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Topics in Linear Isometries of Function Algebras†

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Abstract: After a brief presentation of Banach-Stone's and Kahane-Zelazko's results on weighted composition operators, the attention is concentrated on the iteration of surjective and non-surjective linear isometries acting on complex Banach algebras. Appealing to results by Grothendieck, Bourgain and Lotz, the set of iterates is replaced by a strongly continuous semigroup of linear isometries.

Key words: Character, composition operator, Gelfand spectrum, strongly continuous semigroup.

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Introduction

Weighted composition operators appear in two different contexts in the study of complex Banach algebras or, more in general, of locally multiplicatively convex, sequentially complete, associative algebras.

In the first one, originating in the ground-breaking research work carried out by Stefan Banach and Marshall H. Stone in the Thirties, they are related to linear isometries between uniform algebras. More than thirty years later, some results obtained by A. M. Gleason and by J.-P. Kahane and W. Zelazko characterized a class of weighted composition operators between unital commutative Banach algebras mapping invertible elements to invertible elements.

The beginning of these lecture-notes is devoted to illustrating the interplay between the two approaches, before concentrating on linear isometries of uniform algebras, extending to a class of non-surjective isometries classical results established by N. Nagasawa and by K. de Leeuw - W. Rudin - J. Wermer for surjective ones.

The iteration of linear isometries will be investigated in some relevant examples: the uniform algebra of all continuous functions on a compact Haus-

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dorff space, the disc algebra $\mathcal{A}^0$ and the algebra H^∞ of all bounded holomorphic functions on the open unit disc Δ of $\mathbb{C}$. In the latter two cases, the Wolf-Denjoy theorem on the iteration of holomorphic self-maps of Δ plays a dual role, both as a research tool and as a model of what the outcome of the iteration should be.

Stepping from discrete to continuum replaces the iteration process by a strongly continuous semigroup T of linear isometries, which in the cases of $\mathcal{A}^0$ and of H^∞ is associated to a continuous flow of holomorphic self-maps of Δ, the *conformal flow* of T.

Since, according to a result by J. Bourgain, H^∞ is a Grothendieck space with the Dunford-Pettis property, T, as any strongly continuous semigroup of linear isometries acting on H^∞, is the restriction to $\mathbb{R}_+$ of a uniformly continuous group of surjective isometries of H^∞, whose conformal flow is a group of Moebius transformations of Δ. As a consequence, it turns out that the group T is almost periodic if, and only if, its conformal flow fixes a point of Δ. This result may be extended to the semigroup T acting on the disc algebra $\mathcal{A}^0$ by a direct inspection of the spectral structure of T.

1. Composition operators and weighted composition operators

Let X be a compact Hausdorff space, and let $C(X)$ denote the algebra of all continuous functions $f : X \to \mathbb{C}$ with the uniform norm

$$\|f\| = \sup\{|f(x)| : x \in X\}$$

and pointwise composition.

A uniform algebra on X is a closed subalgebra of $C(X)$ which contains the constants and separate points of X.

Let $\mathcal{A}$ and $\mathcal{B}$ be uniform algebras on two compact Hausdorff spaces X and Y.

A *composition operator* $C \in \mathcal{L}(\mathcal{A}, \mathcal{B})$ is a continuous linear map represented by

$$C : f \mapsto f \circ \phi$$

for all $f \in \mathcal{A}$, where ϕ is a continuous map of Y into X.

A *weighted composition operator* $A \in \mathcal{L}(\mathcal{A}, \mathcal{B})$ is a multiplicative perturbation of C expressed by

$$A : f \mapsto u.f \circ \phi,$$

where u is a fixed element of $\mathcal{B}$.

Weighted composition operators appear in different contexts of research.

One of these concerns linear operators acting on commutative unital Banach algebras, mapping invertible elements to invertible elements, and is an application of the following result due to A. Gleason, [14] and J.-P. Kahane and W. Zelazko ([20], but see also [11], [30] and [37]).

Let $\mathcal{A}$ be a unital algebra (which as all algebras considered in these notes, will be assumed to be associative), and let $\lambda : x \mapsto \langle x, \lambda \rangle$ be a homomorphism of the vector space $\mathcal{A}$ into $\mathbb{C}$ such that

$$\langle 1_{\mathcal{A}}, \lambda \rangle = 1. \tag{1.1}$$

The proof of the following lemma can be found, *e.g.*, in [30] (or in [37]).

LEMMA 1.1. *If* (1.1) *holds, and if*

$$x \in \ker \lambda \Longrightarrow x^2 \in \ker \lambda,$$

then

$$\langle xy, \lambda \rangle = \langle x, \lambda \rangle \, \langle y, \lambda \rangle \tag{1.2}$$

for all $x, y \in \mathcal{A}$.

Let the algebra $\mathcal{A}$ considered above be now endowed with a topology: let $\mathcal{A}$ be a unital Banach algebra, or, more in general, a complex, unital, locally mutiplicatively convex, sequentially complete algebra, [24].

The topology of $\mathcal{A}$ is defined by a basis $\{U_\alpha\}$ for the neighbourhoods of 0 consisting of open, convex, symmetric sets containing 0 such that $U_\alpha \, U_\alpha \subset U_\alpha$ for any index α. Setting, for $x \in U_\alpha$,

$$p_\alpha(x) = \inf\{t > 0 : x \in tU_\alpha\},$$

p_α is a continuous seminorm on $\mathcal{A}$ which defines the topology of $\mathcal{A}$ and is such that

$$p_\alpha(x^n) \leq (p_\alpha(x))^n \quad \forall \, x \in \mathcal{A}, \;\; n = 1, 2, \ldots .$$

If the linear form λ considered above is continuous - *i.e.*, if $\lambda \in \mathcal{A}'$, the topological dual of the topological vector space $\mathcal{A}$, - for any index α there is a real constant $c_\alpha \geq 0$ such that

$$|\langle x, \lambda \rangle| \leq c_\alpha \, p_\alpha(x) \quad \forall \, x \in \mathcal{A}.$$

Under the hypotheses of Lemma 1.1, $\ker \lambda$ is a closed right ideal of $\mathcal{A}$.

Denoting by $\mathcal{A}^{-1}$ the set of all invertible elements of $\mathcal{A}$, then

$$\exp(zx) = \sum_{n=0}^{+\infty} \frac{1}{n!} z^n x^n \in \mathcal{A}^{-1} \quad \forall\, z \in \mathbb{C},\, x \in \mathcal{A}. \tag{1.3}$$

The entire function $f : \mathbb{C} \ni z \mapsto \langle \exp(zx), \lambda \rangle$ is such that, for any α,

$$\begin{aligned} |f(z)| &\leq \sum_{n=0}^{+\infty} \frac{|z|^n}{n!} \, |\langle x^n, \lambda \rangle| \leq 1 + c_\alpha \sum_{n=1}^{+\infty} \frac{|z|^n}{n!} p_\alpha(x^n) \\ &\leq 1 + c_\alpha \sum_{n=1}^{+\infty} \frac{(|z| p_\alpha(x))^n}{n!} \leq \max\{1, c_\alpha\} \mathrm{e}^{|z|\, p_\alpha(x)}. \end{aligned} \tag{1.4}$$

By (1.3), if

$$\mathcal{A}^{-1} \cap \ker \lambda = \emptyset, \tag{1.5}$$

then $f(z) \neq 0$ for all $z \in \mathbb{C}$.

The proof of the following result from complex function theory, can be found, *e.g.*, in [30] (or in [37]),

LEMMA 1.2. *Let h be an entire function. If $h(0) = 1$, if $h'(0) = 0$ and if there are two real numbers a and $b \geq 1$ such that*

$$0 < |h(z)| < b\, \mathrm{e}^{a|z|} \quad \forall\, z \in \mathbb{C},$$

then $h \equiv 1$.

If $\langle x, \lambda \rangle = 0$, and therefore $f'(0) = 0$, Lemma 1.2 implies that $f(z) = 1$ for all $z \in \mathbb{C}$. As a consequence,

$$\langle x^2, \lambda \rangle = f''(0) = 0,$$

and therefore, by Lemma 1.1, (1.1) and (1.2) hold.

In conclusion the following theorem holds:

THEOREM 1.1. *Let $\mathcal{A}$ be a unital, locally multiplicatively convex and sequentially complete complex algebra and let λ be a continuous linear form on $\mathcal{A}$. If $\ker \lambda$ contains no invertible element of $\mathcal{A}$, there is a continuous character χ of $\mathcal{A}$*[1] *such that*

$$\langle x, \lambda \rangle = \langle 1_{\mathcal{A}}, \lambda \rangle \, \langle x, \chi \rangle \quad \forall\, x \in \mathcal{A}.$$

In particular, if (1.1) holds, λ is a character of $\mathcal{A}$.

[1] *i.e.* a continuous homomorphism of the algebra $\mathcal{A}$ into $\mathbb{C}$.

Let $\sigma(x) \subset \mathbb{C}$ be the spectrum of $x \in \mathcal{A}$. Theorem 1.1 yields:

Corollary. *If the continuous linear form* λ *on* $\mathcal{A}$ *is such that*

$$\langle x, \lambda \rangle \in \sigma(x) \quad \forall\, x \in \mathcal{A},$$

and satisfies (1.1), *then* λ *is a character.*

Let now $\mathcal{A}$, $\mathcal{B}$ be unital, abelian Banach algebras, let $\Sigma(\mathcal{A})$, $\Sigma(\mathcal{B})$ be the Gelfand spectra of $\mathcal{A}$, $\mathcal{B}$, and let $\mathcal{L}(\mathcal{A}, \mathcal{B})$ be the Banach space of all continuous linear maps of the Banach space $\mathcal{A}$ into the Banach space $\mathcal{B}$.

If $A \in \mathcal{L}(\mathcal{A}, \mathcal{B})$ is such that

$$A\left(\mathcal{A}^{-1}\right) \subset \mathcal{B}^{-1}, \tag{1.6}$$

then, for any character $\chi \in \Sigma(\mathcal{B})$,

$$f \in \mathcal{A}^{-1} \Longrightarrow \langle Af, \chi \rangle \neq 0.$$

Since, by Theorem 1.1

$$\mathcal{A} \ni f \mapsto \frac{\langle Af, \chi \rangle}{\langle A1, \chi \rangle}$$

is a character, $\phi(\chi)$, of $\mathcal{A}$, the following theorem holds.

Theorem 1.2. *For any* $A \in \mathcal{L}(\mathcal{A}, \mathcal{B})$ *satisfying* (1.6) *there is a continuous map* $\phi : \Sigma(\mathcal{B}) \to \Sigma(\mathcal{A})$ *such that*

$$\langle Af, \chi \rangle = \langle A1_{\mathcal{A}}, \chi \rangle . \langle f, \phi(\chi) \rangle \quad \forall\, f \in \mathcal{A},\ \chi \in \Sigma(\mathcal{B}).$$

For an extension of this theorem to non-unital algebras, see [38].

2. Weighted composition operators and isometries

Much earlier than the Gleason-Kahane-Zelazko papers, weighted composition operators made their appearance in the context of linear isometries between the uniform algebras $C(X)$ and $C(Y)$ of all continuous scalar valued functions on two compact Hausdorff spaces X and Y.

If X and Y are homeomorphic, $C(X)$ and $C(Y)$ are isometricaly isomorphic. An answer to the opposite question is provided by the classical Banach-Stone theorem, [2], [33], [11]:

Theorem 2.1. *If $A \in \mathcal{L}(C(X), C(Y))$ is a surjective isometry, then A is represented by the weighted composition operator*

$$Af = u\,.\,f \circ \phi \quad \forall\, f \in C(X),$$

where ϕ is a homeomorphism of Y onto X, and $u \in C(Y)$ is a unimodular function.

As a consequence, A maps unimodular functions to unimodular functions.

A link between Banach-Stone's and Gleason-Kahane-Zelazko's approaches is offered by the following lemma.

Lemma 2.1. *If $A \in \mathcal{L}(C(X), C(Y))$ is an isometry of $C(X)$ into $C(Y)$ mapping unimodular functions in $C(X)$ to unimodular functions in $C(Y)$, then*

$$A\left(C(X)^{-1}\right) \subset C(Y)^{-1}.$$

Proof. Let $f \in C(X)^{-1}$ and let $u = f/|f|$. Then

$$\left|u - \frac{f}{\|f\|}\right| < 1 \quad \Rightarrow \quad |u\,\|f\| - f| < \|f\| \quad \Rightarrow \quad \|u\,\|f\| - f\| < \|f\|.$$

Let $y \in Y$ be such that $(Af)(y) = 0$. Then

$$\|f\|\,(Au)(y) = (A(\|f\|\,u - f))\,(y),$$

and therefore

$$\|f\| = \left|\big(A(\|f\|\,u - f)\big)(y)\right| \leq \big\|A(\|f\|\,u - f)\big\| = \big\|\,\|f\|\,u - f\big\| < \|f\|.$$

Contradiction. ∎

Surjective linear isometries between two uniform algebras $\mathcal{A}$ and $\mathcal{B}$ are described by the following theorem, [1], [8], [26], [17].

Theorem 2.2. *If $A \in \mathcal{L}(\mathcal{A}, \mathcal{B})$ is a surjective isometry, there exist an algebra homomorphism C of $\mathcal{A}$ onto $\mathcal{B}$ and a unimodular function $u \in \mathcal{B}^{-1}$ such that*

$$Af = u\,.\,Cf \quad \forall\, f \in \mathcal{A}.$$

One of the main purposes of these Notes is that of describing C under suitable hypotheses on the uniform algebras $\mathcal{A}$ and $\mathcal{B}$.

3. A theorem by A. Bernard, and some examples

Let: $\mathcal{A}$ be a uniform algebra on a compact Hausdorff space X; $B = B(\mathcal{A})$, be the open unit ball of $\mathcal{A}$; $\mathfrak{U}(\mathcal{A})$ be the set of all unimodular functions in $\mathcal{A}$:

$$\mathfrak{U}(\mathcal{A}) = \{u \in \mathcal{A} : |u(x)| = 1 \quad \forall\, x \in X\}.$$

Theorem 3.1. (A. Bernard) *If $\mathfrak{U}(\mathcal{A})$ generates $\mathcal{A}$, the closure $\overline{B}$ of B is the closed convex hull of $\mathfrak{U}(\mathcal{A})$.*

Proof. [13] The set of all finite linear combinations of elements of $\mathfrak{U}(\mathcal{A})$ is dense in $\mathcal{A}$. Let n be a positive integer, $f = \sum_{n=1}^{N} c_n u_n$, with $c_n \in \mathbb{C}$, $u_n \in \mathfrak{U}(\mathcal{A})$ and $\|f\|_{\mathcal{A}} < 1$. Then $u = u_1 \cdots u_N \in \mathfrak{U}(\mathcal{A})$, and

$$\left| \frac{f(x) + \mathrm{e}^{i\theta} u(x)}{1 + \mathrm{e}^{i\theta} u(x) \overline{f(x)}} \right| = 1 \quad \forall\, x \in X,\ \theta \in \mathbb{R},$$

$$\frac{f(x) + \mathrm{e}^{i\theta} u(x)}{1 + \mathrm{e}^{i\theta} u(x) \overline{f(x)}} = f(x) + \big(u(x) - |f(x)|^2\big)\mathrm{e}^{i\theta} - u(x)^2 f(x)\big(u(x) - |f(x)|^2\big)\mathrm{e}^{2i\theta} + \cdots$$

uniformly with respect to θ. Hence

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{f(x) + \mathrm{e}^{i\theta} u(x)}{1 + \mathrm{e}^{i\theta} u(x) \overline{f(x)}} d\theta = f(x)$$

and there exists a sequence of Riemann sums converging uniformly to f, *i.e.* a sequence of finite convex combinations of elements of $\mathfrak{U}(\mathcal{A})$ converging uniformly to f. ∎

We will be concerned with uniform algebras generated by their unimodular functions. Here are some relevant examples.

3.1. The algebra $C(X)$

Theorem 3.2. (R. Phelps) *The closed unit ball $\overline{B(C(X))}$ of $C(X)$ is the closed convex hull of $\mathfrak{U}(C(X))$.*

Proof. (R. B. Burckel) Let $f \in C(X)$, with $\|f\| < 1$, and let f_θ be defined by

$$f_\theta(x) = \frac{f(x) - w}{1 - w\overline{f(x)}},$$

where $x \in X$, $\theta \in \mathbb{R}$ and $w = \mathrm{e}^{2\pi i\theta}$. Then $f_\theta \in \mathfrak{U}(C(X))$, and

$$f_\theta(x) = f(x) + \big(|f(x)|^2 - 1\big) \sum_{n=1}^{+\infty} w^n \overline{f(x)}^{n-1}.$$

Choose now $\theta = k/m$ with k and m positive integers and $k \leq m$. Then

$$\begin{aligned}\frac{1}{m}\sum_{k=1}^{m} f_{k/m}(x) &= f(x) + \big(|f(x)|^2 - 1\big) \sum_{n=1}^{+\infty} \left(\frac{1}{m}\sum_{k=1}^{m} \mathrm{e}^{\frac{2\pi k i}{m}}\right)^n \overline{f(x)}^{n-1} \\ &= f(x) + \big(|f(x)|^2 - 1\big) \sum_{n=m}^{+\infty} \left(\frac{1}{m}\sum_{k=1}^{m} \mathrm{e}^{\frac{2\pi k i}{m}}\right)^n \overline{f(x)}^{n-1},\end{aligned}$$

because

$$\sum_{k=1}^{m} \left(\frac{1}{n}\mathrm{e}^{\frac{2\pi k i}{m}}\right)^n = \sum_{k=1}^{m} \left(\frac{1}{n}\mathrm{e}^{\frac{2\pi n i}{m}}\right)^k,$$

and

$$\sum_{k=1}^{m} \left(\frac{1}{n}\mathrm{e}^{\frac{2\pi n i}{m}}\right)^k = 1 \text{ or } 0$$

according as m divides n or not. Then,

$$\begin{aligned}\left|\frac{1}{m}\sum_{k=1}^{m} f_{k/m}(x) - f(x)\right| &\leq \Big(1 - |f(x)|^2\Big) \sum_{n=m}^{+\infty} |f(x)|^{n-1} \\ &= \Big(1 - |f(x)|^2\Big) |f(x)|^{m-1} \sum_{n=0}^{+\infty} |f(x)|^n \\ &= \frac{1 - |f(x)|^2}{1 - |f(x)|} |f(x)|^{m-1} = \big(1 + |f(x)|\big)\, |f(x)|^{m-1} \\ &\leq 2\,|f(x)|^{m-1} \leq 2\|f\|^{m-1} \quad \forall\, x \in X.\end{aligned}$$

Since $\|f\| < 1$, f is the uniform limit of the convex combinations

$$\frac{1}{m}\sum_{k=1}^{m} f_{k/m}(x)$$

as $m \to +\infty$. ∎

The above proof, given by R. Burckel in [6], is an adaptation of an elegant proof, due to L. A. Harris [15], of a theorem by B. Russo and H. Dye [31] - which may be seen as a non-commutative counterpart of Phelps' theorem - whereby the closure of the unit ball B of any unital C^*-algebra $\mathcal{A}$ is the closed convex hull of the set $\mathfrak{U}$ of all unitary elements in the algebra.

Here is an outline of Harris' proof:

Any $x \in B$ defines the Möbius transformation T_x mapping $y \in B$ to

$$T_x(y) = (1 - x\,x^*)^{-1/2}(y + x)(1 + x^* y)^{-1}(1 - x^*\,x)^{1/2}.$$

T_x is a bi-holomorphic map of B onto B, such that

$$T_x(0) = x, \tag{3.1}$$

which has a holomorphic extension to a neighbourhood of the closure $\overline{B}$ of B, [16].

The function f defined by

$$f : z \mapsto f(z) = T_x(z1)$$

is holomorphic in a neighbourhood of $\overline{\Delta}$, maps the unit circle $\partial\Delta$ into the connected component $\mathfrak{U}_1$ of the identity $1_{\mathcal{A}}$ in $\mathfrak{U}$.

By the Cauchy formula (for the theory of operator-valued holomorphic functions, see, *e.g.*, [12]) and by (3.1),

$$x = f(0) = \frac{1}{2\pi}\int_0^{2\pi} f\left(\mathrm{e}^{i\theta}\right) d\theta.$$

Thus, x is the uniform limit of Riemann sums, which turn out to be convex combinations of elements of the identity component of $1_{\mathcal{A}}$ in $\mathfrak{U}$.

A different proof of the Russo-Dye theorem, which does not use function theory, can also be found in [15].

3.2. The disc algebra The disc algebra $\mathcal{A}^0$ is defined by

$$\mathcal{A}^0 = \left\{ f \in C(\partial\Delta) : \int_0^{2\pi} \mathrm{e}^{in\theta} f\left(\mathrm{e}^{i\theta}\right) d\theta = 0 \ \text{ for } \ n = 1, 2, \dots \right\}. \tag{3.2}$$

Any $f \in \mathcal{A}^0$ can be continuously extended as a holomorphic function on the unit disc Δ *via* Poisson integral

$$f(z) = \int_{-\pi}^{\pi} f\left(\mathrm{e}^{it}\right) dm_z(t)$$

where $z = re^{i\theta}$, $0 < r < 1$, and

$$dm_z(t) = \frac{1}{2\pi} \operatorname{Re} \frac{e^{it} + z}{e^{it} - z}\, dt = \frac{1}{2\pi} \frac{1 - r^2}{1 - 2r\cos(t-\theta) + r^2}\, dt. \tag{3.3}$$

The Poisson integral yields an alternate definition of the disc algebra:

$$\mathcal{A}^0 = \left\{ f \in C(\overline{\Delta}) : f_{|\Delta} \text{holomorphic} \right\},$$

and shows that *the disc algebra is a Dirichlet algebra* [2].

Let $\iota \in \mathcal{A}^0$ be the *coordinate function*: $\iota : z \mapsto \iota(z) = z$.

Let $\Sigma(\mathcal{A}^0)$ be the Gelfand spectrum of $\mathcal{A}^0$ endowed with the Gelfand topology.

If $\chi \in \Sigma(\mathcal{A}^0)$, then $\langle \iota, \chi \rangle \in \overline{\Delta}$.

If p is an analytic polynomial of degree N,

$$p = \sum_{n=0}^{N} c_n \iota^n, \quad \text{with } c_n \in \mathbb{C},$$

and if $\chi \in \Sigma(\mathcal{A}^0)$, then

$$\langle p, \chi \rangle = \sum_{n=0}^{N} c_n \langle \iota^n, \chi \rangle = \sum_{n=0}^{N} c_n \langle \iota, \chi \rangle^n = p(\langle \iota, \chi \rangle).$$

Since analytic polynomials are dense in $\Sigma(\mathcal{A}^0)$, then

$$\langle f, \chi \rangle = f(\langle \iota, \chi \rangle) \quad \forall\, \chi \in \Sigma(\mathcal{A}^0)$$

and for any $f \in \mathcal{A}^0$.

PROPOSITION 3.1. $\Sigma(\mathcal{A}^0) = \overline{\Delta}$. *The Gelfand topology is the relative topology in* $\mathbb{C}$ [3]. *The Shilov boundary* $\partial\mathcal{A}^0$ *of* $\mathcal{A}^0$ *is the unit circle* $\partial\Delta$.

Every character χ has a unique representing measure, *i.e.* a unique probability measure whose support is contained in $\partial\mathcal{A}^0$. The representing measure of $z \in \Delta$ is given by the Poisson kernel (3.3); if $z \in \partial\Delta$, the representing measure is the Dirac measure δ_z.

Let E be a closed subset of $\partial\Delta$ with Lebesgue measure zero.

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[2] A uniform algebra on a compact Husdorff space X is, by definition, a Dirichlet algebra if $\operatorname{Re}\mathcal{A} = \{\operatorname{Re} f : f \in \mathcal{A}\}$ is dense in $C_{\mathbb{R}}(X)$.

[3] Alternate proof: the algebra $\mathcal{A}^0$ is generated by ι. Therefore $\Sigma(\mathcal{A}^0)$ is the spectrum of ι: $\sigma(\iota) = \overline{\Delta}$.

THEOREM 3.3. (P. Fatou, [10], [17]) *There exists a function in $\mathcal{A}^0$ which vanishes precisely on E.*

THEOREM 3.4. (W. Rudin, [28], [17]) *Let F be a continuous complex-valued function on E. There exists a function $f \in \mathcal{A}^0$ whose restriction to E is F. Moreover, f can be so chosen that $|f|$ is bounded on Δ by the maximum of $|F|$ on E.*

3.3. BLASCHKE PRODUCTS Let $\{z_n\}$ be a sequence in Δ such that $\sum(1-|z_n|) < \infty$, and let m be the number of times 0 occurs in the sequence. Then the Blaschke product

$$B(z) = z^m \prod_{z_n \neq 0} \frac{-\overline{z_n}}{|z_n|} \frac{z - z_n}{1 - \overline{z_n} z}$$

converges at all $z \in \Delta$, thus defining a bounded holomorphic function on Δ, whose zeros are the points z_n with multiplicities equal to the number of times they occur in the sequence $\{z_n\}$.

Furthermore $|B(z)| \leq 1$ at all $z \in \Delta$, and

$$|B(\mathrm{e}^{i\theta}| = 1 \tag{3.4}$$

a.e. on $\partial\Delta$.

Let $K \subset \partial\Delta$ be the set of all accumulation points of $\{z_n\}$. The Blaschke product B extends analytically on the complement of $K \cup \{1/\overline{z_n}\}$ in $\mathbb{C}$ and across each arc of $\partial\Delta \backslash K$, but $|B|$ does not extend continuosly from Δ to any point of K.

Hence

$$B \in \mathcal{A}^0 \Longleftrightarrow \{z_n\} \text{ is finite,}$$

in which case (3.4) holds for all $\theta \in \mathbb{R}$.

Since polynomials are dense in $\mathcal{A}^0$, the closed unit ball of $\mathcal{A}^0$ is the closed convex hull of the set of finite Blaschke products; [13].

3.4. BOUNDED HOLOMORPHIC FUNCTIONS The set H^∞ of all bounded holomorphic functions on Δ is a unital, commutative Banach algebra for pointwise composition in Δ and norm

$$\|f\| = \sup\{|f(z)| : z \in \Delta\}.$$

Let $L^\infty = L^\infty(\partial\Delta)$ be the space of all complex-valued, bounded, measurable functions on $\partial\Delta$, where functions equal to each other almost everywhere for the Lebesgue measure are identified. Then $L^\infty(\partial\Delta)$ is a commutative unital C^* algebra for pointwise composition and for the essential sup norm. Hence $L^\infty(\partial\Delta)$ can be identified with $C(M)$, where $M = \Sigma(L^\infty) = \partial L^\infty$, endowed with the Gelfand topology, is a compact Hausdorff space which turns out to be totally disconnected[4]. For any $f \in H^\infty$ there exists a unique $F \in L^\infty(\partial\Delta)$ such that

$$F\left(\mathrm{e}^{i\theta}\right) = \lim_{r\uparrow 1} f\left(r\mathrm{e}^{i\theta}\right) \quad a.e..$$

Furthermore, $\|F\| = \|f\|$, and

$$f(z) = \int_{-\pi}^{\pi} F\left(\mathrm{e}^{it}\right) dm_z(t), \tag{3.5}$$

where dm_z is given by (3.3) for all $z \in \Delta$.

If $F \in L^\infty(\partial\Delta)$ is such that

$$\int_{-\pi}^{\pi} \mathrm{e}^{in\theta} F\left(\mathrm{e}^{i\theta}\right) d\theta = 0 \qquad n = 1, 2, \ldots,$$

(3.5) defines $f \in H^\infty$.

The algebra H^∞ is a closed subalgebra of L^∞. The restriction map $\Sigma(L^\infty) \to \Sigma(H^\infty)$ defines a homeomorphism of $M = \Sigma(L^\infty)$ into $\Sigma(H^\infty)$ whose image is ∂H^∞, [17]. Thus H^∞ is a uniform algebra on M.

Lemma 3.1. *If $h \in C_{\mathbb{R}}(M)$ there is $f \in H^{\infty-1}$ such that*

$$\log|f| = h. \tag{3.6}$$

Sketch of proof. We extend h to a function h on Δ by the integral

$$h(z) = \int_{-\pi}^{\pi} h\left(\mathrm{e}^{it}\right) dm_z(t) \;\; (z \in \Delta),$$

thus obtaining a real-valued, bounded harmonic function on Δ for which

$$\lim_{r\uparrow 1} h\left(r\mathrm{e}^{i\theta}\right) = h\left(\mathrm{e}^{i\theta}\right) \; a.e..$$

[4]Actually M is extremely disconnected, *i.e.* the closure of any open set in M is open; [17], pp. 170-171 (where M is called extremally disconnected), or [13], p. 207.

If k is the conjugate harmonic function of h, the function

$$f = \mathrm{e}^{h+ik}$$

is contained in $H^{\infty -1}$, and satisfies (3.6). ∎

Thus H^∞ is a logmodular algebra[5], and therefore every character $\chi \in \Sigma\,(H^\infty)$ has a unique representing measure supported in M (see, *e.g.*, [34], pp. 170-172).

Note incidentally that, since every Dirichlet algebra is logmodular, any character in a Dirichlet algebra $\mathcal{A}$ has a unique representing measure supported in $\partial\mathcal{A}$.

The question arises whether H^∞ is actually a Dirichlet algebra. The answer to this question turns out to be negative, as a consequence of an observation of A. Gleason (see, *e.g.*, [17], p. 182) whereby, if M is a totally disconnected, compact Haudorff space, the only Dirichlet algebra on M is $C(M)$.

Let $R : \Sigma\,(H^\infty) \to \overline{\Delta}$ map $\chi \in \Sigma\,(H^\infty)$ to $\langle \iota, \chi \rangle$.

THEOREM 3.5. *The map R is continuous, and*

$$R\,(\Sigma\,(H^\infty)) = \overline{\Delta};$$

$R_{|R^{-1}(\Delta)}$ *is one-to-one, and R^{-1} maps Δ homeomorphically onto an open subset D of $\Sigma\,(H^\infty)$ whose complement is connected*[6].

According to L. Carleson's *corona theorem* the image D of Δ is dense in $\Sigma\,(H^\infty)$ (see, *e.g.*, [13]).

The structure of $\Sigma\,(H^\infty)$ is extremely complicated. Here are a few preliminary, elementary results (whose proofs can be found in [17]), which are useful in the following.

LEMMA 3.2. *Let $\alpha \in \partial\Delta$ and $f \in H^\infty$ be such that there is a sequence $\{z_n\} \subset \Delta$ converging to α for which $\{f(z_n)\}$ converges to some $\zeta \in \mathbb{C}$. Then there is $\chi \in \Sigma\,(H^\infty)$ for which $\langle \iota, \chi \rangle = \alpha$ and $\langle f, \chi \rangle = \zeta$.*

For $\alpha \in \partial\Delta$, let Σ_α be the "fiber"

$$\Sigma_\alpha = \{\chi \in \Sigma\,(H^\infty) : \langle \iota, \chi \rangle = \alpha\} = R^{-1}(\alpha).$$

[5] *i.e.* the set $\log|H^{\infty -1}| = \{\log|f(t)| : f \in H^{\infty -1}\}$ is dense in $L_{\mathbb{R}}{}^\infty$; see, *e.g.*, [18], [13].

[6] For a proof, see, *e.g*, [17], [13].

LEMMA 3.3. *A function $f \in H^\infty$ is continuously extendable to $\Delta \cup \{\alpha\}$ if, and only if, its Gelfand transform $\hat{f}$ is constant on Σ_α.*

For any $\theta \in \mathbb{R}$, the map $z \mapsto e^{i\theta} z$ defines a "rotation" mapping homeomorphically $\Sigma(H^\infty)$ and Δ respectively onto themselves, and induces a homeomorphism of Σ_α onto $\Sigma_{e^{i\theta}\alpha}$.

Every fiber Σ_α is connected. Denoting by $\mathcal{A}_\alpha$ the algebra of the restrictions to Σ_α of the Gelfand transforms of all functions in H^∞, then, [17], $\mathcal{A}_\alpha$ is a regular uniform algebra whose Gelfand spectrum and Shilov boundary are respectively

$$\Sigma(\mathcal{A}_\alpha) = \Sigma_\alpha \text{ and } \partial\mathcal{A}_\alpha = \Sigma_\alpha \cap \partial H^\infty.$$

PROPOSITION 3.2. *The set $\mathfrak{U}(H^\infty)$ is the set of all inner functions.*

In other words, $u \in H^\infty$ if, and only if,

$$\lim_{r\uparrow 1} |u(re^{i\theta})| = 1$$

for almost all $\theta \in [0, 2\pi]$.

Proof. The "only if" part follows from D. J. Newman's characterization of the Shilov boundary ∂H^∞ of H^∞ ([17]; [13]), according to which, if u is an inner function, then

$$|\langle u, \chi\rangle| = 1 \ \ \forall \ \ \chi \in \partial H^\infty.$$

Conversely, if $u \in \mathfrak{U}$ is not inner, there is a Borel set $C \subset \partial\Delta$, with positive Lebesgue measure, such that

$$\lim_{r\uparrow 1} |u(r\zeta)| < 1 \quad \forall \zeta \in C.$$

By the Lusin theorem, there is a Borel set $E \subset C$, with positive Lebesgue measure, such that u is continuously extendable to $\Delta \cup E$. By Lemma 3.3, u is constant on the fiber Σ_ζ, and therefore has modulus less that 1 on Σ_ζ for all $\zeta \in E$, contradicting the hypothesis whereby u is unimodular. ∎

The set $\mathfrak{U}(H^\infty)$ satisfies the hypotheses of the A. Bernard theorem because, as was shown by D.E. Marshall, the Blaschke products generate H^∞, [23], [13].

4. Linear isometries of uniform algebras

Here is an elementary lemma which will play a role in the following considerations.

LEMMA 4.1. ([41]) *Let m be a positive, regular Borel measure on a compact Hausdorff space X with $\|m\| \leq 1$. If $f \in L^1(X, m)$ is such that $|f| \leq 1$ a.e. and*

$$\left| \int f \, dm \right| = 1,$$

then $\|m\| = 1$ and $f = \mathrm{e}^{i\theta}$ a.e. for some $\theta \in \mathbb{R}$.

Let now $\mathcal{A}$ be a uniform algebra on a compact Hausdorff X, and let $\mathcal{A}'$ be its topological dual.

For every $\lambda \in \mathcal{A}'$ there is a complex regular Borel measure μ with support in X, with $\|\mu\| = \|\lambda\|$, such that

$$\langle f, \lambda \rangle = \int f \, d\mu \quad \forall \, f \in \mathcal{A}. \tag{4.1}$$

Let $d\mu = h \, d|\mu|$ be the polar decomposition of μ (with h measurable and $|h(x)| = 1$ a.e. on $\operatorname{Supp} \mu$).

If $u \in \mathcal{A}$ is such that $\|u\| = |\langle u, \lambda \rangle| = 1$, then $\|\mu\| = |\mu|(X) = 1$ and

$$hu = \langle u, \lambda \rangle \ \ a.e.$$

Let $\mathcal{B}$ be a uniform algebra on a compact Hausdorff Y, and let $A \in \mathcal{L}(\mathcal{A}, \mathcal{B})$ be a linear isometry.

For $x \in X$ let

$$\Omega(x) = \{ f \in \mathcal{A} : |f(x)| = \|f\| = 1 \},$$

and

$$\Upsilon(x) = \{ y \in Y : |(Af)(y)| = 1 \; \forall \, f \in \Omega(x) \}.$$

The proof of the following proposition has been devised by W. Holsztyński, [19] in the case in which $\mathcal{A} = C(X)$ and $\mathcal{B} = C(Y)$, but can be generalized, with no substantial change, to the case considered here, [41].

PROPOSITION 4.1. *The set $\Upsilon(x)$ is closed and not empty for all $x \in X$.*

Let Q be the closed set of all $y \in Y$ such that $|(Au)(y)| = 1$ for all $u \in \mathfrak{U}(\mathcal{A})$. Since

$$y \in \Upsilon(x) \iff |(Af)(y)| = 1 \quad \forall\, f \in \Omega(x)$$
$$\implies |(Au)(y)| = 1 \quad \forall\, u \in \mathfrak{U}(\mathcal{A}) \iff y \in Q,$$

and therefore

$$\Upsilon(x) \subset Q \quad \forall\, x \in X,$$

then $Q \neq \emptyset$.

For $y \in Y$, let $\lambda \in \mathcal{A}'$ be the continuous linear form on $\mathcal{A}$ defined by

$$\langle f, \lambda \rangle = (Af)(y) \quad \forall\, f \in \mathcal{A},$$

and let μ be the complex, regular Borel measure on X which represents λ.

If, given any two distinct points t' and t'' in X there is $u \in \mathfrak{U}(\mathcal{A})$ such that $u(t') \neq u(t'')$, $\mathrm{Supp}(\mu)$ is reduced to one point, and the following lemma holds, [41].

Lemma 4.2. *If $\mathfrak{U}(\mathcal{A})$ separates points in X, there is a map $\psi : Q \to X$ such that*

$$(Au)(y) = \langle u, \lambda \rangle = (A1_{\mathcal{A}})(y)\, u(\psi(y)) \quad \forall\, u \in \mathfrak{U}(\mathcal{A}),\ y \in Q.$$

An application of Bernard's theorem yields then the following

Theorem 4.1. ([41]) *If $\mathfrak{U}(\mathcal{A})$ generates $\mathcal{A}$, for any isometry $A \in \mathcal{L}(\mathcal{A}, \mathcal{B})$ there exist a closed, non-empty subset $Q \subset Y$ and a continuous map ψ of Q onto X such that*

$$(Af)(y) = (A1_{\mathcal{A}})(y)\, f(\psi(y)) \quad \forall f \in \mathcal{A},\ y \in Q. \tag{4.2}$$

Corollary. *If $\mathfrak{U}(\mathcal{A})$ generates $\mathcal{A}$ and*

$$A(\mathfrak{U}(\mathcal{A})) \subset \mathfrak{U}(\mathcal{B}),$$

then $Q = Y$.

Since $\psi(Q) = X$, if $Af \in \mathfrak{U}(\mathcal{B})$, then $f \in \mathfrak{U}(\mathcal{A})$, *i.e.*

$$A^{-1}\left(\mathfrak{U}(\mathcal{B}) \cap A\left(\mathcal{A}\right)\right) \subset \mathfrak{U}(\mathcal{A}).$$

Hence, if A is surjective, $\mathfrak{U}(\mathcal{B}) \subset A\left(\mathfrak{U}(\mathcal{A})\right)$ and also $\mathfrak{U}(\mathcal{A}) \subset A^{-1}\left(\mathfrak{U}(\mathcal{B})\right)$.

Corollary. *If* $\mathfrak{U}(\mathcal{A})$ *generates* $\mathcal{A}$ *and if the isometry* A *is surjective, then* $A(\mathfrak{U}(\mathcal{A})) = \mathfrak{U}(\mathcal{B})$*, and* A *is expressed by (4.2), where* $Q = Y$ *and* ψ *is a homeomorphism of* Y *onto* X.

Assume now that every $\chi \in \Sigma(\mathcal{B})$ has a unique representing measure m_χ supported in Y.

Letting

$$P = \{\chi \in \Sigma(\mathcal{B}) : \operatorname{Supp} m_\chi \subset Q\}\,,$$

then $Q \subset P$ and

$$Q = Y \quad \Longrightarrow \quad P = \Sigma(\mathcal{B}).$$

The following theorem, which yields a partial description of the linear isometry A was established in [41].

Theorem 4.2. *If every* $\chi \in \Sigma(\mathcal{B})$ *has a unique representing measure supported in* Y*, and if* $\mathfrak{U}(\mathcal{A})$ *generates* $\mathcal{A}$*, there exist a subset* P *of* $\Sigma(\mathcal{B})$*, with* $Q \subset P$*, and a continuous map* $\omega : P \to \Sigma(\mathcal{A})$ *such that* $\omega_{|Q} = \psi$ *and*

$$\langle Af, \chi\rangle = \langle A1_{\mathcal{A}}, \chi\rangle.\langle f, \omega(\chi)\rangle \quad \forall\, f \in \mathcal{A},\ \chi \in P. \tag{4.3}$$

Let $A, \mathcal{A}, \mathcal{B}$ be as before, and suppose furthermore that

$$A\left(\mathcal{A}^{-1}\right) \subset \mathcal{B}^{-1}. \tag{4.4}$$

By the Gleason-Kahane-Zelazko theorem, this inclusion implies that

$$\langle Af, \chi\rangle = \langle A1_{\mathcal{A}}, \chi\rangle.\langle f, \phi(\chi)\rangle \quad \forall\, f \in \mathcal{A},\ \chi \in \Sigma(\mathcal{B}),$$

where $\phi : \Sigma(\mathcal{B}) \to \Sigma(\mathcal{A})$ is continuous.

Since $A1_{\mathcal{A}} \in \mathcal{B}^{-1}$, then ϕ is a continuous extension of ω, and therefore

$$\phi(\partial\mathcal{B}) \supset \partial\mathcal{A}.$$

Hence, if $\Sigma(\mathcal{A}) = \partial\mathcal{A}$ and $A1_{\mathcal{A}} \in \mathfrak{U}(\mathcal{B})$, then

$$|\langle Au, \chi\rangle| = |\langle u, \phi(\chi)\rangle| = 1 \quad \forall\, \chi \in \partial\mathcal{B},\ u \in \mathfrak{U}(\mathcal{A}),$$

proving thereby the following proposition which yields a complete description of A in terms of the characters in $\Sigma(\mathcal{B})$.

Proposition 4.2. *Let* $A, \mathcal{A}, \mathcal{B}$ *be as before, and let* (4.4) *hold. If* $\Sigma(\mathcal{A}) = \partial\mathcal{A}$ *and* $A1_{\mathcal{A}} \in \mathfrak{U}(\mathcal{B})$*, then* $P = \Sigma(\mathcal{B})$ *and*

$$\langle Af, \chi\rangle = \langle A1_{\mathcal{A}}, \chi\rangle.\langle f, \omega(\chi)\rangle \quad \forall\, f \in \mathcal{A},\ \chi \in \Sigma(\mathcal{B}).$$

Corollary. *If $\Sigma(\mathcal{A}) = \partial\mathcal{A}$ and A is an isometric homomorphism, then*

$$\langle Af, \chi\rangle = \langle f, \omega(\chi)\rangle \quad \forall\, f \in \mathcal{A},\ \chi \in \Sigma(\mathcal{B}),$$

where ω is a continuous map of $\Sigma(\mathcal{B})$ onto $\Sigma(\mathcal{A})$.

Let $\mathcal{A} = \mathcal{B}$. If the isometry $A \in \mathcal{L}(\mathcal{A})$ is not surjective, the spectrum $\sigma(A)$ of A is $\sigma(A) = \overline{\Delta}$, and Δ is contained in the residual spectrum of A. As a consequence, $A(\mathcal{A})$ is contained in a proper closed linear subspace of $\mathcal{A}$. Hence,

Lemma 4.3. *If $A\left(\mathcal{A}^{-1}\right)$ contains a non-empty open set, the isometry A is surjective.*

5. Examples

I. The algebras $C(X)$ and $C(Y)$

By Theorem 3.2, Theorem 4.1 yields Holsztyński's theorem, [19]:

Theorem 5.1. *If $A \in \mathcal{L}(C(X), C(Y))$ is an isometry, there exist a closed subset Q of Y and a continuous surjective map $\psi : Q \to X$ such that*

$$(Af)(y) = (A1_{C(X)})(y).f(\psi(y)) \quad \forall\, f \in C(X),\ y \in Q.$$

Furthermore,

$$Q = Y \quad \Longleftrightarrow \quad A\left(\mathfrak{U}(C(X))\right) \subset \mathfrak{U}(C(Y)),$$

and, if A is an isometric homomorphism of $C(X)$ into $C(Y)$, then $Q = Y$ and

$$(Af)(y) = f(\psi(y)) \quad \forall\, f \in C(X),\ y \in Y.$$

Let now $X = Y$, and let the linear isometry A be such that

$$A\left(\mathfrak{U}(C(X))\right) \subset \mathfrak{U}(C(X)),$$

and that

$$\lim_{n \to +\infty} \left(A^n f\right)(y) = f(y) \quad \forall\, f \in C(X),\ y \in X, \tag{5.1}$$

where A^n is the n-th iterate of A:

$$A^n = A \cdots A, \quad (n \text{ times}),$$

expressed by

$$(A^n f)(y) = (A^n 1_{C(X)})(y)\, f(\psi^{\circ n}(y)) \quad \forall\, f \in C(X),\ y \in X,$$

and

$$\psi^{\circ n} = \psi \circ \cdots \circ \psi \quad (n \text{ times}),$$

$$\begin{aligned}(A^n 1_{C(X)})(y) \\ = (A1_{C(X)})(y)\,(A1_{C(X)})(\psi(y))\ \ldots\ (A1_{C(X)})(\psi^{\circ(n-1)}(y)).\end{aligned} \tag{5.2}$$

Since $A^n(1_{C(X)}) \in \mathfrak{U}(C(X))$ for $n = 1, 2, \ldots$, then

$$|(A^n f)(y)| = |f(\psi^{\circ n}(y))| \quad \forall\, f \in C(X),\ y \in X,\ n = 1, 2, \ldots,$$

and (5.1) yields

$$\lim_{n \to +\infty} |f(\psi^{\circ n}(y))| = |f(y)| \quad \forall\, f \in C(X),\ y \in X.$$

Since X is compact, for any $y \in X$ there are $y' \in X$ and a sequence $n_1 < n_2 < \ldots$ of positive integers such that

$$\lim_{j \to +\infty} \psi^{\circ n_j}(y) = y'.$$

Thus, by (5.2),

$$|f(y)| = \lim_{j \to +\infty} |f\,(\psi^{\circ n_j}(y))| = |f(y')| \quad \forall\, f \in C(X),$$

and therefore $y = y'$, whence

$$\lim_{n \to +\infty} \psi^{\circ n}(y) = y \quad \forall\, y \in X,$$

and, in conclusion,

$$\psi(y) = \lim_{n \to +\infty} \psi^{\circ(n+1)}(y) = y \quad \forall\, y \in X. \tag{5.3}$$

Thus

$$Af = A1_{C(X)}\, f \quad \forall\, f \in C(X),$$

and, by (5.3), (5.2) reads

$$(A^n 1_{C(X)})(y) = \big((A1_{C(X)})(y)\big)^n \quad \forall\, y \in X,\ n = 1, 2, \ldots,$$

whence $A1_{C(X)} = 1_{C(X)}$, *i.e.* $Af = f$ for all $f \in C(X)$, proving thereby

Proposition 5.1. *If the isometry $A \in \mathcal{L}(C(X))$ maps unimodular functions to unimodular functions (in particular, if A is surjective or is an isometric homomorphism) and if the sequence A^n of the iterates of A converges to the identity for the weak operator topology, then A is the identity.*

II. The disc algebra

Let $A \in \mathcal{L}\left(\mathcal{A}^0\right)$ be an isometry with

$$A\left(\mathfrak{U}\left(\mathcal{A}^0\right)\right) \subset \mathfrak{U}\left(\mathcal{A}^0\right). \tag{5.4}$$

Then

$$A1_{\mathcal{A}^0} \in \mathfrak{U}\left(\mathcal{A}^0\right),$$

and, since every character in $\Sigma\left(\mathcal{A}^0\right)$ has a unique representing measure, by Theorem 4.2 there is a continuous map $\omega : \overline{\Delta} \to \overline{\Delta}$ such that $\omega(\partial\Delta) = \partial\Delta$, and

$$(Af)(z) = (A1_{\mathcal{A}^0})(z)\, f(\omega(z)) \quad \forall\, f \in \mathcal{A}^0,\ z \in \overline{\Delta}.$$

If ι is the coordinate function, then

$$(A\iota)(z) = (A1_{\mathcal{A}^0})(z)\, \varpi(z),$$

where ϖ, defined by $\varpi(z) = \iota(\omega(z))$, is a unimodular function in $\mathcal{A}^0$.

If $f \in \mathcal{A}^0$ is an analytic polynomial of degree N,

$$f = \sum_{n=0}^{N} c_n\, \iota^n, \quad c_n \in \mathbb{C},$$

then

$$\begin{aligned}(Af)(z) &= (A1_{\mathcal{A}^0})(z) \sum_{n=0}^{N} c_n \iota^n(\varpi(z)) \\ &= (A1_{\mathcal{A}^0})(z) \sum_{n=0}^{N} c_n (\iota(\varpi(z))^n = (A1_{\mathcal{A}^0})(z) f(\varpi(z)).\end{aligned}$$

Since analytic polynomials are dense in $\mathcal{A}^0$,

$$Af = A1_{\mathcal{A}^0}\, f \circ \varpi \quad \forall\, f \in \mathcal{A}^0.$$

If the isometry A is surjective, then

$$A^{-1}f = A^{-1}(1_{\mathcal{A}^0})\, f\circ\tau,$$

where $\tau \in \mathfrak{U}\left(\mathcal{A}^0\right)$.

Being

$$(A^{-1})(z)\,(A^{-1}1_{\mathcal{A}^0})(\tau(z)) = (A1_{\mathcal{A}^0})(z)\,(A^{-1})(\varpi(z)) = 1 \quad \forall\, z \in \overline{\Delta},$$

the holomorphic function $A1_{\mathcal{A}^0}$ is constant: $A1_{\mathcal{A}^0} = c1_{\mathcal{A}^0}$ for some $c \in \partial\Delta$.

Being also

$$f\circ\varpi\circ\tau = f\circ\tau\circ\varpi = f \quad \forall\, f \in \mathcal{A}^0,$$

ϖ is a Möbius transformation, and $\tau = \varpi^{-1}$, proving, in conclusion,

THEOREM 5.2. *If, and only if, $A \in \mathcal{L}\left(\mathcal{A}^0\right)$ is a bijective isometry, then*

$$Af = c\;\, f\circ\varpi \quad \forall\, f \in \mathcal{A}^0,$$

where $c \in \partial\Delta$ and ϖ is a Möbius transformation.

Here is an example of a linear isometry A of $\mathcal{A}^0$ into itself which does not satisfy (5.4).

Let: C be a Cantor set, closed in $\partial\Delta$ and with Lebesgue measure zero; $y_0 \in \partial\Delta\backslash C$; $K = C \cup \{y_0\}$; $\beta : K \to \mathbb{C}$ a continuous function such that $\beta(C) = \partial\Delta$ and $\beta(y_0) = 0$.

By Rudin's Theorem 3.4, there is $\varphi \in \mathcal{A}^0$ such that $\|\varphi\| = 1$ and $\varphi_{|K} = \beta$.

The map

$$A : f \mapsto f\circ\varphi$$

is a continuous endomorphism of $\mathcal{A}^0$, but

$$A\left(\mathfrak{U}\left(\mathcal{A}^0\right)\right) \not\subset \mathfrak{U}\left(\mathcal{A}^0\right).$$

We will see now how Theorem 5.2 and the Wolff-Denjoy theorem, yield some information on the point spectrum of a surjective linear isometry A of $\mathcal{A}^0$ mapping inner functions to inner functions

Note first that if, and only if, $A1_{\mathcal{A}^0}$ is constant ($A1_{\mathcal{A}^0} = c1_{\mathcal{A}^0}$ for some $c \in \partial\Delta$), then $1_{\mathcal{A}^0}$ is an eigenvector of A (with eigenvalue c).

Let now $A \in \mathcal{L}(\mathcal{A}^0)$ be an isometry satisfying (5.4).

Then,

$$Af = A1_{\mathcal{A}^0}\, f\circ\varpi \quad \forall\, f \in \mathcal{A}^0, \tag{5.5}$$

where $\{A1_{\mathcal{A}^0}, \varpi\} \subset \mathfrak{U}(\mathcal{A}^0)$, and ϖ is not constant.

If ϖ is not an elliptic Moebius transformation, by the Wolff-Denjoy theorem (see, *e.g.*, [7] or [25]), the sequence $\{\varpi^{\circ n}(z) : n = 0, 1, 2, \ldots\}$ converges to a point $\zeta \in \partial\Delta$ for all $z \in \Delta$:

$$\lim_{n\to+\infty} \varpi^{\circ n}(z) = \zeta \quad \forall\, z \in \Delta.$$

Let $c \in \partial\Delta$ be an eigenvalue of A and let $g \in \mathcal{A}^0$ be an eigenvector associated to c:

$$A\,g = c\,g.$$

For any $z \in \Delta$ and $n = 1, 2, \ldots$

$$|g(z)| = |c^n g(z)| = |(A^n g)\,(z)| = |g\,(\varpi^{\circ n}(z))|\,,$$

and therefore

$$|g(z)| = \lim_{n\to+\infty} |g\,(\varpi^{\circ n}(z))| = |g(\zeta)|$$

for all $z \in \Delta$.

Hence the holomorphic function g is constant:

$$g = k1_{\mathcal{A}^0} \ \text{ for some } \ k \in \partial\Delta.$$

Since

$$k1_{\mathcal{A}^0} = g = \frac{1}{c}\,Ag = \frac{k}{c}\,A1_{\mathcal{A}^0},$$

the following proposition holds

Proposition 5.2. *If the linear isometry A is represented by* (5.5)*, where $A1_{\mathcal{A}^0} \in \mathfrak{U}(\mathcal{A}^0)$ and $\varpi \in \mathfrak{U}(\mathcal{A}^0)$ is non-constant and is not an elliptic Moebius transformation of Δ, then: c is the only eigenvalue of A; its associated eigenspace is spanned by $1_{\mathcal{A}^0}$;* (5.4) *holds; the unimodular function ϖ in* (5.5) *fixes c and is such that*

$$\lim_{n\to+\infty} \varpi^n(z) = c \quad \forall\, z \in \Delta.$$

If A is a surjective isometry, then $A1_{\mathcal{A}^0} = c1_{\mathcal{A}^0}$ for some $c \in \partial\Delta$ and ϖ in (5.5) is a Moebius transformation. If it is not elliptic, the asymptotic behaviour of its iterates is described by the above proposition.

III. Bounded holomorphic functions

Let $A \in \mathcal{L}(H^\infty)$ be an isometry. As in the case of $\mathcal{A}^0$, by Theorem 4.2 there are $P \subset \Sigma(H^\infty)$ and a continuous map $\omega : P \to \Sigma(H^\infty)$ such that

$$\langle Af, \chi\rangle = \langle A1_{H^\infty}, \chi\rangle . \langle f, \omega(\chi)\rangle \quad \forall\, f \in H^\infty,\ \chi \in P.$$

Moreover, $Q = P \cap \partial H^\infty$ is closed in ∂H^∞, and $\omega(Q) = \partial H^\infty$.

If

$$A\left(\mathfrak{U}(H^\infty)\right) \subset \mathfrak{U}(H^\infty), \tag{5.6}$$

then $Q = \partial H^\infty, P = \Sigma(H^\infty)$, and there is an inner function φ such that

$$(Af)(z) = (A1_{H^\infty})(z)\, f(\varphi(z)) \quad \forall\, f \in H^\infty, z \in \Delta. \tag{5.7}$$

Similar arguments to those developed above for $\mathcal{A}^0$ show that, if the isometry A is surjective, $A1_{H^\infty}$ is constant, and

$$Af = c \,.\, f \circ \varphi \quad \forall\, f \in H^\infty,$$

where $c \in \partial\Delta$ and φ is a Möbius transformation.

Remark. The same example constructed above for $\mathcal{A}^0$ shows the existence of linear isometries of H^∞ which do not satisfy (5.6).

We will now show that, if the iterates of the linear isometry A of H^∞ satisfying (5.6) converge to the identity for the weak operator topology, then A itself is the identity.

The hypothesis implies that

$$\lim_{n\to+\infty} \left(A^n f\right)(z) = f(z) \quad \forall\, f \in H^\infty,\ \forall\, z \in \Delta. \tag{5.8}$$

If

$$\left|\left(A^m 1_{H^\infty}\right)(z)\right| \le a$$

for some $z \in \Delta$, $a \in (0,1)$ and $m \ge 1$, then

$$\left|\left(A^n 1_{H^\infty}\right)(z)\right| \le a < 1$$

when $n \gg 1$, contradicting (5.8). Thus $|(A1_{H^\infty})(z)| = 1$ for all $z \in \Delta$, and therefore, by the maximum principle, there is some constant $c \in \partial\Delta$ such that

$$(A1_{H^\infty})(z) = c \quad \forall\, z \in \Delta.$$

Since, by (5.8),

$$c^n \to 1 \text{ as } n \to +\infty,$$

then $c = 1$ and (5.7) yields

$$Af = f \circ \varphi. \tag{5.9}$$

Thus, again by (5.8),

$$\lim_{n\to+\infty} \varphi^{\circ n}(z) = \lim_{n\to+\infty} \iota(\varphi^{\circ n}(z)) = z$$

for all $z \in \Delta$. The Wolff-Denjoy theoremñ implies that

$$\varphi(z) = z \quad \forall\, z \in \Delta,$$

proving thereby the following theorem.

Theorem 5.3. *The identity operator is the only linear isometry A of H^∞ into itself which satisfies* (5.7), *and whose iterates converge to the identity for the weak operator topology.*

6. Strongly continuous semigroups of linear isometries of H^∞

Let $\mathcal{E}$ be a complex Banach space and let $T : \mathbb{R}_+ \to \mathcal{L}(\mathcal{E})$ be a strongly continuous semigroup. For $t \in \mathbb{R}_+$, let $T''(t)$ be the bi-dual of $T(t)$

The Banach space $\mathcal{E}$ is said to be a *Grothendieck space* if every weak-star convergent sequence in the topological dual $\mathcal{E}'$ of $\mathcal{E}$ converges weakly.

Theorem 6.1. *If $\mathcal{E}$ is a Grothendieck space, $T'' : \mathbb{R}_+ \ni t \mapsto T''(t)$ is a strongly continuous semigroup.*

The Banach space $\mathcal{E}$ is said to have the *Dunford-Pettis property* if, whenever $\{x_n\}$ and $\{\lambda_n\}$ are sequences in $\mathcal{E}$ and in $\mathcal{E}'$ converging weakly to zero, the sequence $\langle x_n, \lambda_n\rangle$ converges.

Theorem 6.2. *Let $\mathcal{E}$ have the Dunford-Pettis property. If T'' is a strongly continuous semigroup, the semigroup T is uniformly continuous.*

Thus:

Theorem 6.3. *If $\mathcal{E}$ is a Grothendieck space with the Dunford-Pettis property every strongly continuous semigroup T on $\mathcal{E}$ is uniformly continuous* [7].

[7] See, [21] and [22] also for bibliographical references.

The link between these results and linear isometries of H^∞ is provided by a result, due to J. Bourgain, [5], whereby H^∞ is a Grothendieck space with the Dunford-Pettis property.

Thus, any strongly continuous semigroup $T : \mathbb{R}_+ \to \mathcal{L}(H^\infty)$ of linear isometries of H^∞ into itself is the restriction to $\mathbb{R}_+$ of a uniformly continuous group on $\mathbb{R}$, with values in $\mathcal{L}(H^\infty)$, which will be denoted by the same symbol T.

Since for any $t < 0$ and any $f \in H^\infty$

$$\|f\| = \|T(-t)\,T(t)\,f\| = \|T(t)\,f\|,$$

T is the restriction to $\mathbb{R}_+$ of a strongly continuous group $T : \mathbb{R} \to \mathcal{L}(H^\infty)$ of surjective isometries of H^∞.

Hence there are a function $c : \mathbb{R} \to \partial\Delta$ and a family $\{\varpi_t : t \in \mathbb{R}\}$ of holomorphic automorphisms of Δ such that

$$T(t)f = c(t)\,f \circ \varpi_t \quad \forall\, t \in \mathbb{R},\ \ f \in H^\infty, \tag{6.1}$$

Being $c(t) = T(t)1_{H^\infty}$, c is a continuous homomorphism of $\mathbb{R}$ into $\partial\Delta$. Therefore there is $\delta \in \mathbb{R}$ such that

$$c(t) = \mathrm{e}^{i\delta t} \quad \forall\, t \in \mathbb{R}. \tag{6.2}$$

Furthermore

$$\varpi_{s+t} = \varpi_s \circ \varpi_t \quad \forall\, s,t \in \mathbb{R}$$

and the map $\varpi : t \to \varpi_t(z)$ is continuous for every $z \in \Delta$; thus, ϖ is a continuous flow of holomorphic automorphisms of Δ: the *conformal flow* of T.

Hence the following theorem holds:

THEOREM 6.4. *Any strongly continuous semigroup of linear isometries of H^∞ is the restriction to $\mathbb{R}_+$ of a uniformly continuous group of surjective isometries.*

The continuous flow ϖ is defined by a one-parameter subgroup $t \mapsto \exp t\Theta$ of $SU(1,1)$ defined by a 2×2 matrix

$$\Theta \begin{pmatrix} i\gamma & c \\ \overline{c} & -i\gamma \end{pmatrix},$$

with $\gamma \in \mathbb{R}$, $c \in \mathbb{C}$.

As was shown, *e.g.*, in [40], if $\gamma^2 - |c|^2$ is positive, negative or zero, then $\varpi_t(z)$ is expressed respectively by:

$$\varpi_t(z) = \frac{\left(\cos(rt) + i\frac{\gamma}{r}\sin(rt)\right) z + \frac{c}{r}\sin(rt)}{\frac{\bar{c}}{r}\sin(rt)z + \cos(rt) - i\frac{\gamma}{r}\sin(rt)}$$

with $r = \sqrt{\gamma^2 - |c|^2}$;

$$\varpi_t(z) = \frac{\left(\cosh(st) + i\frac{\gamma}{s}\sinh(st)\right) z + \frac{c}{s}\sinh(st)}{\frac{\bar{c}}{s}\sinh(st)z + \cosh(st) - i\frac{\gamma}{s}\sinh(st)}$$

with $s = \sqrt{|c|^2 - \gamma^2}$;

$$\varpi_t(z) = \frac{(1 + it\gamma)z + tc}{t\bar{c}z + 1 - it\gamma}.$$

In the first case, the flow ϖ is elliptic, *i.e.* fixes one point in Δ and is periodic with period $2\pi/r$. In the second and third cases, the flow ϖ is respectively hyperbolic and parabolic and has no fixed point in Δ.

In the elliptic case, the periodicity of ϖ and (6.2) show that the group T is almost periodic.

It was shown in [40] that, if the flow ϖ is not elliptic, there is some $k > 0$ such that

$$\|T(t)\iota\| > \frac{1}{2} \quad \forall\, t > k.$$

In conclusion, the following theorem holds, [40].

THEOREM 6.5. *The group T is almost periodic if, and only if, its conformal flow ϖ is elliptic.*

Let now $T : \mathbb{R} \to \mathcal{L}(\mathcal{A}^0)$ be a strongly continuous group of (surjective) linear isometries of $\mathcal{A}^0$. Arguing as in the case of H^∞ one shows that T is represented by

$$T(t)f = \mathrm{e}^{i\delta t} \,.\, f \circ \varpi_t$$

for all $t \in \mathbb{R}$ and all $f \in \mathcal{A}^0$, where $\delta \in \mathbb{R}$ and $\varpi : t \mapsto \varpi_t$ is a continuous flow of Moebius transformations of Δ.

As before we see that, if the continuous flow ϖ is elliptic, the group T is almost periodic.

On the other hand, if the flow ϖ is hyperbolic or parabolic, by Proposition 5.2 the only eigenspace of $T(t)$ is a complex line. Theorem 2 of [3] shows then that the group T is not almost periodic.

7. Some remarks on the non-commutative case

In both the Banach-Stone and the Gleason-Kahane-Zelazko approaches, the existence of weighted composition operators depends on the presence of a wealth of continuous homomorphisms into the complex field, which is granted, via the Gelfand theory, by the commutativity of the algebra. On the other hand, these homomorphisms may be rare - or even not exist at all - in the case of some non-abelian Banach algebra. For example, an elementary computation shows that, for any integer $n > 1$ and for any linear form λ on the matrix algebra $\mathcal{L}(\mathbb{C}^n)$, there is some invertible element $x \in \mathcal{L}(\mathbb{C}^n)$ such that $\langle x, \lambda \rangle = 0$.

A similar result holds for the W^*-algebra $\mathcal{L}(\mathcal{H})$ of all continuous linear operators on any separable, infinite-dimensional complex Hilbert space $\mathcal{H}$ and for a non-vanishing normal linear form λ on $\mathcal{L}(\mathcal{H})$:

Theorem 7.1. ([39]) *If* $\dim \mathcal{H} > 1$, *for any normal linear form* λ *on the* W^**-algebra* $\mathcal{L}(\mathcal{H})$[8] *there is some invertible element* $x \in \mathcal{L}(\mathcal{H})$ *such that* $\langle x, \lambda \rangle = 0$.

Proof. [42] We shall prove that there is no non-vanishing normal linear form λ on $\mathcal{L}(\mathcal{H})$ such that

$$\langle xy, \lambda \rangle = \langle x, \lambda \rangle \langle y, \lambda \rangle \quad \forall\, x, y \in \mathcal{L}(\mathcal{H}). \tag{7.1}$$

The polar decomposition of λ is expressed by $\lambda = R_v\, \mu$, where: μ is a normal positive functional on $\mathcal{L}(\mathcal{H})$, v is a partial isometry on $\mathcal{H}$ and R_v acts on $\gamma \in \mathcal{L}(\mathcal{H})'$ by

$$\langle \bullet, R_v \gamma \rangle = \langle v\, \bullet, \gamma \rangle.$$

Therefore,

$$\langle x, \lambda \rangle = \langle xv, \mu \rangle \quad \forall\, x \in \mathcal{L}(\mathcal{H}).$$

Since μ is a positive normal functional, there is a sequence $\{\xi_n : n = 1, 2, \ldots\}$ of mutually orthogonal elements $\xi_n \in \mathcal{H}$ such that $\sum_{n=1}^{+\infty} \|\xi_n\|^2 < \infty$ and

$$\langle x, \mu \rangle = \sum_{n=1}^{+\infty} (x\xi_n | \xi_n)$$

[8] *i.e.* a linear form which is continuous for the topology defined in $\mathcal{L}(\mathcal{H})$ by the norm-topology on its pre-dual; see, *e.g,*, [32]

for all $x \in \mathcal{L}(\mathcal{H})$ ([32], Corollary 1.15.4). Thus, letting $\eta_n = v\xi_n$, then $\sum_{n=1}^{+\infty} \|\eta_n\|^2 < \infty$ and

$$\langle x, \lambda \rangle = \sum_{n=1}^{+\infty} (x\eta_n|\xi_n) \quad \forall\, x \in \mathcal{L}(\mathcal{H}).$$

Hence, (7.1) becomes

$$\sum_{p=1}^{+\infty} (xy\eta_p|\xi_p) = \sum_{p=1}^{+\infty} (x\eta_p|\xi_p) \sum_{p=1}^{+\infty} (y\eta_p|\xi_p) \quad \forall\, x, y \in \mathcal{L}(\mathcal{H}). \tag{7.2}$$

Note at this point that the set $\{\xi_n\}$ contains infinite non-vanishing elements because, if that set contains only $N > 0$ non-vanishing elements $\xi_1, \ldots, \xi_N$, there are $x, y \in \mathcal{L}(\mathcal{H})$ such that

$$\sum_{p=1}^{N} (x\eta_p|\xi_p) \sum_{p=1}^{N} (y\eta_p|\xi_p) \neq 0$$

and

$$y^* x^* \xi_p = 0 \text{ for } p = 1, \ldots, N,$$

contradicting (7.2).

Choosing now x and y with eigenvectors η_n and eigenvalues ζ_n and τ_n for $n = 1, 2, \ldots$, *i.e.*,

$$x\eta_n = \zeta_n \eta_n \text{ and } y\eta_n = \tau_n \eta_n,$$

with

$$\sum_{n=1}^{+\infty} |\zeta_n|^2 < \infty, \quad \sum_{n=1}^{+\infty} |\tau_n|^2 < \infty, \tag{7.3}$$

(7.2) reads

$$\sum_{p=1}^{+\infty} \zeta_p \tau_p \,(\,\eta_p|\xi_p) = \sum_{p=1}^{+\infty} \zeta_p \,(\eta_p|\xi_p) \sum_{p=1}^{+\infty} \tau_p \,(\eta_p|\xi_p) \tag{7.4}$$

for every choice of the sequences $\{\zeta_p\}$ and $\{\tau_p\}$ satisfying (7.3).

Fixing $n \geq 1$, $\zeta_n = \tau_n = 1$ and $\zeta_m = \tau_m = 0$ whenever $m \neq n$, (7.4) becomes

$$(\eta_n|\xi_n)\ (1 - (\eta_n|\xi_n)) = 0,$$

and implies that either $(\eta_n|\xi_n) = 0$ or $(\eta_n|\xi_n) = 1$. Hence $(\eta_n|\xi_n) = 0$ when $n \gg 0$.

Choose n and m in such a way that $(\eta_n|\xi_n) = 1$ and $(\eta_m|\xi_m) = 0$ and choose x, y in $\mathcal{L}(\mathcal{H})$ such that: $x\eta_p y\eta_p = 0$ whenever $p \neq n, m$, and

$$\begin{cases} x\,\xi_n = a_{11}\xi_n + a_{12}\xi_m, \\ x\,\xi_m = a_{21}\xi_n + a_{22}\xi_m, \end{cases} \qquad \begin{cases} y\,\eta_n = b_{11}\xi_n + b_{12}\xi_m, \\ y\,\eta_m = b_{21}\xi_n + b_{22}\xi_m, \end{cases}$$

with $a_{11}, \ldots, b_{22} \in \mathbb{C}$.

Then, $xy\eta_p = 0$ for all $p \neq n, m$, and

$$\begin{cases} xy\,\eta_n = c_{11}\xi_n + c_{12}\xi_m, \\ xy\,\eta_m = c_{21}\xi_n + c_{22}\xi_m. \end{cases}$$

The matrix

$$\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$$

is the product of the matrices

$$\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, \quad \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Since

$$\langle x, \lambda \rangle = \sum_{p=1}^{+\infty} (x\eta_p|\xi_p) = a_{11}\,(\xi_n|\xi_n) + a_{22}\,(\xi_m|\xi_m) = a_{11}\,(\xi_n|\xi_n)\,,$$

$$\langle y, \lambda \rangle = \sum_{p=1}^{+\infty} (y\eta_p|\xi_p) = b_{11}\,(\xi_n|\xi_n) + b_{22}\,(\xi_m|\xi_m) = b_{11}\,(\xi_n|\xi_n)\,,$$

$$\begin{aligned} \langle xy, \lambda \rangle &= \sum_{p=1}^{+\infty} (xy\eta_p|\xi_p) = c_{11}\,(\xi_n|\xi_n) + c_{22}\,(\xi_m|\xi_m) \\ &= c_{11}\,(\xi_n|x_n) = (b_{11}a_{11} + b_{12}a_{21})\,(\xi_n|\xi_n)\,, \end{aligned}$$

choosing $b_{12}a_{21} \neq 0$ we contradict (7.1), proving thereby Theorem 7.1. ∎

An attempt to extend the Banach-Stone and the Gleason-Kahane-Zelazko approaches to the non-commutative case faces the problem how to replace the continuous maps into $\mathbb{C}$ which play a relevant role in Theorem 2.2 and Theorem 4.2. A preliminary exploration in this direction is carried out in

[42], where some linear idempotents are singled out among all continuous homomorphisms with a higher dimensional range. Here are some scattered findings of this exploration.

Let $\mathcal{A}$ be a unital Banach algebra, let $\lambda \in \mathcal{A}'$ and let $\tilde{\lambda} \in \mathcal{L}(\mathcal{A})$ be defined by

$$\tilde{\lambda} : x \mapsto \langle x, \lambda \rangle \, 1_{\mathcal{A}} \quad \forall \, x \in \mathcal{A}.$$

Then, $\tilde{\lambda}$ is an idempotent of $\mathcal{L}(\mathcal{A})$ mapping $\mathcal{A}^{-1}$ into itself if, and only if, λ is a continuous character of $\mathcal{A}$.

If $\tilde{\lambda}$ is an idempotent of $\mathcal{L}(\mathcal{A})$ mapping $\mathcal{A}^{-1}$ into itself, then $\tilde{\lambda}$ is a module-map of $\mathcal{A}$ as a two-sided $\ker(\tilde{\lambda} - I)$-module.

Let $\mathcal{B}$ be a unital Banach algebra, let $E(\mathcal{B})$ be the set of all invertible elements of $\mathcal{B}$ which have logarithms, and let $f : \mathbb{C} \to \mathcal{B}$ be a holomorphic map for which

$$f(0) = 1_{\mathcal{B}}, \quad f'(0) = 0, \quad f(\mathbb{C}) \subset E(\mathcal{B})$$

and there is $a \geq 1$ such that

$$0 < \|f(z)\| \leq a e^{|z|} \quad \forall \, z \in \mathbb{C}.$$

If, for any two distinct points x, y of $f(\mathbb{C})$ there is $\lambda \in \mathcal{B}'$ for which $0 \notin \langle E(\mathcal{B}), \lambda \rangle$ and $\langle x, \lambda \rangle \neq \langle y, \lambda \rangle$, then

$$f(z) = 1_{\mathcal{B}} \quad \forall \, z \in \mathbb{C}.$$

As a consequence of this result, if $\mathcal{A}$ is a unital Banach algebra, and if $\Lambda \in \mathcal{L}(\mathcal{A}, \mathcal{B})$ is such that

$$\Lambda(\mathcal{A}^{-1}) \subset E(\mathcal{B}) \quad \text{and} \quad \Lambda(1_{\mathcal{A}}) = 1_{\mathcal{B}},$$

then

$$\Lambda(x) = 0 \implies \Lambda(x^n) = 0 \quad \forall \, n = 1, 2, \ldots.$$

References

[1] R. Arens, I.M. Singer, Function values as boundary integrals, *Proc. Amer. Math. Soc.* **5** (1954), 735 – 745.

[2] S. Banach, "Théorie des Opérations Linéaires", Monografje Matematyczne, PWN, Warsaw, 1932; Dover, New York, 1955.

[3] H. Bart, S. Goldberg, Characterizations of almost periodic strongly continuous groups and semigroups, *Math. Ann.* **236** (1978), 105 – 116.

[4] A. Bernard, Algèbres quotients d'algèbres uniformes, *C.R. Acad. Sci. Paris, Sér. A* **272** (1971), 1101 – 1104.

[5] J. Bourgain, Propriétés de relèvement et projections dans les espaces L^1/H^1_0 et H^∞, *C.R. Acad. Sci. Paris, Sér. A-Math.* **291** (11) (1980), 607 – 609.

[6] R.B. Burckel, "Characterizations of $C(X)$ among its Subalgebras", Lecture Notes in Pure and Applied Mathematics, 6, Marcel Dekker, Inc., New York, 1972.

[7] R.B. Burckel, Iterating analytic self-maps of discs, *Amer. Math. Monthly* **88** (1981), 396-407.

[8] K. de Leeuw, W. Rudin, J. Wermer, The isometries of some function spaces, *Proc. Amer. Math. Soc.* **11** (1960), 694 – 698.

[9] S. Eilenberg, Banach space methods in topology, *Ann. of Math.* **43** (1942), 568 – 579.

[10] P. Fatou, Séries trigonométriques et séries de Taylor, *Acta Math.* **30** (1906), 335 – 400.

[11] R.J. Fleming, J.E. Jamison, "Isometries on Banach Spaces: Function Spaces", Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics, 129, Chapman and Hall/CRC, Boca Raton, 2003.

[12] T. Franzoni, E. Vesentini, "Holomorphic Maps and Invariant Distances", Notas de Matemática (Mathematical Notes), 69, North Holland Publishing Co., Amsterdam-New York, 1980.

[13] J.B. Garnett, "Bounded Analytic Functions", Pure and Applied Mathematics, 96, Academic Press, Inc., New York-London, 1981.

[14] A. Gleason, A characterization of maximal ideals, *J. Analyse Math.* **19** (1967), 171 – 172.

[15] L.A. Harris, Banach algebras with involution and Möbius transformations, *J. Funct. Anal.* **11** (1972), 1 – 16.

[16] L.A. Harris, Bounded symmetric homogeneous domains in infinite dimensional spaces, in "Proceedings in Infinite Dimensional Holomorphy", T.L. Hayden and T.J. Suffridge (eds), University of Kentucky, 1973, Lecture Notes in Mathematics, 364, Springer, Berlin, 1974, 13 – 40.

[17] K. Hoffman, "Banach Spaces of Analytic Functions", Prentice-Hall, Englewood Cliffs, N.J., 1962.

[18] K. Hoffman, Analytic functions and logmodular Banach algebras, *Acta Math.* **108** (1962), 271 – 317.

[19] W. Holsztyński, Continuous mappings induced by isometries of spaces of continuous functions, *Studia Math.* **26** (1966), 133 – 136.

[20] J.-P. Kahane, W. Zelazko, A caracterization of maximal ideals in commutative Banach algebras, *Studia Math.* **29** (1968), 339 – 343.

[21] H.P. Lotz, Uniform convergence of operators on L^∞ and similar spaces, *Math. Z.* **190** (1985), 207 – 220.

[22] H.P. Lotz, Semigroups on L^∞ and H^∞, in "One-Parameter Semigroups of Positive Operators", R. Nagel (ed.), Lecture Notes in Math., 1184, Springer-Verlag, Berlin-Heidelberg-New York-Tokyo, 1986, 54 – 59.

[23] D.E. Marshall, Blaschke products generate H^∞, *Bull. Amer. Math. Soc.* **82** (1976), 494 – 496.

[24] E. Michael, Locally multiplicatively-convex topological algebras, *Mem. Amer. Math. Soc.* **11** (1952).

[25] J. Milnor, "Dynamics in One Complex Variable", Annals of Mathematics Studies, 160, Princeton University Press, Princeton, N.J., 2006.

[26] N. Nagasawa, Isomorphisms between commutative Banach algebras with an application to rings of analytic functions, *Kodai Math. Sem. Rep.* **11** (1959), 162 – 188.

[27] R.R. Phelps, Extreme points in function algebras, *Duke Math. J.* **32** (1965), 267 – 278.

[28] W. Rudin, Boundary values of continuous analytic functions, *Proc. Amer. Math. Soc.* **7** (1956), 808 – 811.

[29] W. Rudin, "Real and Complex Analysis", McGraw Hill, New York, 1966.

[30] W. Rudin, "Functional Analysis", McGraw Hill, New York, 1973.

[31] B. Russo, H. Dye, A note on unitary operators in C^*-algebras, *Duke Math. J.* **33** (1966), 413 – 416.

[32] S. Sakai, "C^*-Algebras and W^*-Algebras", Springer-Verlag, New York-Heidelberg-Berlin, 1971.

[33] M.H. Stone, Applications of the theory of Boolean algebras to general topology, *Trans. Amer. Math. Soc.* **41** (1937), 375 – 481.

[34] E.L. Stout, "The Theory of Uniform Algebras", Bogden and Quigley, Inc., Tarrytown-on-Hudson, N.Y., 1971.

[35] E. Vesentini, On the Banach-Stone theorem, *Adv. Math.* **112** (1995), 135 – 146.

[36] E. Vesentini, Periodic points and non-wandering points of continuous dynamical systems, *Adv. Math.* **134** (1998), 308 – 327.

[37] E. Vesentini, Weighted composition operators and the Gleason-Kahane-Zelazko theorem, *Adv. Math.* **191** (2005), 423 – 445.

[38] E. Vesentini, The Gleason-Kahane-Zelazko theorem and function algebras, *Rend. Lincei Mat. Appl.* (9),**16** (2005), 87 – 108; *Errata*, ibd, (9) **16** (2005), 211.

[39] E. Vesentini, Weighted composition operators and locally convex algebras, *Science in China, Ser. A, Mathematics* **48** (2005), 32 – 46.

[40] E. Vesentini, Periodic and almost periodic linear isometries of Hardy spaces, *Rend. Acc. Naz. Sci. XL Mem. Mat. Appl. (5)* **29** (1) (2005), 455 – 472.

[41] E. Vesentini, Linear isometries of some function algebras, *Rend. Lincei Mat. Appl.* **18** (2007), 235 – 255.

[42] E. Vesentini, Characters and idempotents acting on Banach algebras, *Tuebinger Berichte zur Funktionalanalysis* **16** (2007/2008), 451 – 473.