EXTRACTA MATHEMATICAE Vol. **23**, Núm. 3, 265 – 277 (2008)

On Three-Dimensional Trans-Sasakian Manifolds

U. C. De, Avijit Sarkar

Department of Mathematics, University of Kalyani, Kalyani 741235, *West Bengal, India, e-mail: uc_de@yahoo.com*

Department of Mathematics, University of Burdwan, Burdwan 713104, *West Bengal, India, e-mail: avjaj@yahoo.co.in*

Presented by Oscar García Prada   Received February 23, 2008

Abstract: The object of the present paper is to study 3-dimensional trans-Sasakian manifolds which are locally ϕ-symmetric and have η-parallel Ricci tensor. Also 3-dimensional trans-Sasakian manifolds of constant curvature have been considered. An example of a three-dimensional locally ϕ-symmetric trans-Sasakian manifold is given.

Key words: trans-Sasakian manifold, scalar curvature, locally ϕ-symmetric, η-parallel Ricci tensor, constant curvature.

AMS *Subject Class.* (2000): 53C25.

1. Introduction

Trans-Sasakian manifolds arose in a natural way from the classification of almost contact metric structures by D. Chinea and C. Gonzales [3], and they appear as a natural generalization of both Sasakian and Kenmotsu manifolds. Again in the Gray-Hervella classification of almost Hermite manifolds [7], there appears a class W_4 of Hermitian manifolds which are closely related to locally conformally Kähler manifolds. An almost contact metric structure on a manifold M is called a trans-Sasakian structure [13] if the product manifold $M \times R$ belongs to the class W_4. The class $C_6 \oplus C_5$ ([10], [11]) coincides with the class of trans-Sasakian structures of type (α, β). In [11], the local nature of the two subclasses C_5 and C_6 of trans-Sasakian structures is characterized completely. In [4], some curvature identities and sectional curvatures for C_5, C_6 and trans-Sasakian manifolds are obtained. It is known that ([8]) trans-Sasakian structures of type (0,0), $(0, \beta)$ and $(\alpha, 0)$ are cosymplectic, β-Kenmotsu and α-Sasakian respectively. In [15], it is proved that trans-Sasakian structures are generalized quasi-Sasakian structures [12]. Thus, trans-Sasakian structures also provide a large class of generalized quasi-Sasakian structures.

The local structure of trans-Sasakian manifolds of dimension $n \geq 5$ has been completely characterized by J. C. Marrero [10]. He proved that a trans-

Sasakian manifold of dimension $n \geq 5$ is either cosymplectic or α-Sasakian or β-Kenmotsu manifold. But so far, it is not too much known about the 3-dimensional case.

This paper deals just on 3-dimensional connected trans-Sasakian manifolds. In Section 2 some preliminary results are recalled and explicit formulae for Ricci tensor and curvature tensor [6] of 3-dimensional trans Sasakian manifolds are given. In Section 3 we characterize 3-dimensional locally ϕ-symmetric trans-Sasakian manifolds and prove that a 3-dimensional connected trans-Sasakian manifold of type (α, β) is locally ϕ-symmetric if and only if the scalar curvature of the manifold is constant where α and β are constants. This result is an extension of an analogous result concerning Kenmotsu manifolds obtained by the first author [5]. Section 4 of our paper deals with a 3-dimensional trans-Sasakian manifold with η-parallel Ricci tensor. In this section we also show that a 3-dimensional connected trans-Sasakian manifold of type (α, β) has η-parallel Ricci tensor if and only if the scalar curvature of the manifold is constant where α and β are constants. In Section 5, we show that a 3-dimensional compact connected trans-Sasakian manifold of constant curvature is either α-Sasakian or β-Kenmotsu. This is the most important result obtained in this paper. Finally in the last section we construct an example of a three-dimensional locally ϕ-symmetric trans-Sasakian manifold.

2. Preliminaries

Let M be a connected almost contact metric manifold with an almost contact metric structure (ϕ, ξ, η, g), that is, ϕ is an $(1,1)$ tensor field, ξ is a vector field, η is an 1-form and g is compatible Riemannian metric such that

$$\phi^2(X) = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta\phi = 0, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

$$g(X, \phi Y) = -g(\phi X, Y), \quad g(X, \xi) = \eta(X), \tag{2.3}$$

for all $X, Y \in T(M)$ [1]. The fundamental 2-form Φ of the manifold is defined by

$$\Phi(X, Y) = g(X, \phi Y), \tag{2.4}$$

for $X, Y \in T(M)$.

An almost contact metric structure (ϕ, ξ, η, g) on a connected manifold M is called trans-Sasakian structure [13] if $(M \times R, J, G)$ belongs to the class

W_4 [7], where J is the almost complex structure on $M \times R$ defined by

$$J(X, fd/df) = (\phi X - f\xi, \eta(X)d/dt),$$

for all vector fields X on M, a smooth function f on $M \times R$ and the product metric G on $M \times R$. This may be expressed by the condition [2]

$$(\nabla_X \phi)Y = \alpha\big(g(X,Y)\xi - \eta(Y)X\big) + \beta\big(g(\phi X, Y)\xi - \eta(Y)\phi X\big), \tag{2.5}$$

for smooth functions α and β on M. Here we say that the trans-Sasakian structure is of type (α, β). From (2.5) it follows that

$$\nabla_X \xi = -\alpha\phi X + \beta\big(X - \eta(X)\xi\big), \tag{2.6}$$

$$(\nabla_X \eta)Y = -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y). \tag{2.7}$$

An explicit example of 3-dimensional proper trans-Sasakian manifold is constructed in [10]. In [6], the Ricci operator, Ricci tensor and curvature tensor for 3-dimensional trans-Sasakian manifolds are studied and their explicit formulae are given.

From [6] we know that for a 3-dimensional trans-Sasakian manifold

$$2\alpha\beta + \xi\alpha = 0, \tag{2.8}$$

$$S(X,\xi) = \big(2(\alpha^2 - \beta^2) - \xi\beta\big)\eta(X) - X\beta - (\phi X)\alpha, \tag{2.9}$$

$$\begin{aligned} S(X,Y) =& \Big(\frac{r}{2} + \xi\beta - (\alpha^2 - \beta^2)\Big) g(X,Y) \\ & - \Big(\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2)\Big)\eta(X)\eta(Y) \\ & - \big(Y\beta + (\phi Y)\alpha\big)\eta(X) - \big(X\beta + (\phi X)\alpha\big)\eta(Y), \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} R(X,Y)Z =& \Big(\frac{r}{2} + 2\xi\beta - 2(\alpha^2 - \beta^2)\Big)\big(g(Y,Z)X - g(X,Z)Y\big) \\ & - g(Y,Z)\Big[\Big(\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2)\Big)\eta(X)\xi \\ & \qquad - \eta(X)(\phi \mathrm{grad}\alpha - \mathrm{grad}\beta) + \big(X\beta + (\phi X)\alpha\big)\xi\Big] \\ & + g(X,Z)\Big[\Big(\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2)\Big)\eta(Y)\xi \\ & \qquad - \eta(Y)(\phi \mathrm{grad}\alpha - \mathrm{grad}\beta) + \big(Y\beta + (\phi Y)\alpha\big)\xi\Big] \end{aligned} \tag{2.11}$$

$$
\begin{aligned}
&- \Big[\big(Z\beta + (\phi Z)\alpha\big)\eta(Y) + \big(Y\beta + (\phi Y)\alpha\big)\eta(Z) \\
&\quad + \Big(\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2)\Big)\eta(Y)\eta(Z)\Big]X \\
&+ \Big[\big(Z\beta + (\phi Z)\alpha\big)\eta(X) + \big(X\beta + (\phi X)\alpha\big)\eta(Z) \\
&\quad + \Big(\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2)\Big)\eta(X)\eta(Z)\Big]Y,
\end{aligned}
$$

where S is the Ricci tensor of type $(0,2)$, R is the curvature tensor of type $(1,3)$ and r is the scalar curvature of the manifold M.

3. Locally ϕ-symmetric three-dimensional trans-Sasakian manifolds

Definition 3.1. A trans-Sasakian manifold is said to be locally ϕ-symmetric if

$$\phi^2(\nabla_W R)(X,Y)Z = 0,$$

for all vector fields W, X, Y, Z orthogonal to ξ.

This notion was introduced for Sasakian manifolds by Takahashi [14].

Let M be a 3-dimensional connected trans-Sasakian manifold. Then its curvature tensor is given by (2.11). Differentiating (2.11) we get

$$
\begin{aligned}
&(\nabla_W R)(X,Y)Z \\
&= \Big[\frac{dr(W)}{2} + 2(\nabla_W(\xi\beta)) - 4\big(d\alpha(W) - d\beta(W)\big)\Big]\big[g(Y,Z)X - g(X,Z)Y\big] \\
&\quad - g(Y,Z)\Big[\Big(\frac{dr(W)}{2} + \big(\nabla_W(\xi\beta)\big) - 6\big(d\alpha(W) - d\beta(W)\big)\Big)\eta(X)\xi \\
&\qquad + \Big(\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2)\Big)\big((\nabla_W\eta)(X)\xi + \eta(X)(\nabla_W\xi)\big) \\
&\qquad - (\nabla_W\eta)(X)\big(\phi(\mathrm{grad}\alpha) - \mathrm{grad}\beta\big) - \eta(X)\big(\nabla_W(\phi(\mathrm{grad}\alpha) - \mathrm{grad}\beta)\big) \\
&\qquad + \big(\nabla_W\big(X\beta + (\phi X)\alpha\big)\big)\xi + \big(X\beta + (\phi X)\alpha\big)\nabla_W\xi\Big] \\
&\quad + g(X,Z)\Big[\Big(\frac{dr(W)}{2} + \big(\nabla_W(\xi\beta)\big) - 6\big(d\alpha(W) - d\beta(W)\big)\Big)\eta(Y)\xi \\
&\qquad + \Big(\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2)\Big)\big((\nabla_W\eta)(Y)\xi + \eta(Y)(\nabla_W\xi)\big) \\
&\qquad - (\nabla_W\eta)(Y)\big(\phi(\mathrm{grad}\alpha) - \mathrm{grad}\beta\big) - \eta(Y)\big(\nabla_W\big(\phi(\mathrm{grad}\alpha) - \mathrm{grad}\beta\big)\big) \\
&\qquad + \big(\nabla_W\big(Y\beta + (\phi X)\alpha\big)\big)\xi + \big(Y\beta + (\phi Y)\alpha\big)\nabla_W\xi\Big] \qquad (3.1)
\end{aligned}
$$

$$
\begin{aligned}
&- \Big[\big(\nabla_W\big(Z\beta + (\phi Z)\alpha\big)\big)\eta(Y) + \big(Z\beta + (\phi Z)\alpha\big)(\nabla_W\eta)Y \\
&\quad + \big(\nabla_W\big(Y\beta + (\phi Y)\alpha\big)\big)\eta(Z) + \big(Y\beta + (\phi Y)\alpha\big)(\nabla_W\eta)Z \\
&\quad + \Big(\frac{dr(W)}{2} + \big(\nabla_W(\xi\beta)\big) - 6\big(d\alpha(W) - d\beta(W)\big)\Big)\eta(Y)\eta(Z) \\
&\quad + \Big(\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2)\Big)\big((\nabla_W\eta)Y\eta(Z) + \eta(Y)(\nabla_W\eta)Z\big)\Big]X \\
&+ \Big[\big(\nabla_W\big(Z\beta + (\phi Z)\alpha\big)\big)\eta(X) + \big(Z\beta + (\phi Z)\alpha\big)(\nabla_W\eta)X \\
&\quad + \big(\nabla_W\big(X\beta + (\phi X)\alpha\big)\big)\eta(Z) + \big(X\beta + (\phi X)\alpha\big)(\nabla_W\eta)Z \\
&\quad + \Big(\frac{dr(W)}{2} + \big(\nabla_W(\xi\beta)\big) - 6\big(d\alpha(W) - d\beta(W)\big)\Big)\eta(X)\eta(Z) \\
&\quad + \Big(\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2)\Big)\big((\nabla_W\eta)X\eta(Z) + \eta(X)(\nabla_W\eta)Z\big)\Big]Y.
\end{aligned}
$$

Suppose that α and β are constants and X, Y, Z, W are orthogonal to ξ. Then using $\phi\xi = 0$ and (3.1), we get

$$\phi^2(\nabla_W R)(X,Y)Z = \Big(\frac{dr(W)}{2}\Big)\big(g(Y,Z)X - g(X,Z)Y\big). \tag{3.2}$$

Hence from (3.2) we get $\phi^2(\nabla_W R)(X,Y)Z = 0$ if and only if the scalar curvature r is constant. Thus we can state the following:

Theorem 3.1. *A 3-dimensional connected trans-Sasakian manifold of type (α,β) is locally ϕ-symmetric if and only if the scalar curvature is constant provided α and β are constants.*

The above theorem is just an extension of an analogous result concerning Kenmotsu manifolds obtained by the first author in the paper [5].

4. $\eta-$parallel Ricci tensor

Definition 4.1. The Ricci tensor S of a trans-Sasakian manifold is said to be η-parallel if it satisfies

$$(\nabla_X S)(\phi Y, \phi Z) = 0, \tag{4.1}$$

for all vector fields X, Y and Z.

This notion was introduced in the context of Sasakian manifolds by Kon [9].

Let M be a 3-dimensional connected trans-Sasakian manifold. Then its Ricci tensor is given by (2.10)

In (2.10) replacing X by ϕX, Y by ϕY and using (2.1) we get for a trans-Sasakian manifold of dimension three

$$S(\phi X, \phi Y) = \Big(\frac{r}{2} + \xi\beta - (\alpha^2 - \beta^2)\Big)\big(g(X,Y) - \eta(X)\eta(Y)\big). \tag{4.2}$$

Now we see that

$$\begin{aligned}(\nabla_Z S)(\phi X, \phi Y) =& \nabla_Z S(\phi X, \phi Y) - S(\nabla_Z \phi X, \phi Y) - S(\phi X, \nabla_Z \phi Y)\\ =& \nabla_Z S(\phi X, \phi Y) - S\big((\nabla_Z\phi)X, \phi Y\big) - S(\phi\nabla_Z X, \phi Y)\\ & - S\big(\phi X, (\nabla_Z\phi)Y\big) - S(\phi X, \phi\nabla_Z Y).\end{aligned} \tag{4.3}$$

Using (2.5), (2.10) and (4.2) in (4.3) we have

$$\begin{aligned}&(\nabla_Z S)(\phi X, \phi Y)\\ &= \Big(\frac{1}{2}dr(Z) + \nabla_Z(\xi\beta) - 2\alpha d\alpha(Z) + 2\beta d\beta(Z)\Big)\big(g(X,Y) - \eta(X)\eta(Y)\big)\\ &\quad + \Big(\frac{r}{2} + \xi\beta - (\alpha^2 - \beta^2)\Big)\Big(\nabla_Z g(X,Y) - \big(\nabla_Z\eta(X)\big)\eta(Y)\\ &\qquad - \eta(X)\big(\nabla_Z\eta(Y)\big)\Big)\\ &\quad - S\Big(\alpha\big(g(Z,X)\xi - \eta(X)Z\big) + \beta\big(g(\phi Z, X)\xi - \eta(X)\phi Z\big), \phi Y\Big)\\ &\quad - \Big(\frac{r}{2} + \xi\beta - (\alpha^2 - \beta^2)\Big)\big(g(\nabla_Z X, Y) - \eta(\nabla_Z X)\eta(Y)\big)\\ &\quad - S\Big(\phi X, \alpha\big(g(Z,Y)\xi - \eta(Y)Z\big) + \beta\big(g(\phi Z, Y)\xi - \eta(Y)\phi Z\big)\Big)\\ &\quad - \Big(\frac{r}{2} + \xi\beta - (\alpha^2 - \beta^2)\Big)\big(g(X, \nabla_Z Y) - \eta(X)\eta(\nabla_Z Y)\big).\end{aligned} \tag{4.4}$$

By virtue of (2.9) and (2.10) we obtain from (4.4)

$$\begin{aligned}&(\nabla_Z S)(\phi X, \phi Y)\\ &= \Big(\frac{1}{2}dr(Z) + \nabla_Z(\xi\beta) - 2\alpha d\alpha(Z) + 2\beta d\beta(Z)\Big)\big(g(X,Y) - \eta(X)\eta(Y)\big)\\ &\quad + \Big(\frac{r}{2} + \xi\beta - (\alpha^2 - \beta^2)\Big)\Big(\nabla_Z g(X,Y) - \big(\nabla_Z\eta(X)\big)\eta(Y) - \eta(X)\big(\nabla_Z\eta(Y)\big)\Big)\\ &\quad + \alpha g(Z,X)\big((\phi Y)\beta + (\phi^2 Y)\alpha\big)\end{aligned} \tag{4.5}$$

$$
\begin{aligned}
&+\alpha\eta(X)\Big(\Big(\frac{r}{2}+\xi\beta-(\alpha^2-\beta^2)\Big)g(Z,\phi Y)-\big((\phi Y)\beta+(\phi^2 Y)\alpha\big)\eta(Z)\Big)\\
&+\beta g(\phi Z,X)\big((\phi Y)\beta+(\phi^2 Y)\alpha\big)\\
&+\beta\eta(X)\Big(\Big(\frac{r}{2}+\xi\beta-(\alpha^2-\beta^2)\Big)\big(g(Z,Y)-\eta(Z)\eta(Y)\big)\Big)\\
&-\Big(\frac{r}{2}+\xi\beta-(\alpha^2-\beta^2)\Big)\big(g(\nabla_Z X,Y)-\eta(\nabla_Z X)\eta(Y)\big)\\
&+\alpha g(Z,Y)\big((\phi X)\beta+(\phi^2 X)\alpha\big)\\
&+\alpha\eta(Y)\Big(\Big(\frac{r}{2}+\xi\beta-(\alpha^2-\beta^2)\Big)g(Z,\phi X)-\big((\phi X)\beta+(\phi^2 X)\alpha\big)\eta(Z)\Big)\\
&+\beta g(\phi Z,Y)\big((\phi X)\beta+(\phi^2 X)\alpha\big)\\
&+\beta\eta(Y)\Big(\Big(\frac{r}{2}+\xi\beta-(\alpha^2-\beta^2)\Big)\big(g(Z,X)-\eta(Z)\eta(X)\big)\Big)\\
&-\Big(\frac{r}{2}+\xi\beta-(\alpha^2-\beta^2)\Big)\big(g(X,\nabla_Z Y)-\eta(X)\eta(\nabla_Z Y)\big).
\end{aligned}
$$

The above relation can be written as

$$
\begin{aligned}
&(\nabla_Z S)(\phi X,\phi Y)\\
&\quad=\Big(\frac{1}{2}dr(Z)+\nabla_Z(\xi\beta)-2\alpha d\alpha(Z)+2\beta d\beta(Z)\Big)\big(g(X,Y)-\eta(X)\eta(Y)\big)\\
&\quad\quad+\Big(\frac{r}{2}+\xi\beta-(\alpha^2-\beta^2)\Big)\Big(\nabla_Z g(X,Y)-\big(\nabla_Z\eta(X)\big)\eta(Y)-\eta(X)\big(\nabla_Z\eta(Y)\big)\\
&\quad\quad\quad+\alpha\eta(X)g(Z,\phi Y)+\beta\eta(X)\big(g(Z,Y)-\eta(Z)\eta(Y)\big)-g(\nabla_Z X,Y)\\
&\quad\quad\quad+\eta(\nabla_Z X)\eta(Y)+\alpha\eta(Y)g(Z,\phi X)+\beta\eta(Y)\big(g(Z,X)-\eta(Z)\eta(X)\big)\\
&\quad\quad\quad-g(X,\nabla_Z Y)+\eta(X)\eta(\nabla_Z Y)\Big) \qquad (4.6)\\
&\quad\quad+\big((\phi Y)\beta+(\phi^2 Y)\alpha\big)\big(\alpha g(Z,X)+\beta g(\phi Z,X)-\alpha\eta(X)\eta(Z)\big)\\
&\quad\quad+\big((\phi X)\beta+(\phi^2 X)\alpha\big)\big(\alpha g(Z,Y)+\beta g(\phi Z,Y)-\alpha\eta(Y)\eta(Z)\big).
\end{aligned}
$$

Suppose that α and β are constants. Then using (2.7) in (4.6), we obtain

$$(\nabla_Z S)(\phi X,\phi Y)=\frac{1}{2}dr(Z)\big(g(X,Y)-\eta(X)\eta(Y)\big). \qquad (4.7)$$

Hence, from (4.7) we can state the following:

Theorem 4.1. *A 3-dimensional connected trans-Sasakian manifold of type (α,β) has η-parallel Ricci tensor if and only if the scalar curvature of the manifold is constant provided α and β are constants.*

From Theorem 3.1 and Theorem 4.1 we can state the following:

Corollary 4.1. *A 3-dimensional connected trans-Sasakian manifold of type (α,β) has η-parallel Ricci tensor if and only if it is locally ϕ-symmetric provided α and β are constants.*

5. Three-dimensional trans-Sasakian manifold with constant curvature

Let M be a 3-dimensional compact connected trans-Sasakian manifold. If the manifold is of constant curvature then the Ricci tensor of type $(0,2)$ of the manifold is given by

$$S(X,Y) = 2\lambda g(X,Y), \tag{5.1}$$

where λ is a constant. Putting $Y=\xi$ in (5.1) and using (2.9), we get

$$X\beta + (\phi X)\alpha + [2(\lambda - \alpha^2 + \beta^2) + \xi\beta]\eta(X) = 0. \tag{5.2}$$

For $X=\xi$, (5.2) yields

$$\xi\beta = -(\lambda - \alpha^2 + \beta^2). \tag{5.3}$$

By virtue of (5.2) and (5.3) it follows that

$$X\beta + (\phi X)\alpha + (\lambda - \alpha^2 + \beta^2)\eta(X) = 0. \tag{5.4}$$

The gradient of the function β is related to the exterior derivative $d\beta$ by the formula

$$d\beta(X) = g(\mathrm{grad}\beta, X). \tag{5.5}$$

Using (5.5) in (5.4) we obtain

$$d\beta(X) + g(\mathrm{grad}\alpha, \phi X) + (\lambda - \alpha^2 + \beta^2)\eta(X) = 0. \tag{5.6}$$

Differentiating (5.6) covariantly with respect to Y we get

$$\begin{aligned}(\nabla_Y d\beta)(X) &+ g(\nabla_Y \mathrm{grad}\alpha, \phi X) + g(\mathrm{grad}\alpha, (\nabla_Y\phi)X)\\ &+ Y(\beta^2 - \alpha^2)\eta(X) + (\lambda - \alpha^2 + \beta^2)(\nabla_Y\eta)(X) = 0.\end{aligned} \tag{5.7}$$

Interchanging X and Y in (5.7), we get

$$\begin{aligned}(\nabla_X d\beta)(Y) + g(\nabla_X \mathrm{grad}\alpha, \phi Y) + g(\mathrm{grad}\alpha, (\nabla_X \phi)Y) \\ + X(\beta^2 - \alpha^2)\eta(Y) + (\lambda - \alpha^2 + \beta^2)(\nabla_X \eta)(Y) = 0.\end{aligned} \tag{5.8}$$

Subtracting (5.7) from (5.8) we get

$$\begin{aligned}g(\nabla_X \mathrm{grad}\alpha, \phi Y) - g(\nabla_Y \mathrm{grad}\alpha, \phi X) + \big((\nabla_X \phi)Y - (\nabla_Y \phi)X\big)\alpha \\ + [X(\beta^2 - \alpha^2)\eta(Y) - Y(\beta^2 - \alpha^2)\eta(X)] \\ + (\lambda - \alpha^2 + \beta^2)\big((\nabla_X \eta)(Y) - (\nabla_Y \eta)(X)\big) = 0.\end{aligned} \tag{5.9}$$

From (2.7) and (2.4) we get

$$(\nabla_X \eta)(Y) - (\nabla_Y \eta)(X) = \alpha\big(\Phi(X, Y) - \Phi(Y, X)\big) = 2\alpha\Phi(X, Y). \tag{5.10}$$

Using (5.10) in (5.9) we have

$$\begin{aligned}g(\nabla_X \mathrm{grad}\alpha, \phi Y) - g(\nabla_Y \mathrm{grad}\alpha, \phi X) + \big((\nabla_X \phi)Y - (\nabla_Y \phi)X\big)\alpha \\ + [X(\beta^2 - \alpha^2)\eta(Y) - Y(\beta^2 - \alpha^2)\eta(X)] \\ + 2(\lambda - \alpha^2 + \beta^2)\alpha\Phi(X, Y) = 0.\end{aligned} \tag{5.11}$$

Let $\{E_0, E_1, E_2\}$ be a local ϕ-basis, that is, an orthonormal frame such that $E_0 = \xi$ and $E_2 = \phi E_1$. In (2.5) putting $X = E_1$, $Y = E_2$ we get

$$\begin{aligned}(\nabla_{E_1}\phi)E_2 &= \alpha\big(g(E_1, E_2)\xi - \eta(E_2)E_1\big) + \beta\big(g(\phi E_1, E_2)\xi - \eta(E_2)\phi E_1\big) \\ &= \beta g(\phi E_1, E_2)\xi = \beta\xi.\end{aligned} \tag{5.12}$$

Similarly,

$$(\nabla_{E_2}\phi)E_1 = -\beta\xi. \tag{5.13}$$

Now,

$$\Phi(E_1, E_2) = g(E_1, \phi E_2) = g(E_1, \phi^2 E_1) = -1. \tag{5.14}$$

In (5.11) putting $X = E_1$ and $Y = E_2$ and using (5.12), (5.13) and (5.14) we obtain

$$g(\nabla_{E_1}\mathrm{grad}\alpha, E_1) + g(\nabla_{E_2}\mathrm{grad}\alpha, E_2) = 2\beta\xi\alpha - 2\alpha(\lambda - \alpha^2 + \beta^2). \tag{5.15}$$

Also (2.8) can be written as

$$g(\mathrm{grad}\alpha, \xi) = -2\alpha\beta. \tag{5.16}$$

Differentiating (5.16) covariantly with respect to ξ we get

$$g(\nabla_\xi \mathrm{grad}\alpha, \xi) + g(\mathrm{grad}\alpha, \nabla_\xi \xi) = -2\beta\xi\alpha - 2\alpha\xi\beta. \tag{5.17}$$

In view of (5.3) we can write the above relation as

$$g(\nabla_\xi \mathrm{grad}\alpha, \xi) = -2\beta\xi\alpha + 2\alpha(\lambda - \alpha^2 + \beta^2). \tag{5.18}$$

From (5.15) and (5.18) we get $\triangle\alpha = 0$, where $\triangle$ is the Laplacian defined by

$$\triangle\alpha = \sum_{i=0}^{2} g(\nabla_{E_i} \mathrm{grad}\alpha, E_i).$$

Since M is compact we get α is constant.

Now let us consider the following two cases:

Case-I: In this case we suppose that α is non-zero constant then by (2.8), $\beta = 0$ every where on M.

Case-II: In this case let $\alpha = 0$. Then from (5.4)

$$X\beta + (\lambda + \beta^2)\eta(X) = 0,$$

that is,

$$g(\mathrm{grad}\beta, X) + (\lambda + \beta^2)g(X, \xi) = 0.$$

Therefore,

$$\mathrm{grad}\beta + (\lambda + \beta^2)\xi = 0. \tag{5.19}$$

Differentiating (5.19) covariantly with respect to X we have

$$\nabla_X \mathrm{grad}\beta + (X\beta^2)\xi + (\lambda + \beta^2)\nabla_X \xi = 0.$$

Using (2.6) we get from above

$$\nabla_X \mathrm{grad}\beta + (X\beta^2)\xi + (\lambda + \beta^2)\big(-\alpha\phi X + \beta(X - \eta(X)\xi)\big) = 0.$$

Now taking inner product of the above equation with X, we have

$$\begin{aligned} g(\nabla_X \mathrm{grad}\beta, X) = &- g\big((X\beta^2)\xi, X\big) \\ &- (\lambda + \beta^2)\big(g(-\alpha\phi X, X) + \beta g(X - \eta(X)\xi, X)\big). \end{aligned} \tag{5.20}$$

Therefore putting $X = E_i$ and taking summation over i, $i = 0, 1, 2$, we get from above

$$\triangle\beta = -2\beta(\xi\beta + \lambda + \beta^2). \tag{5.21}$$

For $\alpha = 0$, (5.3) yields $\xi\beta = -(\lambda + \beta^2)$, which in view of (5.21) gives $\triangle\beta = 0$. Hence β=constant, M being compact. This leads to the following:

THEOREM 5.1. *If a 3-dimensional compact connected trans-Sasakian manifold is of constant curvature then it is either α-Sasakian or β-Kenmotsu.*

6. EXAMPLE OF A LOCALLY ϕ-SYMMETRIC THREE-DIMENSIONAL TRANS-SASAKIAN MANIFOLD

We consider the three-dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3, z \neq 0\}$, where (x, y, z) are the standard coordinates in $\mathbb{R}^3$. The vector fields

$$e_1 = z\frac{\partial}{\partial x}, \qquad e_2 = z\frac{\partial}{\partial y}, \qquad e_3 = z\frac{\partial}{\partial z}$$

are linearly independent at each point of M. Let g be the Riemannian metric defined by

$$g(e_1, e_3) = g(e_2, e_3) = g(e_1, e_2) = 0, \qquad g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1.$$

Let η be the 1-form defined by $\eta(Z) = g(Z, e_3)$ for any $Z \in \chi(M)$. Let ϕ be the (1,1) tensor field defined by $\phi(e_1) = -e_2$, $\phi(e_2) = e_1$, $\phi(e_3) = 0$. Then using the linearity of ϕ and g we have

$$\eta(e_3) = 1, \qquad \phi^2 Z = -Z + \eta(Z)e_3, \qquad g(\phi Z, \phi W) = g(Z, W) - \eta(Z)\eta(W),$$

for any $Z, W \in \chi(M)$. Thus for $e_3 = \xi$, (ϕ, ξ, η, g) defines an almost contact metric structure on M. Now, by direct computations we obtain

$$[e_1, e_2] = 0, \qquad [e_2, e_3] = -e_2, \qquad [e_1, e_3] = -e_1.$$

The Riemannian connection ∇ of the metric g is given by the Koszul's formula which is

$$\begin{aligned} 2g(\nabla_X Y, Z) =& Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ &- g\big(X, [Y, Z]\big) - g\big(Y, [X, Z]\big) + g\big(Z, [X, Y]\big). \end{aligned} \tag{6.1}$$

Using (6.1) we have

$$\begin{aligned}
2g(\nabla_{e_1}e_3, e_1) &= 2g(-e_1, e_1),\\
2g(\nabla_{e_1}e_3, e_2) &= 0 = 2g(-e_1, e_2),\\
2g(\nabla_{e_1}e_3, e_3) &= 0 = 2g(-e_1, e_3).
\end{aligned}$$

Hence, $\nabla_{e_1}e_3 = -e_1$. Similarly, $\nabla_{e_2}e_3 = -e_2$ and $\nabla_{e_3}e_3 = 0$.
(6.1) further yields

$$\begin{aligned}
\nabla_{e_1}e_2 &= 0, & \nabla_{e_1}e_1 &= e_3,\\
\nabla_{e_2}e_2 &= e_3, & \nabla_{e_2}e_1 &= 0,\\
\nabla_{e_3}e_2 &= 0, & \nabla_{e_3}e_1 &= 0.
\end{aligned}$$

We see that

$$\begin{aligned}
(\nabla_{e_1}\phi)e_1 &= \nabla_{e_1}\phi e_1 - \phi\nabla_{e_1}e_1 = -\nabla_{e_1}e_2 - \phi e_3 = -\nabla_{e_1}e_2 = 0\\
&= 0(g(e_1, e_1)e_3 - \eta(e_1)e_1) - 1(g(\phi e_1, e_1)e_3 - \eta(e_1)\phi e_1).
\end{aligned} \tag{6.2}$$

$$\begin{aligned}
(\nabla_{e_1}\phi)e_2 &= \nabla_{e_1}\phi e_2 - \phi\nabla_{e_1}e_2 = \nabla_{e_1}e_1 - 0 = e_3\\
&= 0\big(g(e_1, e_2)e_3 - \eta(e_2)e_1\big) - 1\big(g(\phi e_1, e_2)e_3 - \eta(e_2)\phi e_1\big).
\end{aligned} \tag{6.3}$$

$$\begin{aligned}
(\nabla_{e_1}\phi)e_3 &= \nabla_{e_1}\phi e_3 - \phi\nabla_{e_1}e_3 = 0 + \phi e_1 = -e_2\\
&= 0\big(g(e_1, e_3)e_3 - \eta(e_3)e_1\big) - 1\big(g(\phi e_1, e_3)e_3 - \eta(e_3)\phi e_1\big).
\end{aligned} \tag{6.4}$$

By (6.2), (6.3) and (6.4) we see that the manifold satisfies (2.5) for $X = e_1$, $\alpha = 0$, $\beta = -1$, and $e_3 = \xi$. Similarly it can be shown that for $X = e_2$ and $X = e_3$ the manifold also satisfies (2.5) for $\alpha = 0$, $\beta = -1$, and $e_3 = \xi$. Hence the manifold is a trans-Sasakian manifold of type $(0, -1)$. With the help of the above results it can be verified that

$$\begin{aligned}
&R(e_1, e_2)e_3 = 0, && R(e_2, e_3)e_3 = -e_2, && R(e_1, e_3)e_3 = -e_1,\\
&R(e_1, e_2)e_2 = -e_1, && R(e_2, e_3)e_2 = e_3, && R(e_1, e_3)e_2 = 0,\\
&R(e_1, e_2)e_1 = e_2, && R(e_2, e_3)e_1 = 0, && R(e_1, e_3)e_1 = e_3.
\end{aligned}$$

From which it follows that $\phi^2(\nabla_W R)(X, Y)Z = 0$. Hence the 3-dimensional trans-Sasakian manifold is locally ϕ-symmetric.

Also from the above expressions of the curvature tensor we obtain the scalar curvature $r = -3$. Hence we note that here α, β and r all are constants. Hence from Theorem 3.1 it follows that the manifold under consideration is locally ϕ-symmetric.

Acknowledgements

The authors are thankful to the referee for his valuable suggestions and remarks in the improvement of the paper.

References

[1] D. E. Blair, "Contact Manifolds in Riemannian Geometry", Lecture Notes in Mathematics, Vol. 509, Springer–Verlag, Berlin-New York, 1976.

[2] D. E. Blair, J. A. Oubiña, Conformal and related changes of metric on the product of two almost contact metric manifolds, *Publ. Mat.* **34** (1) (1990), 199 – 207.

[3] D. Chinea, C. Gonzales, A classification of almost contact metric manifolds, *Ann. Mat. Pura Appl. (4)* **156** (1990), 15 – 36.

[4] D. Chinea, C. Gonzales, Curvature relations in trans-Sasakian manifolds, (Spanish), in "Proceedings of the XIIth Portuguese-Spanish Conference on Mathematics, Vol. II, (Portuguese), Braga, 1987," Univ. Minho, Braga, (1987), 564 – 571.

[5] U. C. De, On ϕ-symmetric Kenmotsu manifolds, *Int. Electron. J. Geom.* **1** (1) (2008), 33 – 38.

[6] U. C. De, M. M. Tripathi, Ricci tensor in 3-dimensional trans-Sasakian manifolds, *Kyungpook Math. J.* **43** (2) (2003),247 – 255.

[7] A. Gray, L. M. Hervella, The sixteen classes of almost Hermitian manifolds and their linear invariants, *Ann. Mat, Pura Appl. (4)* **123** (1980), 35 – 58.

[8] D. Janssens, L. Vanhecke, Almost contact structures and curvature tensors, *Kodai Math. J.* **4** (1) (1981), 1 – 27.

[9] M. Kon, Invariant submanifolds in Sasakian manifolds, *Math. Ann.* **219** (3) (1976), 277 – 290.

[10] J. C. Marrero, The local structure of trans-Sasakian manifolds, *Ann. Mat. Pura Appl. (4)* **162** (1992), 77 – 86.

[11] J. C. Marrero, D. Chinea, On trans-Sasakian manifolds, (Spanish), in "Proceedings of the XIVth Spanish-Portuguese Conference on Mathematics, Vol. I-III, (Spanish), Puerto de la Cruz, 1989", Univ. La Laguna, La Laguna, 1990, 655 – 659.

[12] R. S. Mishra, Almost contact metric manifolds, Monograph 1, Tensor Society of India, Lucknow, 1991.

[13] J. A. Oubiña, New classes of almost contact metric structures, *Publ. Math. Debrecen* **32** (3-4) (1985), 187 – 193.

[14] T. Takahashi, Sasakian ϕ-symmetric spaces, *Tohoku Math. J. (2)* **29** (1) (1977), 91 – 113.

[15] M. M. Tripathi, Trans-Sasakian manifolds are generalized quasi-Sasakian, *Nepali Math. Sci. Rep.* **18** (1-2) (1999/2000), 11 – 14.