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Quillen-Suslin Rings

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Abstract: In this paper we introduce the *Quillen-Suslin rings* and investigate its relation with some other classes of rings as Hermite rings (each stably free module is free), *PSF* rings (each finitely generated projective module is stably free), *PF* rings (each finitely generated projective module is free), etc. Quillen-Suslin rings are induced by the famous Serre's problem formulated by J.P. Serre in 1955 ([30]) and solved independently by Quillen ([28]) and Suslin ([31]) in 1976. The solution is known as *the Quillen-Suslin theorem* and states that every finitely generated projective module over the polynomial ring $K[x_1, \ldots, x_n]$ is free, where K is a field. There are algorithmic proofs and some generalizations of this important theorem that we will also study in this paper. In particular, we will consider extended modules and rings, and the Bass-Quillen conjecture.

Key words: Quillen-Suslin theorem, Hermite rings, extended modules and rings, Bass-Quillen conjecture.

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