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Unbounded Operators: Functional Calculus, Generation, Perturbations†

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Abstract: These lectures review the H^∞-functional calculus of sectorial operators and related classes of unbounded operators. Their theory is related to the well established theory of C_0-semigroups and cosine functions. In most cases the existence of a bounded H^∞-calculus is equivalent to certain quadratic estimates arising from harmonic analysis. In addition to describing the topic in general, the lectures include recent results in perturbation theory for functional calculus and for differentiable semigroups.

Key words: Sectorial operator, functional calculus, quadratic estimates, strip type, operator logarithm, semigroup, cosine function, perturbation, triangular operator.

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