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The Unit Ball of $\mathcal{L}_s({}^2l^2_\infty)$

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Abstract: We classify the extreme, exposed and smooth points of the unit ball of the space of symmetric bilinear forms on the 2-dimensional real spaces l^2_∞.

Key words: Symmetric bilinear forms, extreme points, exposed points, smooth points.

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1. Introduction

We write B_E for the closed unit ball of a real Banach space E and the dual space of E is denoted by E^*; $x \in B_E$ is called an *extreme point* of B_E if $y, z \in B_E$ with $x = \frac{1}{2}(y+z)$ implies $x = y = z$; $x \in B_E$ is called an *exposed point* of B_E if there is a $f \in E^*$ so that $f(x) = 1 = \|f\|$ and $f(y) < 1$ for every $y \in B_E \setminus \{x\}$; $x \in B_E$ is called a *smooth point* of B_E if there is a unique $f \in E^*$ so that $f(x) = 1 = \|f\|$. It is easy to see that every exposed point of B_E is an extreme point. We denote by $\exp B_E$, $\mathrm{ext} B_E$ and $\mathrm{sm} B_E$ the sets of exposed, extreme and smooth points of B_E, respectively. We recall that a bilinear function $L : E \times E \to \mathbb{R}$ is a symmetric bilinear form if $L(x, y) = L(y, x)$ for every $x, y \in E$. We denote by $\mathcal{L}_s({}^2E)$ the Banach space of all symmetric bilinear forms from E into $\mathbb{R}$ endowed with the norm $\|L\| = \sup_{\|x\|=\|y\|=1} |L(x, y)|$. A mapping $P : E \to \mathbb{R}$ is a continuous 2-homogeneous polynomial if there exists a *unique* continuous symmetric bilinear form L on the product $E \times E$ such that $P(x) = L(x, x)$ for every $x \in E$. We denote by $\mathcal{P}({}^2E)$ the Banach space of all continuous 2-homogeneous polynomials from E into $\mathbb{R}$ endowed with the norm $\|P\| = \sup_{\|x\|=1} |P(x)|$. For more details on polynomials and symmetric bilinear maps, see [6]. It is well-known that there is an isomorphism between $\mathcal{L}_s({}^2E)$ and $\mathcal{P}({}^2E)$. If E is a (real or complex) Hilbert space, then there is an isometric isomorphism between $\mathcal{L}_s({}^2E)$ and $\mathcal{P}({}^2E)$ via $L \to \hat{L}$, where $\hat{L}(x) = L(x, x)$ for every $x \in E$. Thus L is an extreme (exposed, smooth, respectively) point in the unit ball of $\mathcal{L}_s({}^2E)$ if and only if $\hat{L}$ is an extreme (exposed, smooth, respectively) point in the unit ball of $\mathcal{P}(^2E)$. Some work has been done in analyzing the geometry of spaces of polynomials on real Banach spaces ([1]−[5], [7]−[10]). Note that there is no isometry between $\mathcal{L}_s(^2l_\infty^2)$ and $\mathcal{P}(^2l_\infty^2)$. Thus it is natural to consider the problem of analyzing the geometry of spaces of symmetric bilinear forms on real Banach spaces. In this paper, we classify the extreme, exposed and smooth points of the unit ball of the space $\mathcal{L}_s(^2l_\infty^2)$.

2. The results

Theorem 1. *Let $L \in \mathcal{L}_s(^2l_\infty^2)$ with $L\big((x_1,x_2),(y_1,y_2)\big) = ax_1y_1 + bx_2y_2 + c(x_1y_2 + x_2y_1)$ for $(x_1,x_2),(y_1,y_2) \in l_\infty^2$. Then we have*

$$\|L\| = \max\big\{|a+b| + 2|c|,\ |a-b|\big\}.$$

Proof. By symmetric bilinearity of L, we have

$$\|L\| = \max\Big\{\max_{|x|\le 1,|y|\le 1}\big|L\big((1,x),(1,y)\big)\big|,\ \max_{|x|\le 1,|y|\le 1}\big|L\big((1,x),(y,1)\big)\big|,\ \max_{|x|\le 1,|y|\le 1}\big|L\big((x,1),(y,1)\big)\big|\Big\}.$$

It follows that

$$\begin{aligned}\max_{|x|\le 1,|y|\le 1}&\big|L\big((1,x),(1,y)\big)\big|\\ &= \max\Big\{\max_{|y|\le 1}\big|L\big((1,\pm 1),(1,y)\big)\big|,\ \max_{|x|\le 1}\big|L\big((1,x),(1,\pm 1)\big)\big|\Big\}\\ &= \max\big\{|a-b|,\ |a+b|+2|c|\big\}.\end{aligned}$$

Similarly,

$$\begin{aligned}\max_{|x|\le 1,|y|\le 1}\big|L\big((1,x),(y,1)\big)\big| &= \max_{|x|\le 1,|y|\le 1}\big|L\big((x,1),(y,1)\big)\big|\\ &= \max\big\{|a-b|,\ |a+b|+2|c|\big\},\end{aligned}$$

which completes the proof. ∎

Theorem 2. *We have*

$$\operatorname{ext} B_{\mathcal{L}_s(^2l_\infty^2)} = \Big\{\pm x_1y_1,\ \pm x_2y_2,\ \pm\frac{1}{2}\big[\,x_1y_1 - x_2y_2 \pm (x_1y_2 + x_2y_1)\big]\Big\}.$$

Proof. Let $L \in S_{\mathcal{L}_s({}^2 l_\infty^2)}$ with $L\big((x_1,x_2),(y_1,y_2)\big) = ax_1y_1 + bx_2y_2 + c(x_1y_2 + x_2y_1)$ for $(x_1,x_2),(y_1,y_2) \in l_\infty^2$. By Theorem 1, we have $|a| \le 1$, $|b| \le 1$, $|c| \le \frac{1}{2}$.

Claim 1: If $c = 0$ and $L \in \mathrm{ext} B_{\mathcal{L}_s({}^2 l_\infty^2)}$ then $L\big((x_1,x_2),(y_1,y_2)\big) = \pm\, x_1y_1$ or $\pm\, x_2y_2$.

Otherwise $ab \ne 0$. Put

$$A\big((x_1,x_2),(y_1,y_2)\big) := (a + \mathrm{sign}\,(a)\epsilon_0)x_1y_1 + (b - \mathrm{sign}\,(b)\epsilon_0)x_2y_2\,,$$
$$B\big((x_1,x_2),(y_1,y_2)\big) := (a - \mathrm{sign}\,(a)\epsilon_0)x_1y_1 + (b + \mathrm{sign}\,(b)\epsilon_0)x_2y_2\,.$$

Clearly, we have $A, B \in S_{\mathcal{L}_s({}^2 l_\infty^2)}$ and $L = \frac{1}{2}(A+B)$, which is a contradiction of the hypothesis that $L \in \mathrm{ext} B_{\mathcal{L}_s({}^2 l_\infty^2)}$. Thus $a = 0$ or $b = 0$, which shows the claim 1.

Claim 2: If $|c| = \frac{1}{2}$, then

$$L\big((x_1,x_2),(y_1,y_2)\big) = \pm\frac{1}{2}\big[\, x_1y_1 \,-\, x_2y_2 \,\pm\, (x_1y_2 + x_2y_1)\,\big]$$

and $L \in \mathrm{ext} B_{\mathcal{L}_s({}^2 l_\infty^2)}$.

By Theorem 1, we have $b = -a$, $|a| = \frac{1}{2}$. For simplicity, we may assume that $a = c = \frac{1}{2}$, $b = -\frac{1}{2}$. Suppose that $L = \frac{1}{2}(A+B)$ for some $A, B \in S_{\mathcal{L}_s({}^2 l_\infty^2)}$. We may write

$$A\big((x_1,x_2),(y_1,y_2)\big) := \alpha x_1y_1 + \beta x_2y_2 + \gamma(x_1y_2 + x_2y_1)\,,$$
$$B\big((x_1,x_2),(y_1,y_2)\big) := \alpha' x_1y_1 + \beta' x_2y_2 + \gamma'(x_1y_2 + x_2y_1)\,,$$

for some $\alpha, \alpha', \beta, \beta', \gamma, \gamma' \in \mathbb{R}$. Since $\frac{1}{2}(\gamma + \gamma') = \frac{1}{2}$, $|\gamma| \le \frac{1}{2}$ and $|\gamma'| \le \frac{1}{2}$, we have $\gamma = \gamma' = \frac{1}{2}$. By Theorem 1, we have $\beta = -\alpha$, $\beta' = -\alpha'$, $\frac{1}{2}(\alpha + \alpha') = \frac{1}{2}$. Since $|\alpha| \le \frac{1}{2}$ and $|\alpha'| \le \frac{1}{2}$, we have $\alpha = \frac{1}{2} = \alpha'$, which show $A = B = L$, so $L \in \mathrm{ext} B_{\mathcal{L}_s({}^2 l_\infty^2)}$. Thus we may that $0 < |c| < \frac{1}{2}$.

Claim 3: If $|a+b| + 2|c| < 1$, then $L \notin \mathrm{ext} B_{\mathcal{L}_s({}^2 l_\infty^2)}$

Indeed, put

$$A\big((x_1,x_2),(y_1,y_2)\big) := ax_1y_1 + bx_2y_2 + (c + \epsilon_0)(x_1y_2 + x_2y_1)\,,$$
$$B\big((x_1,x_2),(y_1,y_2)\big) := ax_1y_1 + bx_2y_2 + (c - \epsilon_0)(x_1y_2 + x_2y_1)\,,$$

where

$$\epsilon_0 := \min\left\{|c|\,, \frac{1 - (|a+b| + 2|c|)}{2}\right\} > 0\,.$$

By Theorem 1, we have $A, B \in S_{\mathcal{L}_s({}^2l_\infty^2)}$, $A \neq L$, $L = \frac{1}{2}(A+B)$, so $L \notin \mathrm{ext} B_{\mathcal{L}_s({}^2l_\infty^2)}$. Suppose that $|a+b| + 2|c| = 1$.

Claim 4: If $|a-b| < 1$, then $L \notin \mathrm{ext} B_{\mathcal{L}_s({}^2l_\infty^2)}$.

Indeed, put

$$\begin{aligned} A\big((x_1,x_2),(y_1,y_2)\big) &:= (a+\epsilon_0)x_1y_1 + (b-\epsilon_0)x_2y_2 + c(x_1y_2 + x_2y_1)\,, \\ B\big((x_1,x_2),(y_1,y_2)\big) &:= (a-\epsilon_0)x_1y_1 + (b+\epsilon_0)x_2y_2 + c(x_1y_2 + x_2y_1)\,, \end{aligned}$$

where

$$\epsilon_0 := \frac{1-|a-b|}{2} > 0\,.$$

By Theorem 1, we have $A, B \in S_{\mathcal{L}_s({}^2l_\infty^2)}$, $A \neq L$, $L = \frac{1}{2}(A+B)$, so $L \notin \mathrm{ext} B_{\mathcal{L}_s({}^2l_\infty^2)}$.

For simplicity, we may assume that $a > 0$.

Claim 5: If $|a-b| = 1$, then

$$L\big((x_1,x_2),(y_1,y_2)\big) = ax_1y_1 + (-1+a)x_2y_2 + (1-a)(x_1y_2 + x_2y_1)$$

for $\frac{1}{2} < a < 1$ and $L \notin \mathrm{ext} B_{\mathcal{L}_s({}^2l_\infty^2)}$.

By Theorem 1, we have

$$L\big((x_1,x_2),(y_1,y_2)\big) = ax_1y_1 + (-1+a)x_2y_2 + (1-a)(x_1y_2 + x_2y_1)$$

for $\frac{1}{2} < a < 1$. Put

$$\begin{aligned} A\big((x_1,x_2),(y_1,y_2)\big) &:= (a+\epsilon_0)x_1y_1 + (-1+a+\epsilon_0)x_2y_2 \\ &\qquad + (1-a-\epsilon_0)(x_1y_2 + x_2y_1)\,, \\ B\big((x_1,x_2),(y_1,y_2)\big) &:= (a-\epsilon_0)x_1y_1 + (-1+a-\epsilon_0)x_2y_2 \\ &\qquad + (1-a+\epsilon_0)(x_1y_2 + x_2y_1)\,, \end{aligned}$$

where

$$\epsilon_0 := \min\left\{1-a,\ a-\frac{1}{2}\right\} > 0\,.$$

By Theorem 1, we have $A, B \in S_{\mathcal{L}_s({}^2l_\infty^2)}$, $A \neq L$, $L = \frac{1}{2}(A+B)$, so $L \notin \mathrm{ext} B_{\mathcal{L}_s({}^2l_\infty^2)}$. Therefore we complete the proof. ∎

Remark. We note that $\mathrm{ext} B_{\mathcal{L}_s({}^2l_\infty^2)} \subset \mathrm{ext} B_{\mathcal{L}_s({}^2l_\infty)}$.

Indeed, by Theorem 2, it's enough to show that x_1y_1 and $\frac{1}{2}x_1y_1 - \frac{1}{2}x_2y_2 + \frac{1}{2}(x_1y_2 + x_2y_1)$ are extreme points of $B_{\mathcal{L}_s({}^2l_\infty)}$.

Claim 1: $L := x_1y_1$ is an extreme point of $B_{\mathcal{L}_s({}^2l_\infty)}$.

Suppose that $L = \frac{1}{2}(A+B)$ for some $A, B \in S_{\mathcal{L}_s({}^2l_\infty)}$. We may write

$$A\big((x_n),(y_n)\big) := \sum_{j=1}^{\infty} a_{jj}x_jy_j + \sum_{1\le k<s} a_{ks}(x_ky_s + x_sy_k)\,,$$

$$B\big((x_n),(y_n)\big) := \sum_{j=1}^{\infty} b_{jj}x_jy_j + \sum_{1\le k<s} b_{ks}(x_ky_s + x_sy_k)\,.$$

It suffices to show that $a_{11} = 1$ and $a_{jj} = a_{ks} = 0$ for every $j > 1$, $1 \le k < s$. Since $1 = L(e_1, e_1) = \frac{1}{2}(a_{11} + b_{11})$, $|a_{11}| \le 1$, $|b_{11}| \le 1$, we have $a_{11} = 1 = b_{11}$. Let $j > 1$ be fixed. Note that

$$1 \ge |A(x_1e_1 + x_je_j, y_1e_1 + y_je_j)| = |x_1y_1 + a_{jj}x_jy_j + a_{1j}(x_1y_j + x_jy_1)|$$

for every $x_1e_1 + x_je_j, y_1e_1 + y_je_j \in B_{l_\infty}$. By Theorem 1, we have $a_{jj} = 0 = a_{1j}$. Suppose that $2 \le k < s$. Since

$$1 \ge |A(e_1 + e_k + e_s, e_1 \pm e_k \pm e_s)| = |1 \pm 2a_{ks}|\,,$$

we have $a_{ks} = 0$.

Claim 2: $L := \frac{1}{2}x_1y_1 - \frac{1}{2}x_2y_2 + \frac{1}{2}(x_1y_2 + x_2y_1)$ is an extreme point of $B_{\mathcal{L}_s({}^2l_\infty)}$.

Suppose that $L = \frac{1}{2}(A+B)$ for some $A, B \in S_{\mathcal{L}_s({}^2l_\infty)}$. We may write

$$A\big((x_n),(y_n)\big) := \sum_{j=1}^{\infty} a_{jj}x_jy_j + \sum_{1\le k<s} a_{ks}(x_ky_s + x_sy_k)\,,$$

$$B\big((x_n),(y_n)\big) := \sum_{j=1}^{\infty} b_{jj}x_jy_j + \sum_{1\le k<s} b_{ks}(x_ky_s + x_sy_k)\,.$$

Since $L \in \operatorname{ext} B_{\mathcal{L}_s({}^2l_\infty^2)}$, we have $a_{11} = a_{12} = \frac{1}{2}$, $a_{22} = -\frac{1}{2}$. It suffices to that $a_{jj} = 0 = a_{1j} = a_{2j}$ for every $j \ge 3$. Let $j \ge 3$ be fixed. Since $1 \ge |A(e_1 + e_2, e_1 + e_2 \pm e_j)| = |1 \pm (a_{1j} + a_{2j})|$, we have $a_{1j} + a_{2j} = 0$. Since $1 \ge |A(e_1 + e_2 + e_j, e_1 + e_2 \pm e_j)| = |1 \pm a_{jj}|$, we have $a_{jj} = 0$. Since $1 \ge |A(e_1 + e_2 + e_j, e_1 - e_2 \pm e_j)| = |1 \pm 2a_{1j}|$, we have $a_{1j} = 0$, so $a_{2j} = 0$.

THEOREM 3. $\exp B_{\mathcal{L}_s({}^2l_\infty^2)} = \operatorname{ext} B_{\mathcal{L}_s({}^2l_\infty^2)}$.

Proof. It suffices to show that every extreme point of $B_{\mathcal{L}_s({}^2l_\infty^2)}$ is an exposed point. Note that $\{x_1y_1, x_2y_2, x_1y_2 + x_2y_1\}$ is a basis for $\mathcal{L}_s({}^2l_\infty^2)$.

Claim 1: $\pm x_1y_1, \pm x_2y_2 \in \exp B_{\mathcal{L}_s({}^2l_\infty^2)}$.

Let $f \in \mathcal{L}_s({}^2l_\infty^2)^*$ be such that

$$f(x_1y_1) = 1\,, \qquad f(x_2y_2) = 0 = f(x_1y_2 + x_2y_1)\,.$$

By Theorem 1, we have $\|f\| = 1 = f(x_1y_1)$. It is easy to show that f exposes x_1y_1.

Claim 2: $\pm\frac{1}{2}\left[x_1y_1 - x_2y_2 \pm (x_1y_2 + x_2y_1)\right] \in \exp B_{\mathcal{L}_s({}^2l_\infty^2)}$.

Let $f \in \mathcal{L}_s({}^2l_\infty^2)^*$ be such that

$$f(x_1y_1) = \frac{2}{3} = f(x_1y_2 + x_2y_1)\,, \qquad f(x_2y_2) = -\frac{2}{3}\,.$$

Clearly $f\big(\frac{1}{2}x_1y_1 - \frac{1}{2}x_2y_2 + \frac{1}{2}(x_1y_2 + x_2y_1)\big) = 1$. We will show that $\|f\| = 1$ and f exposes x_1y_1. By Theorem 1, it follows that

$$\begin{aligned}
\|f\| &= \sup\Big\{|f(ax_1y_1 + bx_2y_2 + c(x_1y_2 + x_2y_1))| \;: \\
&\qquad\qquad ax_1y_1 + bx_2y_2 + c(x_1y_2 + x_2y_1) \in B_{\mathcal{L}_s({}^2l_\infty^2)}\Big\} \\
&\le \frac{2}{3}\sup\Big\{|a-b| + |c| \;: \qquad\qquad\qquad\qquad (*)\\
&\qquad\qquad ax_1y_1 + bx_2y_2 + c(x_1y_2 + x_2y_1) \in B_{\mathcal{L}_s({}^2l_\infty^2)}\Big\} \\
&\le \frac{2}{3}\left(1 + \frac{1}{2}\right) = 1\,.
\end{aligned}$$

Suppose that $f(ax_1y_1 + bx_2y_2 + c(x_1y_2 + x_2y_1)) = 1$ for some $ax_1y_1 + bx_2y_2 + c(x_1y_2 + x_2y_1) \in B_{\mathcal{L}_s({}^2l_\infty^2)}$. By the argument in $(*)$, we have $|a-b| = 1$, $|c| = \frac{1}{2}$. By Theorem 1, we have $|a+b| = 0$, so $a = \frac{1}{2}$, $b = -\frac{1}{2}$, $c = \frac{1}{2}$. We complete the proof. ∎

THEOREM 4. *Let $f \in \mathcal{L}_s({}^2l_\infty^2)^*$. Then we have*

$$\|f\| = \max\left\{|\alpha|\,,\ |\beta|\,,\ \frac{1}{2}(|\alpha - \beta| + |\gamma|)\right\}\,,$$

where $\alpha = f(x_1y_1)$, $\beta = f(x_2y_2)$, $\gamma = f(x_1y_2 + x_2y_1)$.

Proof. By Theorem 2, we have

$$\begin{aligned}\|f\| &= \max\left\{|f(L)| \ : \ L \in \operatorname{ext} B_{\mathcal{L}_s({}^2l_\infty^2)}\right\} \\ &= \max\left\{|f(x_1y_1)|\,,\ |f(x_2y_2)|\,,\ \left|f\left(\frac{1}{2}x_1y_1 - \frac{1}{2}x_2y_2 \pm \frac{1}{2}(x_1y_2 + x_2y_1)\right)\right|\right\} \\ &= \max\left\{|\alpha|\,,\ |\beta|\,,\ \frac{1}{2}(|\alpha-\beta| + |\gamma|)\right\}.\end{aligned}$$

∎

Theorem 5. *We have*

$$\begin{aligned}&\operatorname{sm} B_{\mathcal{L}_s({}^2l_\infty^2)} = \\ &\Big\{ \pm\big[ax_1y_1 + (a-1)x_2y_2\big]\ (0 < a < 1)\,, \\ &\pm\left[ax_1y_1 + bx_2y_2 + \frac{1-(a+b)}{2}(x_1y_2 + x_2y_1)\right]\ (a > 0, b > 0, a+b < 1)\,, \\ &\pm\big[ax_1y_1 + (a-1)x_2y_2 + c(x_1y_2 + x_2y_1)\big]\ (|2a-1| + 2|c| < 1,\ c \neq 0)\Big\}.\end{aligned}$$

Proof. Claim 1: $\pm[ax_1y_1 + (a-1)x_2y_2] \in \operatorname{sm} B_{\mathcal{L}_s({}^2l_\infty^2)}$ $(0 < a < 1)$.

Let $L := ax_1y_1 + (a-1)x_2y_2$ $(0 < a < 1)$ and $f \in \mathcal{L}_s({}^2l_\infty^2)^*$ satisfying $f(L) = 1 = \|f\|$. Let $\alpha = f(x_1y_1)$, $\beta = f(x_2y_2)$, $\gamma = f(x_1y_2 + x_2y_1)$. It follows that

$$1 = f(L) = a\alpha + (a-1)\beta \le a|\alpha| + (1-a)|\beta| \le a + (1-a) = 1\,,$$

so $\alpha = 1$, $\beta = -1$. Since

$$1 = \|f\| = \max\left\{1, \frac{1}{2}(2 + |\gamma|)\right\} \ge \frac{1}{2}(2 + |\gamma|)\,,$$

so $\gamma = 0$. Thus f is uniquely determined.

Claim 2: $\pm[ax_1y_1 + (1-a)x_2y_2] \notin \operatorname{sm} B_{\mathcal{L}_s({}^2l_\infty^2)}$ $(0 \le a \le 1)$.

It follows that if we choose $f, g \in \mathcal{L}_s({}^2l_\infty^2)^*$ satisfying $f(x_1y_1) = 1 = f(x_2y_2)$, $f(x_1y_2 + x_2y_1) = 0$ and $g(x_1y_1) = 1 = g(x_2y_2) = g(x_1y_2 + x_2y_1)$.

Claim 3: $L := \pm\left[ax_1y_1 + bx_2y_2 \pm \frac{1-(a+b)}{2}(x_1y_2 + x_2y_1)\right] \in \operatorname{sm} B_{\mathcal{L}_s({}^2l_\infty^2)}$ $(a > 0,\ b > 0,\ a + b < 1)$.

Let $L := ax_1y_1 + bx_2y_2 \pm \frac{1-(a+b)}{2}(x_1y_2 + x_2y_1)$ and $f \in \mathcal{L}_s({}^2l_\infty^2)^*$ satisfying $f(L) = 1 = \|f\|$. Let $\alpha = f(x_1y_1)$, $\beta = f(x_2y_2)$, $\gamma = f(x_1y_2 + x_2y_1)$. It follows that

$$\begin{aligned} 1 = f(L) &= a\alpha + b\beta \pm \frac{1-(a+b)}{2}\gamma \\ &\le a|\alpha| + b|\beta| + \frac{1-(a+b)}{2}|\gamma| \\ &\le (a+b) + 2\,\frac{1-(a+b)}{2} = 1\,, \end{aligned}$$

so $\alpha = 1 = \beta$, $\gamma = \pm 2$. Thus f is uniquely determined.

Claim 4: $\pm[ax_1y_1 + (a-1)x_2y_2 + c(x_1y_2 + x_2y_1)] \notin \operatorname{sm} B_{\mathcal{L}_s({}^2l_\infty^2)}$ $(|2a-1| + 2|c| = 1)$.

Clearly $ab < 0$. We may assume that $a > 0$, $b < 0$, $c > 0$. If $a + b \geq 0$, the claim follows that if we choose $f, g \in \mathcal{L}_s({}^2l_\infty^2)^*$ satisfying $f(x_1y_1) = 1$, $f(x_2y_2) = -1$, $f(x_1y_2+x_2y_1) = 0$ and $g(x_1y_1) = 1 = g(x_2y_2)$, $g(x_1y_2+x_2y_1) = 2$. If $a + b < 0$, the claim follows that if we choose $f, g \in \mathcal{L}_s({}^2l_\infty^2)^*$ satisfying $f(x_1y_1) = 1$, $f(x_2y_2) = -1$, $f(x_1y_2 + x_2y_1) = 0$ and $g(x_1y_1) = -1 = g(x_2y_2)$, $g(x_1y_2 + x_2y_1) = 2$.

Claim 5: $\pm[ax_1y_1 + (a-1)x_2y_2 + c(x_1y_2 + x_2y_1)] \in \operatorname{sm} B_{\mathcal{L}_s({}^2l_\infty^2)}$ $(|2a-1| + 2|c| < 1,\ c \neq 0)$.

Let $L := ax_1y_1+(a-1)x_2y_2+c(x_1y_2+x_2y_1)$ for $0 < a < 1$, $|2a-1|+2|c| < 1$ and $f \in \mathcal{L}_s({}^2l_\infty^2)^*$ satisfying $f(L) = 1 = \|f\|$. Let $\alpha = f(x_1y_1)$, $\beta = f(x_2y_2)$, $\gamma = f(x_1y_2+x_2y_1)$. We will show that $\alpha = 1$, $\beta = -1$, $\gamma = 0$. We may assume that $\gamma \neq 0$. Then we will get a contradiction.

Case 1: $0 < a \leq \frac{1}{2}$.

Note that $|c| < a$. First, we claim that $\beta < 0$. Otherwise. Assume that $\beta > 0$. It follows that

$$\begin{aligned} 1 = \ & f(L) = a\alpha + (a-1)\beta + c\gamma \ \le \ a|\alpha| + (a-1)|\beta| + |c||\gamma| \\ < \ & a|\alpha| + (a-1)|\beta| + a|\gamma| \quad (\text{because } |c| < a,\ \gamma \neq 0) \\ \le \ & a\Big(|\alpha| + \Big(1 - \frac{1}{a}\Big)|\beta| + |\gamma|\Big) \\ \le \ & a(|\alpha| - |\beta| + |\gamma|) \quad \big(\text{because } 1 - \tfrac{1}{a} \le -1\big) \\ \le \ & a(|\alpha - \beta| + |\gamma|) \\ \le \ & 2a \ \ (\text{by Theorem 4}) \\ \le \ & 1\,, \end{aligned}$$

which is a contradiction. Thus $\beta \leq 0$. If $\beta = 0$, it follows that

$$\begin{aligned}
1 = f(L) &= a\alpha + c\gamma \\
&\leq a|\alpha| + |c||\gamma| \\
&< a|\alpha| + a|\gamma| \quad (\text{because } |c| < a,\ \gamma \neq 0) \\
&= a(|\alpha - \beta| + |\gamma|) \\
&\leq 2a \ \ (\text{by Theorem 4}) \\
&\leq 1\,,
\end{aligned}$$

which is a contradiction. Thus $\beta < 0$. We claim that $\alpha > 0$. Otherwise. Assume that $\alpha < 0$. If $|\alpha| \geq |\beta|$, it follows that

$$\begin{aligned}
1 = f(L) &= a\alpha + (a-1)\beta + c\gamma \\
&\leq -a|\alpha| + (1-a)|\beta| + |c||\gamma| \quad (\text{because } \beta > 0) \\
&< -a|\alpha| + (1-a)|\beta| + a|\gamma| \quad (\text{because } |c| < a,\ \gamma \neq 0) \\
&\leq -a|\alpha| + (1-a)|\beta| + a(2 - |\alpha| + |\beta|) \\
&\qquad\qquad (\text{because } |\alpha - \beta| + |\gamma| = |\alpha| - |\beta| + |\gamma| \leq 2) \\
&= 2a - 2a|\alpha| + |\beta| \\
&\leq 1 \ \ (\text{because } |\alpha| \geq |\beta|)\,,
\end{aligned}$$

which is a contradiction. Thus $\alpha \geq 0$. If $\alpha < 0$ and $|\alpha| \leq |\beta|$, it follows that

$$\begin{aligned}
1 = f(L) &= a\alpha + (a-1)\beta + c\gamma \\
&\leq -a|\alpha| + (1-a)|\beta| + |c||\gamma| \quad (\text{because } \beta > 0) \\
&< -a|\alpha| + (1-a)|\beta| + a|\gamma| \quad (\text{because } |c| < a,\ \gamma \neq 0) \\
&\leq -a|\alpha| + (1-a)|\beta| + a(2 + |\alpha| - |\beta|) \\
&\qquad\qquad (\text{because } |\alpha - \beta| + |\gamma| = |\beta| - |\alpha| + |\gamma| \leq 2) \\
&= 2a + (1-2a)|\beta| \\
&\leq 1\,,
\end{aligned}$$

which is a contradiction. Thus $\alpha \geq 0$. If $\alpha = 0$, it follows that

$$\begin{aligned}
1 = f(L) &= (a-1)\beta + c\gamma \\
&\leq (1-a)|\beta| + |c||\gamma| \\
&< (1-a)|\beta| + a|\gamma| \quad (\text{because } |c| < a,\ \gamma \neq 0) \\
&\leq (1-a)|\beta| + a(2-|\beta|) \quad (\text{because } |\alpha-\beta| + |\gamma| = |\beta| + |\gamma| \leq 2) \\
&\leq 2a + (1-2a)|\beta| \\
&\leq 1\,,
\end{aligned}$$

which is a contradiction. Thus $\alpha > 0$. By Theorem 4,

$$\|f\| = 1 = \frac{1}{2}\big(|\alpha-\beta| + |\gamma|\big) = \frac{1}{2}\big(|\alpha| + |\beta| + |\gamma|\big)\,.$$

It follows that

$$\begin{aligned}
1 = f(L) &= a\alpha + (a-1)\beta + c\gamma \\
&\leq a|\alpha| + (1-a)|\beta| + |c||\gamma| \\
&< a|\alpha| + (1-a)|\beta| + a|\gamma| \quad (\text{because } |c| < a,\ \gamma \neq 0) \\
&= a|\alpha| + (1-a)|\beta| + a(2-|\alpha|-|\beta|) \quad (\text{because } |\alpha| + |\beta| + |\gamma| = 2) \\
&\leq 2a + (1-2a)|\beta| \\
&\leq 1\,,
\end{aligned}$$

which is a contradiction. Thus $\gamma = 0$. It follows that

$$1 = f(L) = a\alpha + (a-1)\beta \leq a|\alpha| + (1-a)|\beta| \leq 1\,,$$

which implies that $\alpha = 1$, $\beta = -1$, which complete the proof of Case 1.

Case 2: $\frac{1}{2} < a < 1$.

Note that $|c| < 1-a$. First, we claim that $\beta < 0$. Otherwise. Assume that $\beta > 0$. It follows that

$$\begin{aligned}
1 = f(L) &= a\alpha + (a-1)\beta + c\gamma \\
&\leq a|\alpha| - (1-a)|\beta| + |c||\gamma| \\
&< a|\alpha| - (1-a)|\beta| + (1-a)|\gamma| \quad (\text{because } |c| < 1-a,\ \gamma \neq 0)
\end{aligned}$$

$$\begin{aligned}
&\leq a|\alpha| - (1-a)|\beta| + (1-a)(2-|\alpha-\beta|)\\
&\leq a|\alpha| - (1-a)(|\beta| - 2 + |\alpha-\beta|)\\
&\leq a|\alpha| - (1-a)(|\beta| - 2 + |\alpha| - |\beta|)\\
&= a|\alpha| - (1-a)(-2+|\alpha|)\\
&= a|\alpha| + 2(1-a) - (1-a)|\alpha|\\
&= (2a-1)|\alpha| + 2(1-a)\\
&\leq (2a-1) + 2(1-a)\\
&= 1\,,
\end{aligned}$$

which is a contradiction. Thus $\beta \leq 0$. If $\beta = 0$, it follows that

$$\begin{aligned}
1 = f(L) &= a\alpha + c\gamma\\
&\leq a|\alpha| + |c||\gamma|\\
&< a|\alpha| + (1-a)|\gamma| \quad (\text{because } |c| < 1-a,\ \gamma \neq 0)\\
&= (2a-1)|\alpha| + (1-a)(|\alpha| + |\gamma|)\\
&= (2a-1)|\alpha| + 2(1-a)\\
&\leq 1\,,
\end{aligned}$$

which is a contradiction. Thus $\beta < 0$. We claim that $\alpha > 0$. Otherwise. Assume that $\alpha < 0$. If $|\alpha| \geq |\beta|$, it follows that

$$\begin{aligned}
1 = f(L) &= a\alpha + (a-1)\beta + c\gamma\\
&\leq -a|\alpha| + (1-a)|\beta| + |c||\gamma| \quad (\text{because } \beta > 0)\\
&< -a|\alpha| + (1-a)|\beta| + (1-a)|\gamma| \quad (\text{because } |c| < 1-a,\ \gamma \neq 0)\\
&\leq -a|\alpha| + (1-a)|\beta| + (1-a)(2 - |\alpha| + |\beta|)\\
&\qquad\qquad (\text{because } |\alpha-\beta| + |\gamma| = |\alpha| - |\beta| + |\gamma| \leq 2)\\
&= 2(1-a) - |\alpha| + 2(1-a)|\beta|\\
&\leq 1 \ \ (\text{because } |\alpha| \geq |\beta|)\,,
\end{aligned}$$

which is a contradiction. Thus $\alpha \geq 0$. If $\alpha < 0$ and $|\alpha| \leq |\beta|$, it follows that

$$\begin{aligned}
1 = f(L) &= a\alpha + (a-1)\beta + c\gamma \\
&\le -a|\alpha| + (1-a)|\beta| + |c||\gamma| \quad (\text{because } \beta > 0) \\
&< -a|\alpha| + (1-a)|\beta| + (1-a)|\gamma| \quad (\text{because } |c| < 1-a,\ \gamma \ne 0) \\
&\le -a|\alpha| + (1-a)|\beta| + (1-a)(2 + |\alpha| - |\beta|) \\
&\qquad\qquad\qquad (\text{because } |\alpha-\beta| + |\gamma| = |\beta| - |\alpha| + |\gamma| \le 2) \\
&= 2(1-a) + (1-2a)|\alpha| \\
&\le 2(1-a) \\
&\le 1\,,
\end{aligned}$$

which is a contradiction. Thus $\alpha \ge 0$. If $\alpha = 0$, it follows that

$$\begin{aligned}
1 = f(L) &= (a-1)\beta + c\gamma \\
&\le (1-a)|\beta| + |c||\gamma| \\
&< (1-a)|\beta| + (1-a)|\gamma| \quad (\text{because } |c| < 1-a,\ \gamma \ne 0) \\
&\le (1-a)(|\beta| + |\gamma|) \\
&= (1-a)(|\alpha-\beta| + |\gamma|) \\
&\le 2(1-a) \\
&\le 1\,,
\end{aligned}$$

which is a contradiction. Thus $\alpha > 0$. By Theorem 4,

$$\|f\| = 1 = \frac{1}{2}\big(|\alpha-\beta| + |\gamma|\big) = \frac{1}{2}\big(|\alpha| + |\beta| + |\gamma|\big)\,.$$

It follows that

$$\begin{aligned}
1 = f(L) &= a\alpha + (a-1)\beta + c\gamma \\
&\le a|\alpha| + (1-a)|\beta| + |c||\gamma| \\
&< a|\alpha| + (1-a)|\beta| + (1-a)|\gamma| \quad (\text{because } |c| < 1-a,\ \gamma \ne 0) \\
&= (1-a)(|\alpha| + |\beta| + |\gamma|) + (2a-1)|\alpha| \\
&= (1-a)(|\alpha-\beta| + |\gamma|) + (2a-1)|\alpha| \\
&\le 2(1-a) + (2a-1)|\alpha| \\
&\le 1\,,
\end{aligned}$$

which is a contradiction. Thus $\gamma = 0$. It follows that

$$1 = f(L) = a\alpha + (a-1)\beta \le a|\alpha| + (1-a)|\beta| \le 1\,,$$

which implies that $\alpha = 1$, $\beta = -1$, which complete the proof of Case 2. Therefore, we complete the proof. ∎

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