EXTRACTA MATHEMATICAE Vol. **24**, Núm. 3, 281 – 299 (2009)

Torsion Graph over Multiplication Modules

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Presented by Consuelo Martínez Received October 27, 2009

Abstract: For a commutative ring R, the torsion graph of an R-module M is $\Gamma(M)$ whose vertices are nonzero torsion elements of M, and two distinct vertices x and y are adjacent if and only if $[x : M][y : M]M = 0$. In this article we show that if $S = R \setminus Z(M)$, then $\Gamma(M)$ and $\Gamma(S^{-1}M)$ are isomorphic for a multiplication R-module M. Also we prove that for a multiplication R-module M, if $\Gamma(M)$ is uniquely complemented, then $S^{-1}M$ is von Neumann regular or $\Gamma(M)$ is a star graph.

Key words: Torsion graph, multiplication module, von Neumann regular module.

AMS *Subject Class.* (2000): 13A99, 05C99, 13C99.

1. Introduction

The idea of a zero-divisor graph of a commutative ring was introduced by I. Beck in 1988 [6]. He suppose that all elements of the ring are vertices of the graph and was mainly interested in colorings and then this investigation of coloring of a commutative ring was continued by Anderson and Naseer in [1]. Anderson and Livingston [3], studied the zero-divisor graph whose vertices are the nonzero zero-divisors. Let R be a commutative ring with identity and let $Z(R)$ be the set of zero-divisors of R. The zero-divisor graph of R denoted by $\Gamma(R)$, is a graph with vertices $Z(R)^* = Z(R)\setminus\{0\}$ and for distinct $x, y \in Z(R)^*$ the vertices x and y are adjacent if and only if $xy = 0$. This graph turns out to exhibit properties of the set of the zero divisors of a commutative ring with best way. The zero-divisor graph helps us to study the algebraic properties of rings using graph theoretical tools. We can translate some algebraic properties of a ring to graph theory language and then the geometric properties of graphs help us explore some interesting results in algebraic structures of rings.

The zero-divisor graph of a commutative ring has also been studied by several other authors (e.g., [1, 6, 4]). The zero divisor graph has also been introduced and studied for semigroups in [8], nearrings in [7], and for non-commutative rings, in [10].

Throughout, R is a commutative ring with unity and M is a unitary R-module. In this paper, motivated by the work of [2], we will investigate the concept of torsion-graph for modules as a natural generalization of zero-divisor graph for rings. For $x \in M$ the residual of Rx by M, denoted by $[x : M]$, is a set of elements $r \in R$ such that $rM \subseteq Rx$. The annihilator of an R-module M denoted by $\mathrm{Ann}_R(M)$ is $[0 : M]$. Let $T(M)$ be the set of torsion elements of M. It is clear that if R is an integral domain then $T(M)$ is a submodule of M which is called torsion submodule of M. If $T(M) = 0$ then the module M is said to be torsion-free and it is called a torsion module if $T(M) = M$. An R-module M is a multiplication module if for every R-submodule K of M there is an ideal I of R such that $K = IM$. We will study some properties of $\Gamma(M)$, when M is a multiplication R-module. Here the *torsion graph* $\Gamma(M)$ of M is a simple graph whose vertices are nonzero torsion elements of M and two distinct vertices x and y are adjacent if and only if $[x : M][y : M]M = 0$. Thus, $\Gamma(M)$ is an empty graph if and only if M is a torsion-free R-module. In this paper, we will investigate the interplay of module properties of M in relation to the properties of $\Gamma(M)$. We also think that torsion-graph helps us to study the algebraic properties of modules using graph theoretical tools. A graph G is connected if there is a path between any two distinct vertices. The distance, $d(x, y)$ between connected vertices x, y is the length of the shortest path from x to y ($d(x, y) = \infty$ if there is no such path).

A ring R is called reduced if $\mathrm{Nil}(R) = 0$, and an R-module M is called a reduced module if $rm = 0$ for $r \in R$ and $m \in M$, implies that $rM \cap Rm = 0$. Also a ring R is von Neumann regular if for each $a \in R$, there is an element $b \in R$ such that $a = a^2b$. It is clear that every von Neumann regular ring is reduced. An R-module M is called a von Neumann regular module if every cyclic submodule of M is pure in M. Anderson and Fuller in [5], called the submodule N, a pure submodule of M if $IM \cap N = IN$ for every ideal I of R. And so it is clear that every von Neumann regular modules is reduced.

Let Γ be a graph and $V(\Gamma)$ denotes the vertices of Γ. Let $v \in V(\Gamma)$, as in [2], $w \in V(\Gamma)$ is called a complement of v, if v is adjacent to w and no vertex is adjacent to both v and w; i.e., the edge $v - w$ is not an edge of any triangle in Γ. In this case, we write $v \perp w$. In module-theoretic terms, for multiplication R-module M, this is the same as saying that $v \perp w$ in $\Gamma(M)$ if and only if $v, w \in T(M)^*$ and $\mathrm{Ann}(w)M \cap \mathrm{Ann}(v)M \subset \{0, v, w\}$. Moreover, we will follow the authors in [2], and say that Γ is complemented if every vertex has a complement, and is uniquely complemented if it is complemented and any two complements of vertex set are adjacent to the same vertices. From

[2, Theorem 3.5 and Theorem 3.9], we know that for a ring R with nonzero nilpotent elements, $\Gamma(R)$ is uniquely complemented if and only if $\Gamma(R)$ is a star graph. Moreover, we know that, if R is reduced, then $S^{-1}R$ is a von Neumann regular ring.

In Section 2, as a generalization of [2, Theorem 2.2], we show that if M is a multiplication R-module and $S = R \setminus Z(M)$, then $\Gamma(M) \cong \Gamma(S^{-1}M)$. In Section 3, we investigate the complemented and uniquely complemented torsion graph. We also extend [2, Theorem 3.9], to the multiplication R-modules. And furthermore for a multiplication R-module M, we prove that if $\Gamma(M)$ is complemented, but not uniquely complemented, then $M = M_1 \oplus M_2$, where M_1, M_2 are submodules of M. Also for a reduced multiplication R-module M, we show that if $\Gamma(M)$ is complemented, then $S^{-1}M$ is a von Neumann regular module, where $S = R \setminus Z(M)$, also for a faithful multiplication R-module M with $\mathrm{Nil}(M) \neq 0$, we prove that $\Gamma(M)$ is uniquely complemented if and only if $\Gamma(M)$ is a star graph.

Let R be a ring and M be an R-module, throughout $\mathrm{Nil}(R)$ is an ideal consisting of nilpotent elements of R,

$$\mathrm{Nil}(M) := \bigcap_{N \in \mathrm{Spec}(M)} N\,,$$

$\mathrm{Spec}(M)$ is the set of all prime submodules of M, $T(M)^* = T(M) \setminus \{0\}$, $Z(M) = \{r \in R : rm = 0 \text{ for some } 0 \neq m \in M\}$. We let $\mathbb{Q}$, $\mathbb{Z}$ and $\mathbb{Z}_n$ denote the rings of rational numbers, integers and integers modulo n, respectively.

2. Isomorphisms

Recall that two graphs G and H are isomorphic, denoted by $G \cong H$, if there exists a bijection, say φ, from $V(G)$ to $V(H)$ of vertices such that the vertices x and y are adjacent in G if and only if $\varphi(x)$ and $\varphi(y)$ are adjacent in H.

Let $S = R \setminus Z(M)$. It is clear that the well defined map

$$\begin{array}{rcl} \chi : M & \longrightarrow & S^{-1}M \\ m & \longmapsto & \chi(m) = \frac{ms}{s}\,, \end{array}$$

is a monomorphism. So we can identify M with its image in $S^{-1}M$. Thus if m denotes an element of M, then the same symbol is also used to denote the fraction $\frac{m}{1}$. In this manner M becomes a submodule of $S^{-1}M$.

Let M be an R-module. For $m, m' \in T(M)^*$, we define $m \sim_M m'$ if and only if $\mathrm{Ann}(m)M = \mathrm{Ann}(m')M$. Clearly $\sim$ is an equivalence relation on $T(M)^*$. Let $S = R \setminus Z(M)$ and denote equivalence classes by $[m]_M$, so

$$[m]_M = \left\{m' \in T(M)^* \,:\, m \sim_M m'\right\}$$

and

$$([m]_M)_S = \left\{\frac{m'}{s} \,:\, m' \in [m], s \in S\right\}.$$

Now we would like to show that $\Gamma(S^{-1}M)$ and $\Gamma(M)$ are isomorphic by showing that there is a bijection map between equivalence classes of vertex sets $\Gamma(S^{-1}M)$ and $\Gamma(M)$ such that the corresponding equivalence classes have the same cardinality.

Theorem 2.1. *Let M be a faithful multiplication R-module and $S = R \setminus Z(M)$. Then $\Gamma(M)$ and $\Gamma(S^{-1}M)$ are isomorphic.*

Proof. (Our proof is quite similar to the proof in [2], applied for a ring.) Let $S = R \setminus Z(M)$, $M_S = S^{-1}M$, $R_S = S^{-1}R$ and

$$(T(M)_S)^* = \left\{\frac{m}{s} \,:\, m \in T(M)^*, s \in S\right\}.$$

Denote the equivalence relations defined above on $T(M)^*$ and $T(M_S)^*$ by $\sim_M$ and $\sim_{M_S}$, respectively. For all $m \in T(M)^*$, we have $\mathrm{Ann}_{R_S}(\frac{m}{s}) = \mathrm{Ann}_R(m)_S$ and $[N_S : M_S]M_S = [N : M]_S M_S$. By the above comments $(T(M)_S)^* = T(M_S)^*$, $([m]_M)_S = ([\frac{m}{1}])_{M_S}$ and

$$T(M)^* = \bigcup_{\lambda \in \Lambda} [m_\lambda]_M\,, \qquad T(M_S)^* = \bigcup_{\lambda \in \Lambda} \left[\frac{m_\lambda}{1}\right]_{M_S}$$

(both are disjoint unions). We next show that $\left|[x]_M\right| = \left|[\frac{x}{1}]_{M_S}\right|$ for all $x \in T(M)^*$. It is clear that $[x]_M \subseteq [\frac{x}{1}]_{M_S}$. For the reverse inclusion, let $\frac{m}{s} \in [\frac{x}{1}]_{M_S}$, such that $m \in [x]_M$, $s \in S$, so $\mathrm{Ann}(m)M = \mathrm{Ann}(x)M$ and thus, $\{s^n m \,:\, n \geq 1\} \subseteq [x]_M$. Now let $\left|[x]_M\right|$ be finite, then there exists $i \in I$ such that $s^i m = s^{i+1}m$. So

$$\frac{m}{s} = \frac{ms^i}{s^{i+1}} = \frac{ms^{i+1}}{s^{i+1}} = m \in [x]_M,$$

and therefore $\left|[x]_M\right| = \left|[\frac{x}{1}]_{M_S}\right|$. Now suppose that $\left|[x]_M\right|$ is infinite. We define an equivalence relation $\approx$ on S by $s \approx t$ if and only if $sx = tx$. It is easily verified that the map

$$\begin{array}{rcl} [x]_M \times S/\approx & \longrightarrow & [\frac{x}{1}]_{M_S} \\ (b, [s]) & \longmapsto & \frac{b}{s} \end{array}$$

is well-defined and surjective, because if $(b, [s]) = (a, [t])$, then $a = b$ and $[s] = [t]$. Hence,

$$(s - t)M \subseteq \operatorname{Ann}(x)M = \operatorname{Ann}(a)M = \operatorname{Ann}(b)M$$

and since M is multiplication $sa = ta$ and $sb = tb$, therefore $\frac{a}{t} = \frac{b}{s}$. Thus,

$$\left|\left[\frac{x}{1}\right]\right| \leq \left|[x]_M\right|\left|S/\approx\right|.$$

Also, the map

$$\begin{array}{rcl} S/\approx & \longrightarrow & [x]_M \\ [s] & \longmapsto & sa \end{array}$$

is clearly well-defined and injective. Hence, $\left|S/\approx\right| \leq \left|[x]_M\right|$ and thus,

$$\left|\left[\frac{x}{1}\right]_{M_S}\right| \leq \left|[x]_M\right|^2 = \left|[x]_M\right|,$$

since $\left|[x]_M\right|$ is infinite. Hence, $\left|[x]_M\right| = \left|[\frac{x}{1}]_{M_S}\right|$. Thus, there is a bijection map $\varphi_\alpha : [x_\alpha] \longrightarrow [\frac{x_\alpha}{1}]$ for each $\alpha \in \Lambda$. Now define

$$\begin{array}{rcl} \varphi : T(M)^* & \longrightarrow & T(M_S)^* \\ m & \longmapsto & \varphi(m) = \varphi_\alpha(m). \end{array}$$

Clearly φ is a bijection map. Thus, we need only to show that m and n are adjacent in $\Gamma(M)$ if and only if $\varphi(m)$ and $\varphi(n)$ are adjacent in $\Gamma(M_S)$; i.e.,

$$[m : M][n : M]M = 0 \qquad \Longleftrightarrow \qquad [\varphi(m) : M_S][\varphi(n) : M_S]M_S = 0.$$

Let $m \in [x]_M$, $n \in [y]_M$, $w \in [\frac{x}{1}]_{M_S}$ and $z \in [\frac{y}{1}]_{M_S}$. It is sufficient to show that

$$[m : M][n : M]M = 0 \qquad \Longleftrightarrow \qquad \left[\frac{w}{1} : M_S\right]\left[\frac{z}{1} : M_S\right]M_S = 0.$$

Note that

$$\begin{aligned}
&[m:M][n:M]M=0\\
\Longleftrightarrow\;& m\in \mathrm{Ann}_R(n)M=\mathrm{Ann}_R(y)M\\
\Longleftrightarrow\;& \frac{m}{1}\in \mathrm{Ann}_{R_S}\left(\frac{n}{1}\right)M_S=\mathrm{Ann}_{R_S}\left(\frac{y}{1}\right)M_S=\mathrm{Ann}_{R_S}\left(\frac{z}{1}\right)M_S\\
\Longleftrightarrow\;& \left[\frac{m}{1}:M_S\right]\left[\frac{z}{1}:M_S\right]M_S=0\\
\Longleftrightarrow\;& \frac{z}{1}\in \mathrm{Ann}_{R_S}\left(\frac{m}{1}\right)M_S=\mathrm{Ann}_{R_S}\left(\frac{x}{1}\right)M_S=\mathrm{Ann}\left(\frac{w}{1}\right)M_S\\
\Longleftrightarrow\;& \left[\frac{z}{1}:M_S\right]\left[\frac{w}{1}:M_S\right]M_S=0\,.
\end{aligned}$$

Hence, $\Gamma(M)$ and $\Gamma(M_S)$ are isomorphic as graphs. ∎

COROLLARY 2.2. *Let M and N be multiplication R-modules with $S^{-1}M\cong S^{-1}N$, then $\Gamma(M)\cong\Gamma(N)$. In particular $\Gamma(M)\cong\Gamma(N)$ when $S^{-1}M=S^{-1}N$.*

3. Complemented graph and multiplication module

In this section we prove that, if M is a reduced multiplication R-module and $\Gamma(M)$ is uniquely complemented, then $S^{-1}M$ is von Neumann regular and furthermore we show that if M is a multiplication R-module with $\mathrm{Nil}(M)\neq 0$, then $\Gamma(M)$ is uniquely complemented if and only if $\Gamma(M)$ is a star graph with at most six edges or is an infinite star graph (i.e., $\Gamma(M)$ has an infinite vertices such that there exists a vertex adjacent to every other vertices, and these are only adjacent relation). Finally we show that if M is a multiplication R-module and $\Gamma(M)$ is uniquely complemented, then either $\Gamma(M)$ is a star graph or $S^{-1}M$ is von Neumann regular, where $S=R\setminus Z(M)$.

Let G be a (undirected) graph. We will follow the authors in [4], and define that $a\leq b$ if a and b are not adjacent and each vertex of G adjacent to b is also adjacent to a; and we define $a\sim b$ if and only if $a\leq b$ and $b\leq a$. Thus, $a\sim b$ if and only if a and b are adjacent to exactly the same vertices. Clearly $\sim$ is an equivalence relation on G.

Now let M be a multiplication R-module and $m,n\in T(M)^*$, then $m\sim n$ if and only if $\mathrm{Ann}(m)M\setminus\{m\}=\mathrm{Ann}(n)M\setminus\{n\}$. Also we know that if $m\perp n$, then $[m:M][n:M]M=0$ and $\mathrm{Ann}(m)M\cap\mathrm{Ann}(n)M\subseteq\{0,m,n\}$. Now if $\mathrm{Ann}(m)M\cap\mathrm{Ann}(n)M=\{0,m,n\}$, then

$$[m:M]^2M=[n:M]^2M=[m:M][n:M]M=0\,.$$

On the other hand, since $m\perp n$, $m+n\in\{0,m,n\}$, so $m+n$ is adjacent to m and n, which is a contradiction. Therefore $m\perp n$ if and only if $\mathrm{Ann}(m)M\cap\mathrm{Ann}(n)M\subset\{0,m,n\}$ and $[m:M][n:M]M=0$.

LEMMA 3.1. *Consider the following statements for a multiplication R-module M with $m,m'\in T(M)^*$:*

(a) $m\sim m'$;

(b) $Rm=Rm'$;

(c) $\mathrm{Ann}(m)M=\mathrm{Ann}(m')M$.

Then under the above conditions we have:

(1) *If M is reduced, then statements* (a) *and* (c) *are equivalent.*

(2) *If M is von Neumann regular, then all three statements are equivalent.*

Proof. (1) Let M be reduced, one can easily check that (a) $\Leftrightarrow$ (c).

(2) Since every von Neumann regular module is reduced, so (a) $\Leftrightarrow$ (c). Clearly (b) $\Rightarrow$ (c). We show that (b) $\Leftarrow$ (c). Since M is von Neumann regular $Rm\cap[m:M]M=[m:M]Rm$. So $m=sm$ for some $s\in[m:M]$, hence, $(1-s)m'\in\mathrm{Ann}(m)M=\mathrm{Ann}(m')M$. Therefore $[m':M]m'\in Rm$. Moreover, since M is a von Neumann regular multiplication module $[m':M]m'=Rm'$ and so $Rm'\subseteq Rm$ and similarly $Rm\subseteq Rm'$. Consequently $Rm=Rm'$. ∎

LEMMA 3.2. *Let M be a reduced multiplication R-module and let $m,m',m''\in T(M)^*$. If $m\perp m'$ and $m\perp m''$, then $m'\sim m''$. Thus, $\Gamma(M)$ is uniquely complemented if and only if $\Gamma(M)$ is complemented.*

Proof. Let $m,m',m''\in T(M)^*$. Suppose $m\perp m'$ and $m\perp m''$. It is sufficient to show that $\mathrm{Ann}(m')M=\mathrm{Ann}(m'')M$. Suppose $x\in\mathrm{Ann}(m')M$, so $[x:M][m':M]M=0$. One can easily show that for all $\alpha\in[x:M]$,

$$[\alpha m'':M][m':M]M=0=[\alpha m'':M][m:M]M.$$

So $\alpha m''\in\{0,m,m'\}$. If $\alpha m''=m$ or $\alpha m''=m'$, then $m=0$ or $m'=0$, is a contradiction. Thus, $\alpha m''=0$ for all $\alpha\in[x:M]$. Therefore $x\in\mathrm{Ann}(m'')M$ and so $\mathrm{Ann}(m')M\subseteq\mathrm{Ann}(m'')M$. Similarly $\mathrm{Ann}(m'')M\subseteq\mathrm{Ann}(m')M$. ∎

Theorem 3.3. *Let M be a reduced multiplication R-module. If $\Gamma(M)$ is complemented, then $S^{-1}M$ is von Neumann regular, where $S = R \setminus Z(M)$.*

Proof. Let $0 \neq \frac{x}{s} \in S^{-1}M$, where $x \in M$ and $s \in S$. Let $x \notin T(M)^*$ and

$$x = \sum_{i=1}^{n} \alpha_i m_i \in [x : M]M\,,$$

where $\alpha_i \in [x : M]$ and $m_i \in M$. Suppose that $\alpha = \sum_{i=1}^{n} \alpha_i$. If $\alpha \in Z(M)$, then $\alpha m = 0$ for some non zero element $m \in M$. So $[m : M][x : M]M = 0$, hence, $0 \neq [m : M] \subseteq \mathrm{Ann}(x) = 0$, a contradiction. Therefore $\alpha \in S = R \setminus Z(M)$. Thus,

$$S^{-1}R\Big(\frac{x}{s}\Big) \cap S^{-1}M\Big(\frac{r}{t}\Big) = S^{-1}R\Big(\frac{r}{t}\frac{x}{s}\Big).$$

Therefore $S^{-1}M$ is von Neumann regular.

Next we can suppose that $x \in T(M)^*$. By the hypothesis there is $y \in T(M)^*$ such that $x \perp y$. Hence, $y \in \mathrm{Ann}(x)M$ and so $y = \sum_{i=1}^{m} \beta_i m_i$, $m_i \in M$ and $\beta_i \in \mathrm{Ann}(x)$. Let $\beta = \sum_{i=1}^{m} \beta_i$. We show that $\alpha + \beta \in S$. If $\alpha + \beta \in Z(M)$, then $(\alpha + \beta)m_0 = 0$ for some non zero $m_0 \in M$. So

$$[\alpha m_0 : M][x : M]M = 0 = [y : M][\alpha m_0 : M]M\,.$$

Since M is a reduced module $x \neq \alpha m_0$ and $\alpha m_0 \neq y$. Thus, $\alpha m_0 = 0$ and hence, $\beta m_0 = 0$, so

$$[x : M][m_0 : M]M = 0 = [y : M][m_0 : M]M.$$

By a similar argument we have $m_0 = 0$, a contradiction. Therefore $\alpha + \beta \in S$ and $\frac{x}{s} = \frac{\alpha}{\alpha+\beta}\frac{x}{s}$. So a simple check yields that

$$S^{-1}R\Big(\frac{x}{s}\Big) \cap S^{-1}M\Big(\frac{r}{t}\Big) = S^{-1}R\Big(\frac{r}{t}\frac{x}{s}\Big)\,.$$

Hence, $S^{-1}M$ is von Neumann regular. ∎

Next example shows that $S^{-1}M$ is von Neumann regular but M is not von Neumann regular in spite of $\Gamma(M) \cong \Gamma(S^{-1}M)$.

We know that an r-partite graph is one whose vertex set can be partitioned into r subsets so that no edge has both ends in any of these subsets. A complete r-partite graph is one in which each vertex is joined to every vertex that is in another subset. The complete bipartite graph (i.e., 2-partite graph) with vertex sets having m and n elements, will be denoted by $K_{m,n}$. A complete bipartite graph of the form $K_{1,n}$ is called a star graph.

EXAMPLES 3.4. (a) Let M_1 be an R_1-module and M_2 be an R_2-module, then $M = M_1 \times M_2$ is $R = R_1 \times R_2$ module with this multiplication $R \times M \longrightarrow M$, defined by $(r_1, r_2)(m_1, m_2) = (r_1 m_1, r_2 m_2)$. Now let $M = \mathbb{Z} \times n\mathbb{Z}$ and $R = \mathbb{Z} \times \mathbb{Z}$. It is clear that $\Gamma(M)$ is a complete bipartite graph (i.e., $\Gamma(M)$ may be partitioned into two disjoint vertex sets $V_1 = \{(m_1, 0) : m_1 \in (\mathbb{Z})^*\}$ and $V_2 = \{(0, m_2) : m_2 \in (n\mathbb{Z})^*\}$ and two vertices x and y are adjacent if and only if they are in distinct vertex sets). Therefore $\Gamma(M)$ is complemented. Also M is a faithful multiplication R-module, because $M = R(1, n)$. A simple check yields that M is reduced, thus, $S^{-1}M$ is von Neumann regular, by Theorem 3.3. But M is not von Neumann regular (use $N = R(2, 2n)$ and $I = [N : M]$).

(b) Let $R = \mathbb{Z}_2 \times \mathbb{Z}$ and $M = R$ as an R-module. So M is a faithful multiplication R-module. Clearly M is reduced and $\Gamma(M)$ is an infinite star graph with center $(\bar{1}, 0)$. Thus, $\Gamma(M)$ is complemented and by Theorem 3.3, $S^{-1}M$ is von Neumann regular, but M is not von Neumann regular.

COROLLARY 3.5. *Let M be a cyclic faithful reduced R-module. The following statements are equivalent:*

(1) *$S^{-1}M$ is von Neumann regular, where $S = R \setminus Z(M)$;*
(2) *$\Gamma(M)$ is uniquely complemented;*
(3) *$\Gamma(M)$ is complemented.*

Proof. (1) $\Rightarrow$ (2) Let M be a von Neumann regular R-module and $m \in T(M)^*$. So $[m : M]M \cap Rm = Rm[m : M]$. Since Rm is a weakly cancellation module, $R = [m : M] + \mathrm{Ann}(m)$. Say $M := Rx$ for some $x \in M$. Thus, $Rx = Rm + \mathrm{Ann}(m)x$ and therefore $x = rm + y$ for some $r \in R, y \in \mathrm{Ann}(m)x$. One can easily check that $y \in T(M)^*$ and $y \perp m$, so $\Gamma(M)$ is complemented. Since M is a faithful cyclic R-module, then $S^{-1}M$ is a faithful cyclic $S^{-1}R$-module and therefore by the above comments, $\Gamma(S^{-1}M)$ is complemented. Moreover by Theorem 2.1, $\Gamma(M) \cong \Gamma(S^{-1}M)$, so $\Gamma(M)$ is complemented. Consequently $\Gamma(M)$ is uniquely complemented by Lemma 3.2.

(2) $\Rightarrow$ (3) This is true for any graph.

(3) $\Rightarrow$ (1) By Theorem 3.3. ∎

COROLLARY 3.6. *Let M be a reduced multiplication R-module with $T(M) \neq M$. Then the following statements are equivalent:*

(1) *$S^{-1}M$ is von Neumann regular, where $S = R \setminus Z(M)$;*
(2) *$\Gamma(M)$ is uniquely complemented;*
(3) *$\Gamma(M)$ is complemented.*

Now we investigate some properties of M, when M is a multiplication R-module with $\mathrm{Nil}(M) \neq 0$. In this case, we extend [2, Theorem 3.9] in Theorem 3.12. First we give the following key lemma. Recall that a vertex of a graph is called an end if there is only one other vertex adjacent to it.

LEMMA 3.7. *Let R be a ring and M be a multiplication R-module with* $\mathrm{Nil}(M) \neq 0$*, then:*

(a) *If $\Gamma(M)$ is complemented, then either $8 \leq |M| \leq 16$ or $|M| \geq 17$ and* $\mathrm{Nil}(M) = \{0, x\}$ *for some $0 \neq x \in M$.*

(b) *If $\Gamma(M)$ is uniquely complemented and $|M| \geq 17$, then any complement of the nonzero element $x \in \mathrm{Nil}(M)$ is an end.*

Proof. (a) We subdivide the proof of (a) in the following steps:

Step 1: Let $\Gamma(M)$ be complemented. We show that for all $0 \neq \alpha \in [x : M]$, where $x \in \mathrm{Nil}(M)$, $\alpha^n x = 0$ for some $n \in \mathbb{N}$. Let $S = \{\alpha^n x \,:\, n \in \mathbb{N}\}$, we must show that $0 \in S$. Suppose that $0 \notin S$. Let $\Sigma = \{K \,:\, K \leq M, K \cap S = \emptyset\}$. By Zorn's lemma, let H be a maximal member of Σ. We claim that $[H : M]$ is a prime ideal of R. Clearly $[H : M] \neq R$, let $ab \in [H : M]$ but $a, b \notin [H : M]$ for $a, b \in R$. Hence, $(aM + H), (bM + H) \notin \sum$, so $\alpha^{n_1} x \in S \cap (aM + H)$ and $\alpha^{n_2} x \in S \cap (bM + H)$ for some $n_1, n_2 \in \mathbb{N}$. Therefore $\alpha^{n_1+n_2+1} x \in H \cap S$, is a contradiction. Hence, $[H : M]$ is a prime ideal and by [9, Corollary 2.11], H is a prime submodule of M. Since $x \in \mathrm{Nil}(M)$ we have $\alpha x \in H \cap S$, which is a contradiction and consequently $0 \in S$.

Choose n to be as small as possible $\alpha^n x = 0$. Then $n \geq 1$ and $\alpha^{n-1} x \neq 0$.

Step 2: In this step we claim that $n \leq 3$. Suppose that $n > 3$, so $\alpha x \in T(M)^*$. Since $\Gamma(M)$ is complemented, there exists $y \in T(M)^*$ such that y is a complement of αx. Then

$$[\alpha^{n-1} x : M][y : M]M = 0 = [\alpha^{n-1} x : M][\alpha x : M]M,$$

and so $\alpha^{n-1} x = y$ will be the only possibility. Thus, $\alpha x \perp \alpha^{n-1} x$. Similarly $\alpha^i x \perp \alpha^{n-1} x$ for each $1 \leq i \leq n-2$. Let $m = \alpha^{n-2} x + \alpha^{n-1} x$, then

$$[m : M][\alpha^{n-1} x : M]M = 0 = [m : M][\alpha^{n-2} x : M]M,$$

which is a contradiction, since $\alpha^{n-2} x \perp \alpha^{n-1} x$ and

$$\alpha^{n-2} x + \alpha^{n-1} x \notin \{0, \alpha^{n-1} x, \alpha^{n-2} x\}.$$

Thus, $n \leq 3$.

Step 3: Let $n = 3$, so $\alpha^3 x = 0$ but $\alpha^2 x \neq 0$. We show that either $|M| = 16$ or $|M| = 8$. Similar to Step 2, $\alpha x \perp \alpha^2 x$. Also $\mathrm{Ann}(x)M \subseteq \{0, \alpha^2 x\}$, since if $z \in \mathrm{Ann}(x)M$, then $[z : M][x : M]M = 0$, hence, if $0 \neq z$, z is adjacent to two elements αx and $\alpha^2 x$. Since $\alpha x \perp \alpha^2 x$, therefore $z = \alpha^2 x$. So $\mathrm{Ann}(x)M \subseteq \{0, \alpha^2 x\}$. Now for all $r \in R$,

$$[r\alpha^2 x : M][\alpha x : M]M = 0 = [r\alpha^2 x : M][\alpha^2 x : M]M,$$

hence, $r\alpha^2 x \in \{0, \alpha x, \alpha^2 x\}$. But $r\alpha^2 x = \alpha x$, then $\alpha^2 x = 0$, is a contradiction and so $R\alpha^2 x = \{0, \alpha^2 x\}$. Also

$$\mathrm{Ann}(\alpha^2 x)M \subseteq \left\{0, x, \alpha x, \alpha^2 x, x + \alpha x, x + \alpha^2 x, \alpha x + \alpha^2 x, x + \alpha x + \alpha^2 x\right\},$$

since if $z \in \mathrm{Ann}(\alpha^2 x)M$, then $\alpha^2 z \in \mathrm{Ann}(x)M \subseteq \{0, \alpha^2 x\}$ and so either $\alpha^2 z = 0$ or $\alpha^2 z = \alpha^2 x$. Thus, either

$$[\alpha z : M][\alpha x : M]M = 0 = [\alpha z : M][\alpha^2 x : M]M$$

or

$$[(\alpha z - \alpha x) : M][\alpha x : M]M = 0 = [(\alpha z - \alpha x) : M][\alpha^2 x : M]M.$$

Since $\alpha x \perp \alpha^2 x$, we have either $\alpha z \in \{0, \alpha x, \alpha^2 x\}$ or $(\alpha z - \alpha x) \in \{0, \alpha x, \alpha^2 x\}$. Now let $\alpha^2 z = 0$, so $\alpha z \neq \alpha x$ and therefore either $\alpha z = 0$ or $\alpha(z - \alpha x) = 0$ and so

$$[z : M][\alpha x : M]M = 0 = [z : M][\alpha^2 x : M]M$$

or

$$[(z - \alpha x) : M][\alpha x : M]M = 0 = [(z - \alpha x) : M][\alpha^2 x : M]M,$$

hence, $z \in \{0, \alpha x, \alpha^2 x, \alpha^2 x + \alpha x\}$. Thus, we may assume that $\alpha^2 z = \alpha^2 x$, then $\alpha z - \alpha x \neq \alpha x$. On the other hand $\alpha z - \alpha x \in \{0, \alpha x, \alpha^2 x\}$, so either $\alpha z - \alpha x = 0$ or $(\alpha z - \alpha x) = \alpha^2 x$ and by similar argument $z \in \{x, \alpha^2 x, x + \alpha x, x + \alpha x + \alpha^2 x\}$.

Now if $\alpha^2[x : M]x = 0$, then

$$\mathrm{Ann}(\alpha^2 x)M = \left\{0, x, \alpha x, \alpha^2 x, x + \alpha x, x + \alpha^2 x, \alpha x + \alpha^2 x, x + \alpha x + \alpha^2 x\right\}. \quad \text{(i)}$$

And if $\alpha^2[x : M]x \neq 0$, then

$$\mathrm{Ann}(\alpha^2 x)M = \left\{0, \alpha x, \alpha^2 x, \alpha x + \alpha^2 x\right\}. \quad \text{(ii)}$$

Now we claim that $|M| = 16$ in case (i) and $|M| = 8$ in case (ii). Since $\alpha^2[x : M]M \neq 0$, there are $\gamma \in [x : M]$ and $m \in M$ such that $\alpha^2 \gamma m \neq 0$ and a simple check yields $\alpha^2\gamma m = \alpha^2 x$. Let $m_0 \in M$, so $\alpha^2\gamma m_0 \in R\alpha^2 x = \{0, \alpha^2 x\}$. If $\alpha^2\gamma m_0 = 0$, then $m_0 \in \mathrm{Ann}(\alpha^2 x)M$ and if $\alpha^2\gamma m_0 = \alpha^2 x$, then $m_0 - m \in \mathrm{Ann}(\alpha^2 x)$. Consequently $|M| = 16$ in case (i) and $|M| = 8$ in case (ii).

Step 4: In this step we show that $H = \mathrm{Ann}(\alpha^2 x)M$ is the unique maximal submodule of M. Clearly $H \neq M$ and $R\alpha^2 x \cong \frac{R}{\mathrm{Ann}(\alpha^2 x)}$. Since $R\alpha^2 x = \{0, \alpha^2 x\}$, we have $\mathrm{Ann}(\alpha^2 x)$ is a maximal ideal of R. Hence, by [9, Theorem 2.5], $\mathrm{Ann}(\alpha^2 x)M$ is a maximal submodule. Also

$$\mathrm{Ann}(\alpha^2 x)M \subseteq Rx \subseteq \mathrm{Nil}(M) \subseteq \mathrm{Ann}(\alpha^2 x)M.$$

Therefore $\mathrm{Ann}(\alpha^2 x)M = \mathrm{Nil}(M)$ is the unique maximal submodule of M. If $T(M) \subseteq H = \mathrm{Ann}(\alpha^2 x)M$, then $T(M) = \mathrm{Ann}(\alpha^2 x)M$, so $\Gamma(M)$ is a star graph with 5 edges and center $\alpha^2 x$.

Step 5: Assume that $n = 2$, we show that $[x : M]^2 x = 0$. Let $[x : M]^2 x \neq 0$, so there exist two elements $\alpha, \beta \in [x : M]$ such that $\alpha\beta x \neq 0$. Also there are $m \in M$ and $\gamma \in [x : M]$ such that $\alpha\beta\gamma m \neq 0$, on the other hand $\alpha^2 x = \beta^2 x = \gamma^2 x = 0$ and there is $y \in T(M)^*$ such that $\alpha x \perp y$, a simple check yields that $R\alpha x \subseteq \{0, \alpha x, y\}$ and $y = \alpha\beta x$, hence, $\alpha x \perp \alpha\beta x$. So $R(\alpha x) = \{0, \alpha x, \alpha\beta x\}$ and $\mathrm{Ann}(\alpha x)M = \{0, \alpha x, \alpha\beta x\}$. Also $\alpha\beta\gamma m$ is adjacent to two vertices αx and $\alpha\beta x$, but $\alpha\beta\gamma m \neq \alpha x$, thus, $\alpha\beta\gamma m = \alpha\beta x$. We know that $\alpha\beta m$ is adjacent to two vertices αx and $\alpha\beta x$ but $\alpha\beta m \neq \alpha\beta x$ and $\alpha\beta m \neq \alpha x$, which is a contradiction. Thus, $[x : M]^2 x = 0$.

Step 6: Assume that $n = 2$ and $[x : M]^2 x = 0$. We show that $|M| \leq 12$. By hypothesis $\alpha^2 x = 0$ but $\alpha x \neq 0$, hence, $\alpha[x : M]M \neq 0$, thus, $\alpha\beta m \neq 0$ for some $\beta \in [x : M]$ and $m \in M$. We know that $\Gamma(M)$ is complemented and $x \in T(M)^*$, so there is $y \in T(M)^*$ such that $x \perp y$, but αx is adjacent to two vertices x and y. Hence, either $\alpha x = x$ or $\alpha x = y$. If $\alpha x = x$ then by multiplying in α we have $\alpha x = 0$, a contradiction. Therefore $\alpha x = y$, so $\alpha x \perp x$. Let $z \in \mathrm{Ann}(x)M$ hence, $z \in \{0, x, \alpha x\}$, since $x \perp \alpha x = y$, if $z = x$, then $[x : M]x = 0$ which is a contradiction. Therefore $\mathrm{Ann}(x)M = \{0, \alpha x\}$. Also a simple check yields that $R(\alpha x) = \{0, \alpha x\}$. On the other hand $\alpha m \in T(M)^*$ and so there exists $w \in T(M)^*$ such that $\alpha m \perp w$. But $\alpha\beta m$ is adjacent to two vertices αm and w, therefore $\alpha\beta m = w$ will be the only possibility and so $\alpha\beta m \perp \alpha m$. Also $\alpha\beta m$ is adjacent to two vertices αx and x. Hence, $\alpha\beta m = \alpha x$. Now we show that $\mathrm{Ann}(\alpha x)M = \{0, \alpha m, \alpha x, x, x + \alpha m, x + \alpha x\}$, let $v \in \mathrm{Ann}(\alpha x)M$ so $\alpha v \in \mathrm{Ann}(x)M = \{0, \alpha x\}$. If $\alpha v = 0$, then

$$[v : M][\alpha\beta m : M]M = 0 = [v : M][\alpha m : M]M$$

and if $\alpha v = \alpha x$, then

$$[v - x : M][\alpha\beta m : M]M = 0 = [v - x : M][\alpha m : M]M.$$

Consequently,

$$\mathrm{Ann}(\alpha x)M = \big\{0, \alpha m, \alpha x, x, x + \alpha m, x + \alpha x\big\}$$

and so $\big|\mathrm{Ann}(\alpha x)M\big| \leq 6$. For all $m_0 \in M$, $\alpha\beta m_0 \in R(\alpha x) = \{0, \alpha x\}$. So either $m_0 \in \mathrm{Ann}(\alpha x)M$ or $m_0 - m \in \mathrm{Ann}(\alpha x)M$, since $\alpha\beta m = \alpha x$. Therefore $|M| \leq 12$. And by a similar argument in Step 4, $\mathrm{Ann}(\alpha x)M = \mathrm{Nil}(M)$ is the unique maximal submodule of M and $\Gamma(M)$ is a star graph.

Step 7: Suppose that $n = 1$. If $[x : M]x \neq 0$ by the above steps we have $8 \leq |M| \leq 16$. So we can assume that $[x : M]x = 0$. We show that either $|M| = 9$ or $\mathrm{Nil}(M) = \{0, x\}$ with $2x = 0$ and $|M| \neq 9$. Let $x \in [x : M]M$ so $x = \Sigma_{i=1}^{n}\alpha_i m_i$ where $\alpha_i \in [x : M]$ and $m_i \in M$ for all $1 \leq i \leq n$. Assume that $\alpha_i m_i \neq 0$. Since $\Gamma(M)$ is complemented, then there is $y \in T(M)^*$ such that $x \perp y$, so $Rx \subseteq \{0, x, y\}$. If $x \neq \alpha_i m_i$ for all i, then $\alpha_i m_i \in Rx$ and so $\alpha_i m_i = y$ for all i. Suppose that $\alpha_i m_i = \alpha_1 m_1$, thus $x = \Sigma_{i=1}^{n}\alpha_1 m_1 = (\Sigma_{i=1}^{n}\alpha_1)m_1 = \beta m_1$ where $\beta = \Sigma_{i=1}^{n}\alpha_1 \in [x : M]$. Otherwise $x = \alpha_i m_i$ for some $1 \leq i \leq n$. Hence, we may assume that $x = \alpha m$ for some $\alpha \in [x : M]$ and $m \in M$ such that $\alpha^2 m = 0$ but $0 \neq \alpha m$. We know that $x + x \in Rx \subseteq \{0, x, y\}$, if $x + x \neq 0$, then $Rx = \{0, x, 2x\}$, $x \perp 2x$ and $\mathrm{Ann}(x)M = \{0, x, 2x\}$. And for all $m_0 \in M$, $\alpha m_0 \in Rx$, therefore

$$[m_0 : M][x : M]M = 0 = [m_0 : M][2x : M]$$

or

$$[m_0 - m : M][x : M]M = 0 = [m_0 - m : M][2x : M]$$

or

$$[m_0 - 2m : M][x : M]M = 0 = [m_0 - 2m : M][2x : M].$$

Hence, $|M| = 9$ and by a similar argument in Step 4, $\mathrm{Ann}(x)M$ is the unique maximal submodule of M and $\Gamma(M)$ is a star graph. Now let $|M| \neq 9$ so by the above argument we must have $2x = 0$. We claim that $\mathrm{Nil}(M) = \{0, x\}$. Suppose that z is another nonzero element of $\mathrm{Nil}(M)$, hence, $[z : M]z = 0$ and $z = \beta m'$ for some $\beta \in [z : M]$ and $m' \in M$, such that $\beta^2 m' = 0$. So that $\Gamma(M)$ is complemented there are $x', z' \in T(M)^*$ such that $x \perp x'$ and $z \perp z'$, therefore $Rx \subseteq \{0, x, x'\}$ and $Rz \subseteq \{0, z, z'\}$. Observe that $\alpha\beta m = 0$. Let $0 \neq \alpha\beta m \in Rx$ and $\alpha\beta m \in Rz$, if $\alpha\beta m = x \in Rz$, thus,

$x = z'$, so $x \perp z$ and hence, $\alpha\beta m = 0$ is a contradiction. And if $\alpha\beta m = x'$, then $Rx = \{0, x, \alpha\beta m\} = \mathrm{Ann}(x)M$ and similar to the above argument, $|M| = 9$ which is a contradiction. So $\alpha\beta m = 0$ and similarly $\alpha\beta m' = 0$. Let w be a complement of $x + z$. Clearly $x + z$ is neither x nor z. Also $\alpha w \in Rx \subseteq \{0, x, x'\}$, if $\alpha w = 0$, then x is adjacent to two elements w and $x + z$, a contradiction. While if $\alpha w = x'$, then $Rx = \{0, x, \alpha w\} = \mathrm{Ann}(x)M$ and it implies that $|M| = 9$, a contradiction. Hence, we may assume that $\alpha w = x$ and similarly $\beta w = z$. Then

$$0 \neq x + z = \alpha w + \beta w \in [x : M][w : M]M + [z : M][w : M]M$$

since $w \perp x + z$,

$$[w : M]Rx + [w : M]Ry = 0,$$

and so $x + z = 0$ which is a contradiction. Consequently $\mathrm{Nil}(M) = \{0, x\}$.

(b) Let $0 \neq x \in \mathrm{Nil}(M)$ and $|M| \geq 17$. By the proof of (a) we have $\mathrm{Nil}(M) = \{0, x\}$ for some $x \in M$ such that $x = -x$ and $[x : M]x = 0$. Since $\Gamma(M)$ is complemented, there is $y \in T(M)^*$ such that $x \perp y$. We claim that y is an end. We first show that $x + y$ is also a complement for x. Clearly $x + y \in T(M)^*$ and $[x + y : M][x : M]M = 0$, because $[x : M]x = 0$ and $x \perp y$. If $w \in T(M)^*$ is adjacent to both x and $x + y$, then

$$[x + y : M][w : M]M = 0 = [x : M][w : M]M.$$

Hence, $[w : M]R(x + y) = 0$, so $[y : M][w : M]M = 0$. Moreover $x \perp y$, thus, either $w = x$ or $w = y$. If $w = y$, then $[y : M]y = 0$. Therefore $y \in \mathrm{Nil}(M) = \{0, x\}$, a contradiction. So $x = w$. Thus, $x + y$ is a complement for x. Since $\Gamma(M)$ is uniquely complemented, $x + y \sim y$. Assume that $z \in T(M)^* \setminus \{x\}$ such that z is adjacent to y, hence, z is adjacent to $x + y$. So $[z : M][x : M]M = 0$. Thus, $z = y$, because $x \perp y$. Consequently y is an end. ∎

Remark 3.8. The proof of Lemma 3.7 (a), shows that if M is a faithful multiplication R-module such that $\Gamma(M)$ is complemented and $\big|\mathrm{Nil}(M)\big| > 2$, then $8 \leq |M| \leq 16$ and $\Gamma(M)$ is a star graph with at most 5 edges. So it is uniquely complemented. Also it shows that if $\Gamma(M)$ is not uniquely complemented, then $\mathrm{Nil}(M) = \{0, x\}$, which x is an element of M, such that $x[x : M] = 0$. Hence, $x = \beta m$ for some $m \in M$ and $\beta \in [x : M]$.

Before stating the following proposition we define:

$$D(M) := \big\{m \in M \,:\, [m : M][m' : M]M = 0 \text{ for some nonzero } m' \in M\big\}.$$

PROPOSITION 3.9. *Let $M = M_1 \times M_2$ be a multiplication R-module, $R = R_1 \times R_2$, in which M_1 is a reduced module and* $\mathrm{Nil}(M_2) \neq 0$. *If* $\Gamma(M)$ *is complemented but not uniquely complemented, then M_1 is torsion free,* $\mathrm{Nil}(M_2)$ *is the unique maximal submodule of M_2 with* $\left|\mathrm{Nil}(M_2)\right| = 2$ *and* $\left|M_2\right| = 4$. *Furthermore if $D(M_2) \neq M_2$, then the converse is true.*

Proof. Suppose that $\mathrm{Nil}(M_2) \neq 0$ and $\Gamma(M)$ is complemented but not uniquely complemented. Let $0 \neq b \in \mathrm{Nil}(M_2)$, by the proof of Lemma 3.7, Step 1, $\beta^n b = 0$ for some $n \in \mathbb{N}$ and all $\beta \in [b : M_2]$. Therefore $b \in \mathrm{Nil}(M)$. Since $\Gamma(M)$ is not uniquely complemented, by Remark 3.8, $\left|\mathrm{Nil}(M)\right| = 2$ and $b = \beta m$, for some $\beta \in [b : M]$ and $m \in M$ such that $\beta^2 m = 0$. So $\left|\mathrm{Nil}(M_2)\right| = 2$. Let $\mathrm{Nil}(M_2) = \{0, b\}$ for some nonzero $b \in M_2$. Since $Rb \subseteq \mathrm{Nil}(M_2) = \{0, b\}$, hence, $Rb = \{0, b\}$. First we show that $\mathrm{Ann}(b)M_2 = \{0, b\}$. Suppose that $c \in \mathrm{Ann}(b)M_2 - \{0, b\}$ and $0 \neq m_1 \in M_1$, a simple check yields that $(m_1, b) \in T(M)^*$. So there is $(x, y) \in T(M)^*$ for some $x \in M_1$ and $y \in M_2$ such that (x, y) is a complement of (m_1, b). Hence, if $y = b$, then $(0, c)$ is adjacent to both (m_1, b) and (x, y), which is a contradiction. Thus, $y \neq b$, so

$$[(0, b) : M][(m_1, b) : M]M = 0 = [(0, b) : M][(x, y) : M]M,$$

again a contradiction. Therefore $\mathrm{Ann}(b)M_2 = \{0, b\}$. One can easily show that $\left|M_2\right| = 4$ (similar to the proof of Lemma 3.7). Also similar to Step 4 of the proof of Lemma 3.7, $\mathrm{Nil}(M_2) = \mathrm{Ann}(b)M_2$ will be the unique maximal submodule of M_2. Next we show that $D(M_1) = 0$. If not, let $0 \neq m_1 \in D(M_1)$. Since M_1 is reduced, then there is $0 \neq m'_1 \in D(M_1)$ such that $m_1 \neq m'_1$. Therefore $[m_1 : M_1][m'_1 : M_1]M_1 = 0$. On the other hand there is $m = (x, y) \in T(M)^*$ such that $m \perp (m_1, b)$. If $x \neq 0$, then b is adjacent to both m and (m_1, b), a contradiction. And if $x = 0$, then $y \neq 0$, but $y \in \mathrm{Ann}(b)M_2 = \{0, b\}$, thus,

$$[(m'_1, b) : M][(m_1, b) : M]M = 0 = [(m'_1, b) : M][(x, y) : M]M,$$

a contradiction. Therefore $D(M_1) = 0$. Suppose that $m_1 \in T(M_1)^*$, by hypothesis, there is a $(x_1, y_1) \in T(M)^*$ such that $(m_1, 0) \perp (x_1, y_1)$, hence, $[(m_1, 0) : M][(x_1, y_1) : M]M = 0$. So $[m_1 : M_1][x_1 : M_1]M_1 = 0$. Hence, $m_1 = 0$. Thus, M_1 is torsion free.

Conversely, let $M = M_1 \times M_2$ be a multiplication R-module, $R = R_1 \times R_2$, where M_1 is torsion free and $\mathrm{Nil}(M_2)$ is the unique maximal submodule of M_2, also, $D(M_2) \neq M_2$ and $\left|M_2\right| = 4$. So, by [9, Theorem 2.5], $\mathrm{Nil}(M_2) = D(M_2)$.

Let $D(M_2) = \{0, b\}$, therefore

$$T(M) = \big\{(0, m_2) \,:\, m_2 \in M_2\big\} \cup \big\{(m_1, b) \,:\, m_1 \in M_1)\big\} \\ \cup \big\{(m_1, 0) \,:\, m_1 \in M_1\big\}.$$

Thus, $\Gamma(M)$ is complemented but not uniquely complemented, because $(m_1, 0) \perp (0, b)$ and $(0, b) \perp (m_1, b)$ but $(m_1, 0) \not\sim (m_1, b)$. ∎

Example 3.10. Let $M = \mathbb{Z} \times \frac{\mathbb{Z}_2[x]}{<x^2>}$ as R-module, $R = \mathbb{Z} \times \mathbb{Z}_2[x]$. By Proposition 3.9, $\Gamma(M)$ is complemented but not uniquely complemented.

Theorem 3.11. *Let R be a ring and M be a multiplication R-module. If $\Gamma(M)$ is complemented, but not uniquely complemented, then $M = M_1 \oplus M_2$, where M_1, M_2 are submodules of M.*

Proof. Let $\Gamma(M)$ be complemented, but not uniquely complemented. There is a vertex a with distinct complements z and y and a vertex w which is adjacent to y, but not z. Thus, $[w : M][z : M]M \neq 0$. So there is $\beta \in [z : M]$ such that $\beta w \neq 0$. Also $\beta w \in T(M)^*$, on the other hand, $[\beta w : M][y : M]M = 0$ and $[\beta w : M][a : M]M = 0$. Since $a \perp y$ and $0 \neq \beta w$, we have either $\beta w = y$ or $\beta w = a$ and hence, $[y : M]y = 0$ or $[a : M]a = 0$. Thus, $y \in \mathrm{Nil}(M)$ or $a \in \mathrm{Nil}(M)$. Furthermore, by Remark 3.8, $\mathrm{Nil}(M) = \{0, m\}$. Suppose that $a = m$, since $m \perp y$, we have $[w : M][m : M]M \neq 0$ and so there is $\beta_1 \in [w : M]$ such that $\beta_1 m \neq 0$, also we know that $\beta_1 m \in \mathrm{Nil}(M)$, so $\beta_1 m = m$. Let $v = \beta_1 w - w$. Clearly $[v : M][y : M]M = 0$, let $r \in [m : M]$, $rv = r\beta_1 w - rw = 0$, hence $[m : M]v = 0$. Since $m \perp y$, we have $v \in \{0, y, m\}$. If $v = y$, then $y \in \mathrm{Nil}(M)$, a contradiction. If $v = 0$, then $\beta_1 w = w$ and so $(\beta_1 - 1) \in \mathrm{Ann}(w)$. Thus, $M = Rw \oplus \mathrm{Ann}(w)M$. If $v = m$, then $\beta_1 w - w \in \mathrm{Nil}(M)$. Let $n = \beta_1^2 - \beta_1$. Hence, $n \in [\beta_1 w - w : M]$ and by the proof of Lemma 3.7, Step 1, $(n)^s(\beta_1 w - w) = 0$, for some $s \in \mathbb{N}$. Thus, $n^{s+1} w = 0$. Let

$$r = \frac{1}{2}\left[2n - \frac{4}{2}n^2 + \frac{6}{3}n^3 + \ldots + (-1)^s \frac{2s-2}{s-1} n^{s-1}\right],$$

and suppose that $x = rw$ and $e = \beta_1 w + x(1 - 2\beta_1)$. Clearly

$$(r^2 - r)(1 + 4n)w + nw = 0\,.$$

Suppose that $\alpha = [\beta_1 + r(1 - 2\beta_1)]$. Therefore $\alpha w = \alpha^2 w$ and $\alpha \in [w : M]$. As a similar argument we have $M = R\beta_1 w \oplus \mathrm{Ann}(\beta_1)M$. ∎

Clearly star graphs are uniquely complemented. Next theorem shows that for a multiplication R-module M with $\mathrm{Nil}(M) \neq 0$, if $\Gamma(M)$ is uniquely complemented, then $\Gamma(M)$ is an star graph.

THEOREM 3.12. *Let R be a ring and M be a multiplication R-module with* $\mathrm{Nil}(M) \neq 0$. *If $\Gamma(M)$ is a uniquely complemented graph, then either $\Gamma(M)$ is a star graph with at most six edges or $\Gamma(M)$ is an infinite star graph with center x, where* $\mathrm{Nil}(M) = \{0, x\}$.

Proof. Suppose that $\Gamma(M)$ is uniquely complemented and $\mathrm{Nil}(M) \neq 0$. By the proof of Lemma 3.7 (a), M has a unique maximal submodule. Let H be the maximal submodule. Since $\Gamma(M)$ is complemented, $Rm \neq M$ for all $m \in T(M)$ therefore, by [9, Theorem 2.5], $Rm \subseteq H$, so $T(M) \subseteq H$.

Let $|M| \leq 16$, then by Remark 3.8 $\Gamma(M)$ is a star graph with at most six edges.

Now let $|M| > 16$. Hence by Step 7 of Lemma 3.7 (a), $\mathrm{Nil}(M) = \{0, x\}$ for some $0 \neq x \in M$ and $[x : M]x = 0$.

We first show that $\Gamma(M)$ is an infinite graph. Let c be a complement of x, so $\mathrm{Ann}(c)M = \{0, x\} = \mathrm{Nil}(M)$, by Lemma 3.7 (b). Let $c = \Sigma_{i=1}^{n}(\alpha_i m_i) \in [c : M]M$, where $\alpha_i \in [c : M]$ and $m_i \in M$, for $1 \leq i \leq n$ and suppose that $\alpha = \Sigma_{i=1}^{n}\alpha_i$. We claim that αc is also a complement of x. If z is adjacent to both vertices x and αc, then

$$[\alpha c : M][z : M]M = 0 = [x : M][z : M]M.$$

Therefore $\alpha z \in \mathrm{Ann}(c)M = \{0, x\}$. So either $\alpha z = 0$ or $\alpha z = x$. If $\alpha z = 0$, then $z \in \mathrm{Ann}(c)M$, a contradiction. Thus $\alpha z = x$. Hence $\alpha[z : M]z = x[z : M] = 0$. Therefore $z[z : M] \subseteq \mathrm{Ann}(c)M = \mathrm{Nil}(M)$ and hence $z \in \mathrm{Nil}(M) = \{0, x\}$, again a contradiction. Consequently $\alpha c \perp x$ and so, by Lemma 3.7 (b), $\mathrm{Ann}(\alpha c)M = \{0, x\}$. By a similar argument $\alpha^i c \perp x$ and $\mathrm{Ann}(\alpha^i c)M = \{0, x\}$ for $1 \leq i \leq n$. Hence each $\alpha^i c$ is an end. Next note that $\alpha^i c$ are all distinct. If not, suppose that $\alpha^i c = \alpha^j c$ for some $1 \leq i < j$. Therefore $\alpha^i(1 - \alpha^{j-i})c = 0$, so $(1 - \alpha^{j-i}) \in \mathrm{Ann}(\alpha^i c)$. By the proof of Lemma 3.7 (a), Step 7, $x = \beta m$ for some $\beta \in [x : M]$ and $m \in M$ such that $\beta^2 m = 0$ but $\beta m \neq 0$. Hence $(1 - \alpha^{j-i})m \in \mathrm{Ann}(\alpha^i c)M = \{0, x\}$. So either $m - \alpha^{i-j}m = 0$ or $m - \alpha^{i-j}m = x$. If $m = \alpha^{i-j}m$, then

$$x = \beta m = \beta\alpha^{i-j}m \in \beta\alpha^{i-j-1}Rc \subseteq \alpha^{i-j-1}[x : M][c : M]M = 0,$$

a contradiction. Thus $m - \alpha^{i-j}m = x$. So

$$x - \alpha^{i-j}\beta m = \beta m - \alpha^{i-j}\beta m = \beta x = 0.$$

Hence $x \in \alpha^{i-j-1}\beta Rc = 0$, again a contradiction. Consequently $\Gamma(M)$ is infinite.

We next show that $\Gamma(M)$ is a star graph with center x. By contradiction, suppose that $\Gamma(M)$ is not a star graph. Let $c \in T(M)^*$ be a complement of x, so there is a $a \in T(M)^* \setminus \{x, c\}$ such that $[a : M][x : M]M = 0$ but a is not an end. Hence there is $y \in T(M)^* \setminus \{a, x, c\}$ such that $y \perp a$. Let $c = \Sigma_{i=1}^{n}(\alpha_i m_i)$, where $\alpha_i \in [c : M]$ and $m_i \in M$, for $1 \leq i \leq n$ and let $\alpha = \Sigma_{i=1}^{n}\alpha_i$. We can check that $\alpha y \notin \{0, a, x, c, y\}$. If $\alpha y = 0$, then $[y : M]c = 0$, which is a contradiction with c is an end. If $\alpha y = x$, then $\alpha[y : M][c : M]M = 0$, so $y \in \mathrm{Ann}(\alpha c)M = \{0, x\}$, a contradiction. If $\alpha y = y$, then $\alpha y[x : M] \subseteq [x : M]Rc = 0$, a contradiction. If $\alpha y = c$, then a is adjacent to c, which is a contradiction. At last if $\alpha y = a$, then $\alpha y[y : M] = 0$. So $y[y : M] \in \mathrm{Ann}(\alpha c)M = \mathrm{Nil}(M)$ and therefore $y \in \mathrm{Nil}(M)$, which is a contradiction. Thus $\alpha y \in T(M)^* \setminus \{a, x, c, y\}$. By the hypothesis, there is $z \in T(M)^*$ such that z is a complement of αy. One can also verify that $z \notin \{0, \alpha y, a, x, c, y\}$. (Use $y \notin \mathrm{Nil}(M)$ to show that $z \notin \{c, y\}$ and use $\alpha y \perp z$ to show that $z \notin \{a, x\}$.) Clearly $[x : M][z : M]M \neq 0$. Let $z = \Sigma_{i=1}^{s} r_i m_i$, where $r_i \in [z : M]$ and $m_i \in M$, for $1 \leq i \leq s$ and let $\gamma = \Sigma_{i=1}^{n} r_i$. If $\gamma x = 0$, then $[x : M][z : M]M = 0$, a contradiction. So we must suppose that $\gamma x \neq 0$. Also $[\gamma x : M][c : M]M = 0$, hence $\gamma x \in \mathrm{Ann}(c)M$. Thus $\gamma x = x$. On the other hand, $\alpha y \perp z$, so

$$[\gamma y : M][c : M]M = [y : M]R(\Sigma_{i=1}^{n}(\gamma\alpha_i m_i)) \subseteq [y : M]R\alpha z = 0.$$

Therefore $\gamma y \in \mathrm{Ann}(c)M$. Hence either $\gamma y = 0$ or $\gamma y = x$. So x is adjacent to both y and a. But this is a contradiction that $a \perp y$. Consequently $\Gamma(M)$ is an infinite star graph with center x. ∎

Corollary 3.13. *Let M be a multiplication R module. If $\Gamma(M)$ is uniquely complemented, then either $\Gamma(M)$ is a star graph or $S^{-1}M$ is von Neumann regular, where $S = R \setminus Z(M)$.*

Morovere, for faithful cyclic R-module M, the converse is true.

Proof. Let $\Gamma(M)$ be uniquely complemented. If $\mathrm{Nil}(M) = 0$, then M is a reduced and by Theorem 3.3, $S^{-1}M$ is von Neumann regular. If $\mathrm{Nil}(M) \neq 0$, then by Theorem 3.12. $\Gamma(M)$ is a star graph. Converse is true by Corollary 3.5. ∎

Corollary 3.14. *Let M be a multiplication R module with $T(M) \neq M$. Then $\Gamma(M)$ is uniquely complemented, if and only if either $\Gamma(M)$ is a star graph or $S^{-1}M$ is von Neumann regular, where $S = R \setminus Z(M)$.*

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