the number of straight lines in $X(H_{\frac{m}{2}})$.

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