On Three-Dimensional Trans-Sasakian Manifolds

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Abstract: The object of the present paper is to study 3-dimensional trans-Sasakian manifolds which are locally \( \phi \)-symmetric and have \( \eta \)-parallel Ricci tensor. Also 3-dimensional trans-Sasakian manifolds of constant curvature have been considered. An example of a three-dimensional locally \( \phi \)-symmetric trans-Sasakian manifold is given.

Key words: trans-Sasakian manifold, scalar curvature, locally \( \phi \)-symmetric, \( \eta \)-parallel Ricci tensor, constant curvature.

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1. Introduction

Trans-Sasakian manifolds arose in a natural way from the classification of almost contact metric structures by D. Chinea and C. Gonzales [3], and they appear as a natural generalization of both Sasakian and Kenmotsu manifolds. Again in the Gray-Hervella classification of almost Hermite manifolds [7], there appears a class \( W_4 \) of Hermitian manifolds which are closely related to locally conformally Kähler manifolds. An almost contact metric structure on a manifold \( M \) is called a trans-Sasakian structure [13] if the product manifold \( M \times \mathbb{R} \) belongs to the class \( W_4 \). The class \( C_6 \oplus C_5 \) ([10], [11]) coincides with the class of trans-Sasakian structures of type \( (\alpha, \beta) \). In [11], the local nature of the two subclasses \( C_5 \) and \( C_6 \) of trans-Sasakian structures is characterized completely. In [4], some curvature identities and sectional curvatures for \( C_5 \), \( C_6 \) and trans-Sasakian manifolds are obtained. It is known that ([8]) trans-Sasakian structures of type \( (0,0) \), \( (0,\beta) \) and \( (\alpha,0) \) are cosymplectic, \( \beta \)-Kenmotsu and \( \alpha \)-Sasakian respectively. In [15], it is proved that trans-Sasakian structures are generalized quasi-Sasakian structures [12]. Thus, trans-Sasakian structures also provide a large class of generalized quasi-Sasakian structures.

The local structure of trans-Sasakian manifolds of dimension \( n \geq 5 \) has been completely characterized by J. C. Marrero [10]. He proved that a trans-
Sasakian manifold of dimension \( n \geq 5 \) is either cosymplectic or \( \alpha \)-Sasakian or \( \beta \)-Kenmotsu manifold. But so far, it is not too much known about the 3-dimensional case.

This paper deals just on 3-dimensional connected trans-Sasakian manifolds. In Section 2 some preliminary results are recalled and explicit formulae for Ricci tensor and curvature tensor [6] of 3-dimensional trans Sasakian manifolds are given. In Section 3 we characterize 3-dimensional locally \( \phi \)-symmetric trans-Sasakian manifolds and prove that a 3-dimensional connected trans-Sasakian manifold of type \((\alpha, \beta)\) is locally \( \phi \)-symmetric if and only if the scalar curvature of the manifold is constant where \( \alpha \) and \( \beta \) are constants. This result is an extension of an analogous result concerning Kenmotsu manifolds obtained by the first author [5]. Section 4 of our paper deals with a 3-dimensional trans-Sasakian manifold with \( \eta \)-parallel Ricci tensor. In this section we also show that a 3-dimensional connected trans-Sasakian manifold of type \((\alpha, \beta)\) has \( \eta \)-parallel Ricci tensor if and only if the scalar curvature of the manifold is constant where \( \alpha \) and \( \beta \) are constants. In Section 5, we show that a 3-dimensional compact connected trans-Sasakian manifold of constant curvature is either \( \alpha \)-Sasakian or \( \beta \)-Kenmotsu. This is the most important result obtained in this paper. Finally in the last section we construct an example of a three-dimensional locally \( \phi \)-symmetric trans-Sasakian manifold.

2. Preliminaries

Let \( M \) be a connected almost contact metric manifold with an almost contact metric structure \((\phi, \xi, \eta, g)\), that is, \( \phi \) is an \((1, 1)\) tensor field, \( \xi \) is a vector field, \( \eta \) is an 1-form and \( g \) is compatible Riemannian metric such that

\[
\phi^2(X) = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta\phi = 0, \tag{2.1}
\]

\[
g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.2}
\]

\[
g(X, \phi Y) = -g(\phi X, Y), \quad g(X, \xi) = \eta(X), \tag{2.3}
\]

for all \( X, Y \in T(M) \) [1]. The fundamental 2-form \( \Phi \) of the manifold is defined by

\[
\Phi(X, Y) = g(X, \phi Y), \tag{2.4}
\]

for \( X, Y \in T(M) \).

An almost contact metric structure \((\phi, \xi, \eta, g)\) on a connected manifold \( M \) is called trans-Sasakian structure [13] if \((M \times R, J, G)\) belongs to the class
W₄ [7], where J is the almost complex structure on M × R defined by

\[ J(X, f d/dt) = (\phi X - f \xi, \eta(X)d/dt), \]

for all vector fields X on M, a smooth function f on M × R and the product metric G on M × R. This may be expressed by the condition [2]

\[ (\nabla_X \phi)Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X), \quad \text{(2.5)} \]

for smooth functions \( \alpha \) and \( \beta \) on M. Here we say that the trans-Sasakian structure is of type \((\alpha, \beta)\). From (2.5) it follows that

\[ \nabla_X \xi = -\alpha \phi X + \beta(X - \eta(X)\xi), \quad \text{(2.6)} \]

\[ (\nabla_X \eta)Y = -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y), \quad \text{(2.7)} \]

An explicit example of 3-dimensional proper trans-Sasakian manifold is constructed in [10]. In [6], the Ricci operator, Ricci tensor and curvature tensor for 3-dimensional trans-Sasakian manifolds are studied and their explicit formulae are given.

From [6] we know that for a 3-dimensional trans-Sasakian manifold

\[ 2\alpha \beta + \xi\alpha = 0, \quad \text{(2.8)} \]

\[ S(X, \xi) = (2(\alpha^2 - \beta^2) - \xi\beta)\eta(X) - X\beta - (\phi X)\alpha, \quad \text{(2.9)} \]

\[ S(X, Y) = \left( r^2 + 2\xi\beta - (2\alpha^2 - \beta^2) \right) g(X, Y) \]

\[ - \left( r^2 + 2\xi\beta - 2(\alpha^2 - \beta^2) \right) \eta(X)\eta(Y) \]

\[ - \left( Y\beta + (\phi Y)\alpha \right) \eta(X) - (X\beta + (\phi X)\alpha) \eta(Y), \quad \text{(2.10)} \]

and

\[ R(X, Y)Z = \left( r^2 + 2\xi\beta - 2(\alpha^2 - \beta^2) \right) (g(Y, Z)X - g(X, Z)Y) \]

\[ - g(Y, Z) \left[ \left( r^2 + 2\xi\beta - 2(\alpha^2 - \beta^2) \right) \eta(X) \xi \right. \]

\[ - \eta(X)(\phi \text{grad} \alpha - \text{grad} \beta) + (X\beta + (\phi X)\alpha) \xi \]

\[ + \left. g(X, Z) \left[ \left( r^2 + 2\xi\beta - 2(\alpha^2 - \beta^2) \right) \eta(Y) \xi \right. \right. \]

\[ - \eta(Y)(\phi \text{grad} \alpha - \text{grad} \beta) + (Y\beta + (\phi Y)\alpha) \xi \left. \right] \quad \text{(2.11)} \]
where \( S \) is the Ricci tensor of type \((0,2)\), \( R \) is the curvature tensor of type \((1,3)\) and \( r \) is the scalar curvature of the manifold \( M \).

3. Locally \( \phi \)-symmetric three-dimensional trans-Sasakian manifolds

**Definition 3.1.** A trans-Sasakian manifold is said to be locally \( \phi \)-symmetric if

\[
\phi^2(\nabla_W R)(X, Y)Z = 0,
\]

for all vector fields \( W, X, Y, Z \) orthogonal to \( \xi \).

This notion was introduced for Sasakian manifolds by Takahashi [14].

Let \( M \) be a 3-dimensional connected trans-Sasakian manifold. Then its curvature tensor is given by (2.11). Differentiating (2.11) we get

\[
(\nabla_W R)(X, Y)Z = \left[ \frac{dr(W)}{2} + 2(\nabla_W(\xi\beta)) - 4(d\alpha(W) - d\beta(W)) \right] \left[ g(Y, Z)X - g(X, Z)Y \right]
- g(Y, Z) \left[ \left( \frac{dr(W)}{2} + (\nabla_W(\xi\beta)) - 6(d\alpha(W) - d\beta(W)) \right) \eta(X)\xi 
- \left( \frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2) \right) ((\nabla_W \eta)(X)\xi + \eta(X)(\nabla_W \xi)) 
- (\nabla_W \eta)(X)(\phi(\text{grad}\,\alpha) - \text{grad}\,\beta) - \eta(X)(\nabla_W (\phi(\text{grad}\,\alpha) - \text{grad}\,\beta)) 
+ (\nabla_W (X\beta + (\phi X)\alpha))\xi + (X\beta + (\phi X)\alpha) \nabla_W \xi \right] 
+ g(X, Z) \left[ \left( \frac{dr(W)}{2} + (\nabla_W(\xi\beta)) - 6(d\alpha(W) - d\beta(W)) \right) \eta(Y)\xi 
+ \left( \frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2) \right) ((\nabla_W \eta)(Y)\xi + \eta(Y)(\nabla_W \xi)) 
- (\nabla_W \eta)(Y)(\phi(\text{grad}\,\alpha) - \text{grad}\,\beta) - \eta(Y)(\nabla_W (\phi(\text{grad}\,\alpha) - \text{grad}\,\beta)) 
+ (\nabla_W (Y\beta + (\phi Y)\alpha))\xi + (Y\beta + (\phi Y)\alpha) \nabla_W \xi \right] \right) \quad (3.1)
\]
\[- \left[ (\nabla_W (Z\beta + (\phi Z)\alpha)) \eta(Y) + (Z\beta + (\phi Z)\alpha) (\nabla_W \eta) Y \right.
\quad + (\nabla_W (Y\beta + (\phi Y)\alpha)) \eta(Z) + (Y\beta + (\phi Y)\alpha) (\nabla_W \eta) Z
\quad + \left( \frac{dr(W)}{2} + (\nabla_W (\xi) - 6 (d\alpha(W) - d\beta(W))) \right) \eta(Y) \eta(Z)
\quad + \left( \frac{r}{2} + \xi - 3(\alpha^2 - \beta^2) \right) ((\nabla_W \eta) Y \eta(Z) + \eta(Y) (\nabla_W \eta) Z) \right] X
\quad + \left[ (\nabla_W (Z\beta + (\phi Z)\alpha)) \eta(X) + (Z\beta + (\phi Z)\alpha) (\nabla_W \eta) X
\quad + (\nabla_W (X\beta + (\phi X)\alpha)) \eta(Z) + (X\beta + (\phi X)\alpha) (\nabla_W \eta) Z
\quad + \left( \frac{dr(W)}{2} + (\nabla_W (\xi) - 6 (d\alpha(W) - d\beta(W))) \right) \eta(X) \eta(Z)
\quad + \left( \frac{r}{2} + \xi - 3(\alpha^2 - \beta^2) \right) ((\nabla_W \eta) X \eta(Z) + \eta(X) (\nabla_W \eta) Z) \right] Y.\]

Suppose that \(\alpha\) and \(\beta\) are constants and \(X, Y, Z, W\) are orthogonal to \(\xi\). Then using \(\phi_\xi = 0\) and (3.1), we get
\[
\phi^2 (\nabla_W R)(X, Y) Z = \left( \frac{dr(W)}{2} \right) (g(Y, Z) X - g(X, Z) Y). \quad (3.2)
\]

Hence from (3.2) we get \(\phi^2 (\nabla_W R)(X, Y) Z = 0\) if and only if the scalar curvature \(r\) is constant. Thus we can state the following:

**Theorem 3.1.** A 3-dimensional connected trans-Sasakian manifold of type \((\alpha, \beta)\) is locally \(\phi\)-symmetric if and only if the scalar curvature is constant provided \(\alpha\) and \(\beta\) are constants.

The above theorem is just an extension of an analogous result concerning Kenmotsu manifolds obtained by the first author in the paper [5].

### 4. \eta-parallel Ricci tensor

**Definition 4.1.** The Ricci tensor \(S\) of a trans-Sasakian manifold is said to be \(\eta\)-parallel if it satisfies
\[
(\nabla_X S)(\phi Y, \phi Z) = 0, \quad (4.1)
\]
for all vector fields \(X, Y\) and \(Z\).
This notion was introduced in the context of Sasakian manifolds by Kon [9].

Let $M$ be a 3-dimensional connected trans-Sasakian manifold. Then its Ricci tensor is given by (2.10)

In (2.10) replacing $X$ by $\phi X$, $Y$ by $\phi Y$ and using (2.1) we get for a trans-Sasakian manifold of dimension three

$$S(\phi X, \phi Y) = \left( \frac{r}{2} + \xi \beta - (\alpha^2 - \beta^2) \right) (g(X, Y) - \eta(X)\eta(Y)).$$ (4.2)

Now we see that

$$(\nabla_Z S)(\phi X, \phi Y) = \nabla_Z S(\phi X, \phi Y) - S(\nabla_Z \phi X, \phi Y) - S(\phi X, \nabla_Z \phi Y)$$

$$= \nabla_Z S(\phi X, \phi Y) - S((\nabla_Z \phi) X, \phi Y) - S(\phi \nabla_Z X, \phi Y) - S(\phi X, (\nabla_Z \phi) Y) - S(\phi X, \phi \nabla_Z Y).$$ (4.3)

Using (2.5), (2.10) and (4.2) in (4.3) we have

$$(\nabla_Z S)(\phi X, \phi Y)$$

$$\quad = \left( \frac{1}{2} dr(Z) + \nabla_Z (\xi \beta) - 2\alpha d\alpha(Z) + 2\beta d\beta(Z) \right) (g(X, Y) - \eta(X)\eta(Y))$$

$$\quad + \left( \frac{r}{2} + \xi \beta - (\alpha^2 - \beta^2) \right) \left( \nabla_Z g(X, Y) - (\nabla_Z \eta(X))\eta(Y) - \eta(X)(\nabla_Z \eta(Y)) \right)$$

$$\quad - S\left( \alpha(g(Z, X)\xi - \eta(X)Z) + \beta(g(\phi Z, X)\xi - \eta(X)\phi Z), \phi Y \right)$$

$$\quad - \left( \frac{r}{2} + \xi \beta - (\alpha^2 - \beta^2) \right) (g(\nabla_Z X, Y) - \eta(\nabla_Z X)\eta(Y))$$

$$\quad - S\left( \phi X, \alpha(g(Z, Y)\xi - \eta(Y)Z) + \beta(g(\phi Z, Y)\xi - \eta(Y)\phi Z) \right)$$

$$\quad - \left( \frac{r}{2} + \xi \beta - (\alpha^2 - \beta^2) \right) (g(X, \nabla_Z Y) - \eta(X)\eta(\nabla_Z Y)).$$ (4.4)

By virtue of (2.9) and (2.10) we obtain from (4.4)

$$(\nabla_Z S)(\phi X, \phi Y)$$

$$\quad = \left( \frac{1}{2} dr(Z) + \nabla_Z (\xi \beta) - 2\alpha d\alpha(Z) + 2\beta d\beta(Z) \right) (g(X, Y) - \eta(X)\eta(Y))$$

$$\quad + \left( \frac{r}{2} + \xi \beta - (\alpha^2 - \beta^2) \right) \left( \nabla_Z g(X, Y) - (\nabla_Z \eta(X))\eta(Y) - \eta(X)(\nabla_Z \eta(Y)) \right)$$

$$\quad + \alpha g(Z, X)((\phi Y)\beta + (\phi^2 Y)\alpha)$$ (4.5)
+ αη(X) \left( \left( \frac{r}{2} + \xi \beta - (\alpha^2 - \beta^2) \right) g(Z, \phi Y) - ((\phi Y)\beta + (\phi^2 Y)\alpha) \eta(Z) \right) \\
+ \beta g(\phi Z, X)((\phi Y)\beta + (\phi^2 Y)\alpha) \\
+ \beta \eta(X) \left( \left( \frac{r}{2} + \xi \beta - (\alpha^2 - \beta^2) \right) (g(Z, Y) - \eta(Z)\eta(Y)) \right) \\
\left( \frac{r}{2} + \xi \beta - (\alpha^2 - \beta^2) \right) (g(\nabla Z X, Y) - \eta(\nabla Z X)\eta(Y)) \\
+ \alpha g(Z, Y)((\phi X)\beta + (\phi^2 X)\alpha) \\
+ \beta g(\phi Z, Y)((\phi X)\beta + (\phi^2 X)\alpha) \\
+ \beta \eta(Y) \left( \left( \frac{r}{2} + \xi \beta - (\alpha^2 - \beta^2) \right) (g(Z, X) - \eta(Z)\eta(X)) \right) \\
\left( \frac{r}{2} + \xi \beta - (\alpha^2 - \beta^2) \right) (g(X, \nabla Z Y) - \eta(X)\eta(\nabla Z Y)).

The above relation can be written as

\[(\nabla Z S)(\phi X, \phi Y) = \frac{1}{2} dr(Z) \left( g(X, Y) - \eta(X)\eta(Y) \right) \]

Suppose that \(\alpha\) and \(\beta\) are constants. Then using (2.7) in (4.6), we obtain

\[(\nabla Z S)(\phi X, \phi Y) = \frac{1}{2} dr(Z) \left( g(X, Y) - \eta(X)\eta(Y) \right). \tag{4.7} \]

Hence, from (4.7) we can state the following:
**Theorem 4.1.** A 3-dimensional connected trans-Sasakian manifold of type \((\alpha, \beta)\) has \(\eta\)-parallel Ricci tensor if and only if the scalar curvature of the manifold is constant provided \(\alpha\) and \(\beta\) are constants.

From Theorem 3.1 and Theorem 4.1 we can state the following:

**Corollary 4.1.** A 3-dimensional connected trans-Sasakian manifold of type \((\alpha, \beta)\) has \(\eta\)-parallel Ricci tensor if and only if it is locally \(\phi\)-symmetric provided \(\alpha\) and \(\beta\) are constants.

5. **Three-dimensional trans-Sasakian manifold with constant curvature**

Let \(M\) be a 3-dimensional compact connected trans-Sasakian manifold. If the manifold is of constant curvature then the Ricci tensor of type \((0, 2)\) of the manifold is given by

\[
S(X, Y) = 2\lambda g(X, Y),
\]

where \(\lambda\) is a constant. Putting \(Y = \xi\) in (5.1) and using (2.9), we get

\[
X\beta + (\phi X)\alpha + [2(\lambda - \alpha^2 + \beta^2) + \xi\beta] \eta(X) = 0.
\]

For \(X = \xi\), (5.2) yields

\[
\xi\beta = -(\lambda - \alpha^2 + \beta^2).
\]

By virtue of (5.2) and (5.3) it follows that

\[
X\beta + (\phi X)\alpha + (\lambda - \alpha^2 + \beta^2) \eta(X) = 0.
\]

The gradient of the function \(\beta\) is related to the exterior derivative \(d\beta\) by the formula

\[
d\beta(X) = g(\text{grad} \beta, X).
\]

Using (5.5) in (5.4) we obtain

\[
d\beta(X) + g(\text{grad} \alpha, \phi X) + (\lambda - \alpha^2 + \beta^2) \eta(X) = 0.
\]

Differentiating (5.6) covariantly with respect to \(Y\) we get

\[
(\nabla_Y d\beta)(X) + g(\nabla_Y \text{grad} \alpha, \phi X) + g(\text{grad} \alpha, (\nabla_Y \phi) X) + Y(\beta^2 - \alpha^2) \eta(X) + (\lambda - \alpha^2 + \beta^2)(\nabla_Y \eta)(X) = 0.
\]
Interchanging $X$ and $Y$ in (5.7), we get

$$
(\nabla_X d\beta)(Y) + g(\nabla_X \text{grada} \phi Y) + g(\text{grada}(\nabla_X \phi)Y) \\
+ X(\beta^2 - \alpha^2)\eta(Y) + (\lambda - \alpha^2 + \beta^2)(\nabla_X \eta)(Y) = 0.
$$

(5.8)

Subtracting (5.7) from (5.8) we get

$$
g(\nabla_X \text{grada} \phi Y) - g(\nabla_Y \text{grada} \phi X) + ((\nabla_X \phi)Y - (\nabla_Y \phi)X)\alpha \\
+ [X(\beta^2 - \alpha^2)\eta(Y) - Y(\beta^2 - \alpha^2)\eta(X)] \\
+ (\lambda - \alpha^2 + \beta^2)((\nabla_X \eta)(Y) - (\nabla_Y \eta)(X)) = 0.
$$

(5.9)

From (2.7) and (2.4) we get

$$
(\nabla_X \eta)(Y) - (\nabla_Y \eta)(X) = \alpha (\Phi(X,Y) - \Phi(Y,X)) = 2\alpha \Phi(X,Y).
$$

(5.10)

Using (5.10) in (5.9) we have

$$
g(\nabla_X \text{grada} \phi Y) - g(\nabla_Y \text{grada} \phi X) + ((\nabla_X \phi)Y - (\nabla_Y \phi)X)\alpha \\
+ [X(\beta^2 - \alpha^2)\eta(Y) - Y(\beta^2 - \alpha^2)\eta(X)] \\
+ 2(\lambda - \alpha^2 + \beta^2)\alpha \Phi(X,Y) = 0.
$$

(5.11)

Let $\{E_0, E_1, E_2\}$ be a local $\phi$-basis, that is, an orthonormal frame such that $E_0 = \xi$ and $E_2 = \phi E_1$. In (2.5) putting $X = E_1$, $Y = E_2$ we get

$$
(\nabla_{E_1} \phi)E_2 = \alpha (g(E_1, E_2)\xi - \eta(E_2)E_1) + \beta (g(E_1, E_2)\xi - \eta(E_2)\phi E_1) \\
= \beta g(\phi E_1, E_2)\xi = \beta \xi.
$$

(5.12)

Similarly,

$$
(\nabla_{E_2} \phi)E_1 = -\beta \xi.
$$

(5.13)

Now,

$$
\Phi(E_1, E_2) = g(E_1, \phi E_2) = g(E_1, \phi^2 E_1) = -1.
$$

(5.14)

In (5.11) putting $X = E_1$ and $Y = E_2$ and using (5.12), (5.13) and (5.14) we obtain

$$
g(\nabla_{E_1} \text{grada} E_1) + g(\nabla_{E_2} \text{grada} E_2) = 2\beta \xi \alpha - 2\alpha(\lambda - \alpha^2 + \beta^2).
$$

(5.15)

Also (2.8) can be written as

$$
g(\text{grada} \xi, \xi) = -2\alpha \beta.
$$

(5.16)
Differentiating (5.16) covariantly with respect to $\xi$ we get
\[ g(\nabla_\xi \text{grad}\alpha, \xi) + g(\text{grad}\alpha, \nabla_\xi \xi) = -2\beta \xi \alpha - 2\alpha \xi \beta. \] 
(5.17)

In view of (5.3) we can write the above relation as
\[ g(\nabla_\xi \text{grad}\alpha, \xi) = -2\beta \xi \alpha + 2\alpha(\lambda - \alpha^2 + \beta^2). \] 
(5.18)

From (5.15) and (5.18) we get $\Delta \alpha = 0$, where $\Delta$ is the Laplacian defined by
\[ \Delta \alpha = \sum_{i=0}^{2} g(\nabla_{E_i} \text{grad}\alpha, E_i). \]

Since $M$ is compact we get $\alpha$ is constant.

Now let us consider the following two cases:

CASE-I: In this case we suppose that $\alpha$ is non-zero constant then by (2.8), $\beta = 0$ every where on $M$.

CASE-II: In this case let $\alpha = 0$. Then from (5.4)
\[ X\beta + (\lambda + \beta^2)\eta(X) = 0, \]
that is,
\[ g(\text{grad}\beta, X) + (\lambda + \beta^2)g(X, \xi) = 0. \]

Therefore,
\[ \text{grad}\beta + (\lambda + \beta^2)\xi = 0. \] 
(5.19)

Differentiating (5.19) covariantly with respect to $X$ we have
\[ \nabla_X \text{grad}\beta + (X\beta^2)\xi + (\lambda + \beta^2)\nabla_X \xi = 0. \]

Using (2.6) we get from above
\[ \nabla_X \text{grad}\beta + (X\beta^2)\xi + (\lambda + \beta^2)(-\alpha \phi X + \beta (X - \eta(X)\xi)) = 0. \]

Now taking inner product of the above equation with $X$, we have
\[ g(\nabla_X \text{grad}\beta, X) = -g((X\beta^2)\xi, X) \]
\[ - (\lambda + \beta^2)(g(-\alpha \phi X, X) + \beta g(X - \eta(X)\xi, X)). \] 
(5.20)
Therefore putting $X = E_i$ and taking summation over $i, i = 0, 1, 2$, we get from above
\[
\triangle \beta = -2\beta(\xi \beta + \lambda + \beta^2).
\] (5.21)
For $\alpha = 0$, (5.3) yields $\xi \beta = -(\lambda + \beta^2)$, which in view of (5.21) gives $\triangle \beta = 0$. Hence $\beta =$ constant, $M$ being compact. This leads to the following:

**Theorem 5.1.** If a 3-dimensional compact connected trans-Sasakian manifold is of constant curvature then it is either $\alpha$-Sasakian or $\beta$-Kenmotsu.

6. Example of a locally $\phi$-symmetric three-dimensional trans-Sasakian manifold

We consider the three-dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3, z \neq 0\}$, where $(x, y, z)$ are the standard coordinates in $\mathbb{R}^3$. The vector fields
\[
e_1 = z \frac{\partial}{\partial x}, \quad e_2 = z \frac{\partial}{\partial y}, \quad e_3 = z \frac{\partial}{\partial z}
\]
are linearly independent at each point of $M$. Let $g$ be the Riemannian metric defined by
\[
g(e_1, e_3) = g(e_2, e_3) = g(e_1, e_2) = 0, \quad g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1.
\]
Let $\eta$ be the 1-form defined by $\eta(Z) = g(Z, e_3)$ for any $Z \in \chi(M)$. Let $\phi$ be the (1,1) tensor field defined by $\phi(e_1) = -e_2, \phi(e_2) = e_1, \phi(e_3) = 0$. Then using the linearity of $\phi$ and $g$ we have
\[
\eta(e_3) = 1, \quad \phi^2 Z = -Z + \eta(Z)e_3, \quad g(\phi Z, \phi W) = g(Z, W) - \eta(Z)\eta(W),
\]
for any $Z, W \in \chi(M)$. Thus for $e_3 = \xi$, $(\phi, \xi, \eta, g)$ defines an almost contact metric structure on $M$. Now, by direct computations we obtain
\[
[e_1, e_2] = 0, \quad [e_2, e_3] = -e_2, \quad [e_1, e_3] = -e_1.
\]
The Riemannian connection $\nabla$ of the metric $g$ is given by the Koszul’s formula which is
\[
2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g([X, Y], Z) - g(Y, [X, Z]) + g(Z, [X, Y]).
\] (6.1)
Using (6.1) we have
\[ 2g(\nabla e_1 e_3, e_1) = 2g(-e_1, e_1), \]
\[ 2g(\nabla e_1 e_3, e_2) = 0 = 2g(-e_1, e_2), \]
\[ 2g(\nabla e_1 e_3, e_3) = 0 = 2g(-e_1, e_3). \]
Hence, \( \nabla e_1 e_3 = -e_1 \). Similarly, \( \nabla e_2 e_3 = -e_2 \) and \( \nabla e_3 e_3 = 0 \).

(6.1) further yields
\[
\begin{align*}
\nabla e_1 e_2 &= 0, & \nabla e_1 e_1 &= e_3, \\
\nabla e_2 e_2 &= e_3, & \nabla e_2 e_1 &= 0, \\
\nabla e_3 e_2 &= 0, & \nabla e_3 e_1 &= 0.
\end{align*}
\]
We see that
\[
\begin{align*}
(\nabla e_1 \phi)e_1 &= \nabla e_1 \phi e_1 - \phi \nabla e_1 e_1 = -\nabla e_1 e_2 - \phi e_3 = -\nabla e_1 e_2 = 0 \\
&= 0(g(e_1, e_1) e_3 - \eta(e_1) e_1) - 1(g(\phi e_1, e_1) e_3 - \eta(e_1) \phi e_1). \\
(\nabla e_1 \phi)e_2 &= \nabla e_1 \phi e_2 - \phi \nabla e_1 e_2 = \nabla e_1 e_1 - 0 = e_3 \\
&= 0(g(e_1, e_2) e_3 - \eta(e_2) e_1) - 1(g(\phi e_1, e_2) e_3 - \eta(e_2) \phi e_1).
\end{align*}
\]
(6.4) for
\[
\begin{align*}
(\nabla e_1 \phi)e_3 &= \nabla e_1 \phi e_3 - \phi \nabla e_1 e_3 = 0 + \phi e_1 = -e_2 \\
&= 0(g(e_1, e_3) e_3 - \eta(e_3) e_1) - 1(g(\phi e_1, e_3) e_3 - \eta(e_3) \phi e_1).
\end{align*}
\]
By (6.2), (6.3) and (6.4) we see that the manifold satisfies (2.5) for \( X = e_1 \), \( \alpha = 0 \), \( \beta = -1 \), and \( e_3 = \xi \). Similarly it can be shown that for \( X = e_2 \) and \( X = e_3 \) the manifold also satisfies (2.5) for \( \alpha = 0 \), \( \beta = -1 \), and \( e_3 = \xi \). Hence the manifold is a trans-Sasakian manifold of type \( (0, -1) \). With the help of the above results it can be verified that
\[
\begin{align*}
R(e_1, e_2) e_3 &= 0, & R(e_2, e_3) e_3 &= -e_2, & R(e_1, e_3) e_3 &= -e_1, \\
R(e_1, e_2) e_2 &= -e_1, & R(e_2, e_3) e_2 &= e_3, & R(e_1, e_3) e_2 &= 0, \\
R(e_1, e_2) e_1 &= e_2, & R(e_2, e_3) e_1 &= 0, & R(e_1, e_3) e_1 &= e_3.
\end{align*}
\]
From which it follows that \( \phi^2(\nabla W R)(X, Y) Z = 0 \). Hence the 3-dimensional trans-Sasakian manifold is locally \( \phi \)-symmetric.

Also from the above expressions of the curvature tensor we obtain the scalar curvature \( r = -3 \). Hence we note that here \( \alpha, \beta \) and \( r \) all are constants. Hence from Theorem 3.1 it follows that the manifold under consideration is locally \( \phi \)-symmetric.
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REFERENCES