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Studies on Dendrites and the Periodic-Recurrent Property*

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Abstract: In this paper we evidence the interest of considering three outstanding examples of dendrites with different structures, dendrites F_ω, W and G^3. When a dendrite X contains a topological copy of one of them, then it is derived important properties. For example, if X does not contain a topological copy neither F_ω nor W, then X is a tree. If X does not contain a topological copy of G^3 then we obtain that X verifies the Periodic-Recurrent Property (the PR Property) which for dendrites is relevant under the point of view of Topological Dynamics. As an application of the former results, we give a unified proof of the fact that compact intervals of the real line $[a, b]$ ($a \neq b$), arcs and trees have also the PR Property.

Key words: PR Property, dendrites, Gehman dendrite, continua.

AMS *Subject Class.* (2010): 37B45, 37E25.

1. Introduction, definitions and previous results

The interest on dynamical systems on dendrites has been increasing in last years mainly because its appearance in Julia and Fatou problems in complex dynamics. Such interest has extended also to the setting of general continua of low dimension.

In order to understand the behaviors of dynamical systems (X, f) where X is a dendrite and $f : X \to X$ is continuous (written $f \in C(X)$), we consider three well known examples of different types of dendrites whose structures are representative and relevant. This starting point allow us further consideration of similar problems when X is a general continuum of low dimension like $\sin(\frac{1}{x})$-*continuum, Warsaw circle, dendroids,* etc

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Before doing this, it is necessary some introductory definitions and properties which can be seen in more detail in some references, in particular in [10] and [2].

Definition 1. A continuum is a non empty, compact, connected metric space. A sub-continuum is a continuum that is a subset of a continuum.

Definition 2. A metric space is a Peano space if it is locally connected. A Peano continuum is a Peano space which is a continuum.

Next result is an interesting description of Peano continua.

Theorem 1. (Hahn-Mazurkiewicz) *Every Peano continuum is a continuous image of the closed interval* $[0, 1]$.

Definition 3. A dendrite is a Peano continuum that does not contain simple closed curves. A sub-dendrite is a dendrite that is a subset of a dendrite.

Definition 4. Let (X, τ) be a topological space and $p \in X$. Then,

1. If $X \backslash \{p\}$ is non-connected, then p is a *cut point* of X. If $X \backslash \{p\}$ is connected, p is not a *cut point* of X.
2. Let β be a cardinal number. It is said that p has an *order* less or equal than β ($\mathrm{Ord}(p, X) \leq \beta$) whether for every $U \in \tau$ holding $p \in U$, there is $V \in \tau$ such that $p \in V \subset U$ and $\sharp(\partial(V)) \leq \beta$, (with $\partial(V)$ we denote the *boundary* of V). The order of p is β ($\mathrm{ord}(p, X) = \beta$) if $\mathrm{ord}(p, X) \leq \beta$ and for every $\alpha < \beta$ we have $\mathrm{ord}(p, X) \leq \alpha$.
3. p is a *end point* of X if $Ord(p, X) = 1$. $E(X)$ denotes the set of end points of X.
4. p is a *branching point* of X if $Ord(p, X) \geq 3$. $B(X)$ denotes the set of branching points of X.
5. p is a point of order ω of X if for every $n \in \mathbb{N}$ there is $U_n \in \tau$ such that $p \in U_n$ and that for every $V \subset U_n$ is $\sharp(\partial(V)) > n$.

After such definitions we recall the following interesting characterizations. Their proofs can be seen in [10] or [2].

Theorem 2. (Hahn-Mazurkiewicz) *Let X be a continuum having more than one point. Then it is a dendrite if and only if each point of X is either a cut or an end point.*

THEOREM 3. (HAHN-MAZURKIEWICZ) *Let X be a dendrite, then the set $B(X)$ is countable.*

DEFINITION 5. Let X be a topological space. Then,

1. An arc in X is a topological sub-space of X homeomorphic to $[0,1]$.
2. A graph in X is a continuum obtained as the finite union of arcs, where two of them either are disjoint or intersect in one or both end points.
3. A tree in X is a graph that does not contain simple closed curves.

Further we are introducing three particular dendrites which summarize important properties in relation with two facts; deciding when a dendrite is a tree and finding conditions for a dendrite to contain an homeomorphic copy of other topological structure, in particular of other dendrites.

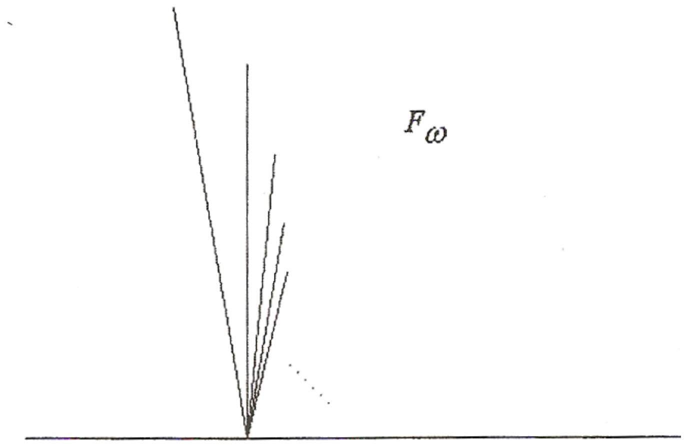

Figure 1: Dendrite F_ω for $n \geq 2$

2. DENDRITES F_ω AND W

F_ω is the graph of the following set in $\mathbb{C}$

$$F_\omega = \{r \exp(\frac{2\pi i}{n}) \in \mathbb{C} : r \in [0, \frac{1}{n}] \text{ for } n \in \mathbb{N}\}$$

which has a unique branching point of order ω (see Figure 1).

Proposition 1. *Let X be a dendrite having a point of order ω. Then X contains a topological copy of F_ω.*

Proof. The homeomorphism can be constructed identifying the points of order ω and enumerating the corresponding branchings to be able of transforming them continuously. ∎

The second dendrite W is

$$W = ([0,1] \times \{0\}) \cup (\cup \{\{\frac{1}{n}\} \times [0, \frac{1}{n}] : n \in \mathbb{N}\})$$

(see Figure 2). W can be seen in $\mathbb{R}^2$ as

$$W = ab_1 \cup (\cup \{a_n b_n : n \in \mathbb{N}\})$$

where $a = (0,0)$ and for every n are $a_n = (\frac{1}{n}, \frac{1}{n})$ and $b_n = (\frac{1}{n}, 0)$, (ab_1 denotes the *segment* joining the points a and b_1 and similarly for $a_n b_n$). First we give

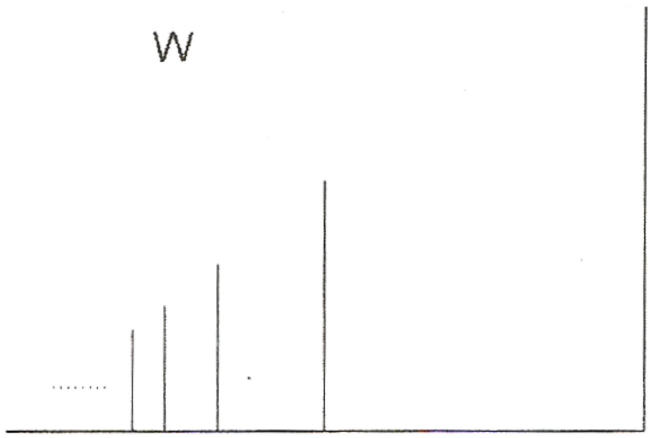

Figure 2: Dendrite W

a consequence of not having a topological copy of W.

Proposition 2. *Let X be a dendrite not containing a topological copy of W. Then for every arc, $J \subset X$, we have $J \cap B(X)$ is a finite set.*

Proof. Let us suppose on contrary that there exists an arc $J \subset X$ such that $J \cap B(X)$ be infinite. Taking a countable number of such points we can define a homeomorphism transforming J into $[0,1]$ and the points $(\frac{1}{n},0)$ into the chosen points. From this it is immediate that X would contain a topological copy of W. ∎

One interesting question is to decide when a dendrite can be a tree.

THEOREM 4. *Let X be a dendrite. Then X is a tree if and only if X does not contain a topological copy neither of F_ω nor of W.*

Proof. If X is a tree, then every sub-continuum is also a tree (see [10]). By the previous definitions, neither F_ω nor W are trees and therefore X can not contain topological copies of them. By other hand, let us suppose X does not contain a topological copy of such dendrites. Then all points of X are of finite order. One possibility is that X contains a finite number of end points. In such case it would be a graph (see [10]) and since it is contained in a dendrite then it would be a tree. Let us suppose that X has an infinite number of end points and choose a convergent non constant sequence $(x_n)_{n=1}^{\infty}$ of end points of X. Let $x = lim_{n\to\infty} x_n$. Since $\text{ord}(x,X)$ is finite, we can suppose that all points belong to the same component of $X\backslash\{x\}$. We introduce now an ordering $\prec_x$ in the following way: $p \prec_x q$ if $p \in xq$ (which means here the unique arc joining x with q). Let $p_n = \inf_{\prec_x}\{x_1,...,x_n\}$. It results that $(p_n)_{n=1}^{\infty}$ is a decreasing sequence and since $p_n \in xx_n$ we have that $lim_{n\to\infty} p_n = x$. Choose now a subsequence $(p_{n_k})_{k=1}^{\infty}$ such that $p_{n_i} \neq p_{n_j}$ for $i \neq j$. Let Y be the smallest continuum containing the points x_{n_k} for $k \in \mathbb{N}$. Then there is an homeomorphism $h: Y \to W$ such that $h(x_{n_k}) = a_k$ and $h(p_{n_k}) = b_k$, therefore X would contain a topological copy of W. It finishes the proof. ∎

COROLLARY 1. *Let X be a dendrite having a finite number of branching points and such that all its points has a finite order. Then X is a tree.*

Proof. The dendrite of the statement can not contain a topological copy neither F_ω nor W. Using the former theorem we obtain that X is a tree. ∎

3. GEHMAN DENDRITES

Fix $n \geq 3$ a positive integer and $\alpha_1,\alpha_2,...,\alpha_i \in \{0,1,...,2n-4\} = \alpha$ chosen numbers. We denote by E_α^n the compact interval composed of numbers $x \in [0,1]$ such that the first i digits of x in base $2n-3$ are exactly $\alpha_1,\alpha_2,...,\alpha_i$, that is,

$$E^n_\alpha = [\frac{\alpha_1}{2n-3} + ... + \frac{\alpha_i}{(2n-3)^i}, \frac{\alpha_1}{2n-3} + ... + \frac{\alpha_{i-1}}{(2n-3)^{i-1}} + \frac{\alpha_{i+1}}{(2n-3)^{i+1}}]$$

When all digits $\alpha_1, \alpha_2, ..., \alpha_i$ are even, we denote by p^n_α the point of the plane whose first coordinate is the center of the interval E^n_α and $\frac{1}{2^i}$ the second. For $i = 0$ we have $E^n = [0, 1]$ and $p^n = (\frac{1}{2}, 1)$.

Definition 6. The set

$$G^n = \overline{\bigcup \{p^n_{\alpha_1,\alpha_2,...,\alpha_{i-1}} p^n_{\alpha_1,\alpha_2,...,\alpha_i} : i \in \mathbb{N}, \alpha_1, \alpha_2, ..., \alpha_i \in \{0, 1, ..., 2n-4\}\}}$$

for arbitrary n is the Gehman dendrite of order n. When $n = 3$, we say that G^3 is simply the Gehman dendrite.

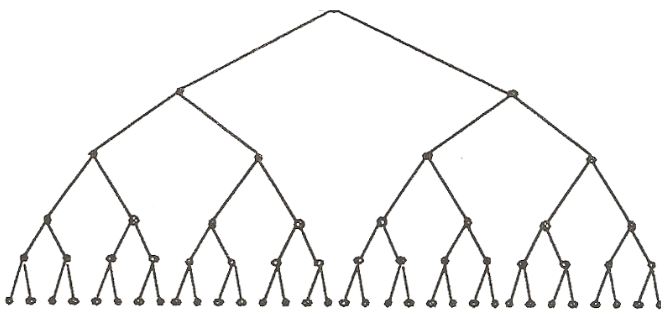

Figure 3: Dendrite G^3

Gehman dendrite G^3 appeared in the literature as an structure called *infinite binary tree* (see [4]). In Figure 3 it is supposed to continue indefinitely at the bottom. Each branching point is connected to two branching points down and the upper point has no predecessors. The end points of such infinite binary tree is a Cantor set C [4]. We may assume that $C = \{0, 1\}^\infty$. It is valuable to consider the homeomorphism g introduced in [7] in the following way. For each $n \geq 1$ let $g_n : \{0, 1\}^n \to \{0, 1\}^n$ be the cyclic permutation given by $g_n(a_1, ..., a_n) = (b_1, ..., b_n)$ where $b_1 = a_1 + 1(\text{mod } 2)$ for $2 \geq i \geq n$, $b_i = a_i + 1(\text{mod } 2)$ if $a_{i-1} = 1$ and $b_{i-1} = 0$ and $b_{i-1} = a_1, ..., a_n)$ for all $n \geq 1$. g is usually called the *binary adding machine* which has particular properties when we consider dynamics on dendrites and which will be used further.

Given a dendrite X it is really difficult to test if it contains a topological copy of G^3. The following result gives us a sufficient condition for it.

THEOREM 5. *Let X be a dendrite. Let S be a non-empty subset of X dense in itself and holding the property that $S \cap X$ is a discrete set for every arc B. Then X contains a topological copy of G^3.*

Proof. Given $p \in X$, a point $x \in X$ is said to be *end with respect to* S, if $x \in (\overline{pz \cap S})$ and for every $s \in S$, is $x \notin (ps \backslash \{s\})$.

The proof is made in several steps,

STATEMENT 1. *Given $p \neq q$ two points in X, such that $q \in S$, there exists $z \in X$ holding $q \in pz$, $z \in (\overline{pz \cap S})$ and $z \notin (ps \backslash \{s\})$. In particular, z is an end point with respect to S.*

To prove it we introduce $T = \{x \in X : x \in (\overline{px \cap S})$ and $q \in px\}$ and define on it a partial ordering, $x \prec y$ if and only if $px \subset py$. With this ordering, T has a maximal element. First observe that $T \neq \emptyset$ since $q \in T$. We use now the Brouwer Maximality Principle to see that a totally ordering sequence of points has an upper bound. Let $(x_n)_{n=1}^{\infty}$ a sequence of elements of T such that $x_n \prec x_{n+1}$ for every $n \in \mathbb{N}$. It means $px_1 \subset px_2 \subset \ldots$ and since X is a dendrite, then there is $x \in X$ such that $px_n \to px$ for an arc $px = \overline{\bigcup\{px_n : n \in \mathbb{N}\}}$. It is clear that $x \in T$ and $x_n \prec x$ for every $n \in \mathbb{N}$. Therefore there is a maximal element z in $(T, \prec)$. If would exist $s \in S$ holding $z \in (ps \backslash \{s\})$, then $q \in pz \subset ps$. It would have $s \in T$, $z \prec s$ and $z \neq s$ which contradicts the maximality of z. This proves that for every $s \in S$ is $z \notin (ps \backslash \{s\})$ and the proof of the first assertion is over.

Now fix $p^\star, q^\star \in S$ and $x_0 \in (p^\star q^\star \backslash \{p^\star, q^\star\})$. For every $n \in \mathbb{N}$ we construct inductively two subsets of X, $E_n = \{p(\alpha) : \alpha \in \{0,1\}^n\}$ and $R_n = \{q(\alpha) : \alpha \in \{0,1\}^n\}$ with the properties:

1. $x_0 \in p(0)p(1)$,
2. For every $\alpha \in \{0,1\}^n$, $p(\alpha)$ is an end point with respect to S,
3. For $n \geq 2$ and $\alpha \in \{0,1\}^{n-1}$ is $p(\alpha, 0) = p(\alpha)$,
4. If $\alpha = (a_1, \ldots, a_{n-1}) \in \{0,1\}^n$, then $q(\alpha) \in (x_0 p(\alpha) \backslash \{x_0, p(\alpha)\})$ and $\text{diam}(p(\alpha)q(\alpha)) < \frac{1}{2^n}$. Moreover if $n \geq 2$ then $q(a_1, \ldots, a_{n-1}) \in (x_0 q(\alpha) \backslash \{q(\alpha)\})$,
5. If $n \geq 2$ and $\alpha = (a_1, \ldots, a_{n-1}) \in \{0,1\}^{n-1}$, then $(x_0 p(\alpha, 0) \cap x_0 p(\alpha, 1)) = x_0 q(\alpha)$.

To start with the construction, we apply first the *Statement* 1 to x_0 and $p^\star \in S$. Then there exists $p(0) \in X$ such that $p^\star \in x_0$, $p(0) \in \overline{x_0 p(0) \cap S}$ and for every $s \in S$ is $p(0) \notin (x_0 s \backslash \{s\})$. Now if we apply again *Statement* 1 to x_0 and $q^\star \in S$ there exists $p(1) \in X$ such that $q^\star \in x_0 p(1)$, $p(1) \in \overline{x_0 p(1) \cap S}$ and for every $s \in S$ is $p(1) \notin (x_0 s \backslash \{s\})$. Clearly $p(0)$ and $p(1)$ are end points with respect to S.

We continue with the procedure and construct E_n and R_n. Once they have been constructed, they hold the following statements,

STATEMENT 2. *Let* $\alpha = (a_1, ..., a_n) \in \{0,1\}^n$ *and* $\beta = (b_1, ..., b_n) \in \{0,1\}^n$ *with* $\alpha \neq \beta$. *Then* $p(\alpha) \neq p(\beta)$.

By the construction of $p(0)$ and $p(1)$, the statement holds for $n = 1$ since $p(0) \neq p(1)$. Assume $n > 1$ and that $p(\alpha) = p(\beta)$. Let $r \in \{1, ...n\}$ such that $(a_1, ...a_{r-1}) = (b_1, ...b_{r-1}) = \gamma$ and $a_r \neq b_r$. From properties 4 and 5 it is immediate to see that for $r = n$ we reach a contradiction. Let us suppose $r < n$. By properties 3 and 4, $q(\gamma, a_r)$ and $q(\gamma, b_r)$ belong to $x_0 p(\alpha) = x_0 p(\beta)$, that is, $q(\gamma, 0)$ and $q(\gamma, 1)$ belong to $x_0 p(\alpha)$. From it can be supposed that $x_0 q(\gamma, 0) \subset x_0 q(\gamma, 1)$. Then $x_0 q(\gamma, 0) \subset (x_0 p(\gamma, 0) \cap x_0 p(\gamma, 1)) = x_0 q(\gamma)$. Therefore $x_0 q(\gamma, 0) \subset x_0 q(\gamma)$, but this contradicts property 4 and as a consequence is $p(\alpha) \neq p(\beta)$.

STATEMENT 3. *If* $\alpha, \beta \in \{0,1\}$ *and* $\alpha \neq \beta$, *then* $p(\alpha) \notin x_0 p(\beta)$.

We suppose that $p(\alpha) \in x_0 p(\beta)$. Since $p(\beta) \in (\overline{x_0 p(\beta) \cap S})$ and $p(\alpha) \neq p(\beta)$ there is $s \in (x_0 p(\beta) \cap S)$ such that $s \notin x_0 p(\alpha)$. Then $s \in (p(\alpha) p(\beta) \backslash p(\alpha))$. Therefore, $p(\alpha) \in (x_0 s \backslash \{s\})$. It contradicts that $p(\alpha)$ is an end point with respect to S.

To finish the construction, we suppose we have constructed the sets E_n and R_{n-1} holding the desired conditions. For every $\alpha \in \{0,1\}^n$ we fix a point $q \in x_0 p(\alpha)$ such that $\text{diam}(qp(\alpha)) < \frac{1}{2^n}$, $qp(\alpha) \cap (\bigcup \{x_0 p(\beta) : \beta \in \{0,1\}^n \backslash \{\alpha\}\}) = \emptyset$ and $q \neq p(\alpha)$. Since $p(\alpha)$ is an end point with respect to S, there is $s \in (qp(\alpha) \cap S) \backslash \{s\})$. By hypothesis $\{s\}$ an isolated point of $x_0 p(\alpha) \cap S$ and by the general characterization of dendrites (see [2]), there exists a connected by arcs neighborhood U of $\{s\}$ such that $U \cap x_0 q = \emptyset$ and $U \cap (x_0 p(\alpha) \cap S) = \{s\}$. Besides, there exists $z \in ((U \cap S) \backslash \{s\})$ and therefore $z \notin x_0 p(\alpha)$. Then we define $q(\alpha) \in x_0 p(\alpha)$ as the point holding $zq(\alpha) \cap x_0 p(\alpha) = \{q(\alpha)\}$. Since U is connected by arcs and $s \in x_0 p(\alpha)$ we have $q(\alpha) \in qp(\alpha)$. Therefore $\text{diam}(p(\alpha) q(\alpha)) < \frac{1}{2^n}$. If $q(\alpha) = p(\alpha)$, then $p(\alpha) \in (x_0 z \backslash \{z\})$, but it contradicts that $p(\alpha)$ is an end point with respect to S. Therefore, $q(\alpha) \neq p(\alpha)$ and $q(\alpha) \in (x_0 p(\alpha) \backslash \{x_0, p(\alpha)\})$.

By the application of *Statement* 1 to x_0 and z, we have $p(\alpha, 1) \in X$ such that $z \in x_0p(\alpha, 1)$ and also that $p(\alpha, 1)$ is an end point with respect to S. We complete the construction taking $p(\alpha, 0) = p(\alpha)$.

Until this point,we have constructed $q(\alpha)$ for every $\alpha \in \{0, 1\}^n$ and $p(\beta)$ for every $\beta \in \{0, 1\}^{n+1}$. The sets were constructed holding the properties $1, 2$ and 5 and the first part of 4. It is left to prove the second part of such property. For this, let $\alpha = (a_1, ..., a_{n-1}) \in \{0, 1\}^{n-1}$ and take $a_n \in \{0, 1\}$ and we want to prove that $q(\alpha) \in (x_0q(\alpha, a_n)\backslash\{q(\alpha, a_n)\})$. By the property 5 we have $q(\alpha)x_0 = p(\alpha, a_n) \cap x_0p(\alpha, 1 - a_n)$ and from the construction of $q(\alpha, a_n)$ is $qp(\alpha, a_n) \cap x_0p(\alpha, a - a_n) = \emptyset$. Then $q(\alpha) \in (x_0p(\alpha, a_n)\backslash qp(\alpha, a_n))$, but $q(\alpha, a_n) \in qp(\alpha, a_n)$. Therefore, $q(\alpha) \in (x_0q(\alpha, a_n)\backslash\{q(\alpha, a_n)\})$ and it finishes the inductive construction.

Let us introduce two sets,

$$\Delta = \bigcup\{\{0, 1\}^n : n \in \mathbb{N}\}$$

and

$$G = (\overline{\bigcup\{x_0p(\alpha) : \alpha \in \Lambda\}})$$

Then G is a dendrite and we want to prove that it is homeomorphic to the Gehman dendrite. It can be proved (see [10]) that it is sufficient to prove that the the set $E(G)$ is homeomorphic to the Cantor set and that all its branching points are of order three.

STATEMENT 4. *$E(G)$ is homeomorphic to the Cantor set.*

In [6], can be seen that $E(G) = \overline{\{p(\alpha) : \alpha \in \Lambda\}}$. Then $E(G)$ is compact and using Theorem 7.14 from [10], it is sufficient to prove that it is totally disconnected and perfect.

To see that $E(G)$ is totally disconnected, suppose by contrary that there exists a connected component A containing more than one point. Since $E(G)$ is closed, then A is a dendrite. That is, there exists an arc $B \subset A$. Let x be the unique point of B for which is $x_0x \cap B = \{x\}$. Let $y \in (B\backslash\{x\})$ and U a connected by arc neighborhood of y holding $U \cap x_0x = \emptyset$. We choose $\alpha \in \Lambda$ such that $p(\alpha) \in U$. Since $x_0x \cup B \cup yp(\alpha)$ is connected, then $x_0p(\alpha) \subset (x_0x \cup B \cup yp(\alpha))$ and then $(x_0p(\alpha)\backslash\{p(\alpha)\}) \subset (x_0x \cup yp(\alpha))$. We obtain that $(x_0p(\alpha)\backslash\{p(\alpha)\})$ is connected and $x_0x \cap yp(\alpha) = \emptyset$. Therefore, $(x_0p(\alpha)\backslash\{p(\alpha)\}) \subset x_0x$. This implies $p(\alpha) \in x_0x$, but $p(\alpha) \in U$ and $U \cap x_0x = \emptyset$, which it is a contradiction and the claim is proved.

Now we claim that $E(G)$ is perfect. Since $E(g)$ is closed, it is sufficient to prove that does not contain isolated points. This is equivalent to prove that the set $\{p(\alpha) : \alpha \in \Lambda\}$ has not isolated points. Let $\alpha = (a_1, ..., a_n) \in \Lambda$ and $\epsilon > 0$. Now apply the following result which is easy to prove: Let A be an arc and $A \subset X$, where X is a dendrite, $p \in A$, $\epsilon > 0$, d a distance in X and $B(p, \epsilon)$ the corresponding ball. Then there exists $\delta > 0$ such that $r^{-1}(A \cap B(p, \delta)\backslash\{p\}) \subset B(p, \epsilon)$ (r denotes *a retraction* on the dendrite X (see [10]). Apply the former result to the arc $x_0p(\alpha)$ and the point $p(\alpha)$ and let δ the corresponding constant. Since $q(\alpha), q(\alpha, 0), q(\alpha, 0, 0), ...$ converges to $p(\alpha)$, then there exists $\beta = (\alpha, 0, ..., 0) \in \Lambda$ such that $d(p(\alpha, q(\beta)) < \delta$. If $A = x_0p(\alpha)$, then $r_A(p(\beta, 1)) = q(\beta)$ (where r_A denotes the *retraction associate to* A), which is a consequence of property 5 since $x_0p(\beta, 0) \cap x_0p(\beta, 1) = x_0q(\beta)$ and to the fact that $q(\beta) \neq p(\beta, 0)$, $p(\beta, 0) \neq p(\beta, 1)$ which implies that $x_0p(\alpha) \cap p(\beta, 1) = \{q(\beta)\}$. Since $q(\beta) \in x_0p(\alpha) \cap (B(p(\alpha, \delta)\backslash\{p(\alpha\})$, we conclude that $p(\beta, 1) \in B(p(\alpha), \epsilon)$ and therefore $E(G)$ has not isolated points.

STATEMENT 5. *All branching points of G are of order three.*

Suppose on the contrary that there exists a point $x \in G$ or order at least four. Then there exist four points $x_1, x_2, x_3, x_4 \in (G\backslash\{x\})$ such that $xx_i \cap xx_j = \{x\}$ if $i \neq j$. Since $x \notin E(G)$ and $E(G)$ is closed, we can suppose $(xx_1 \cup xx_2 \cup xx_3 \cup xx_4) \cup E(G) = \emptyset$. Let y_0 be the unique point in $(xx_1 \cup xx_2 \cup xx_3 \cup xx_4)$ such that $x_0y_0 \cap (xx_1 \cup xx_2 \cup xx_3 \cup xx_4) = \{y_0\}$. We can suppose for example that $y_0 \in xx_4$. Since $x_1, x_2, x_3 \notin E(G)$ there exist $\alpha, \beta, \gamma \in \Lambda$ such that $x_1 \in x_0p(\alpha), x_2 \in x_0p(\beta)$ and $x_3 \in x_0p(\gamma)$. Adding zeros if it is necessary, we can assume that $\alpha = (a_1, ..., a_m), \beta = (b_1, ..., b_m)$ and $\gamma = (c_1, ..., c_m)$. If $\alpha = \beta$, then $\{x, x_1, x_2\} \subset x_0p(\alpha)$. This implies $xx_1 \subset xx_2$ or $xx_2 \subset xx_1$ which is a contradiction. Then it is $\alpha \neq \beta$. Let $r \in \{1, ..., m\}$ the smallest index such that $a_r \neq b_r$ and let $q_1 = x_0$ if $r = 1$ or $q_1 = q(a_1, ..., a_{r-1})$ if $r \geq 2$. We have $(x_0p(\alpha)) \cap x_0p(\beta) = x_0q_1$. It is easy to see that $q_1 = x$ and $x_0p(\alpha) \cap x_0p(\beta) = \{x\}$. In a similar way we obtain $\beta \neq \gamma$ and $x_0p(\beta) \cap x_0p(\gamma) = \{x\}$. Using property 5 we have

$$q(a_1, ..., a_r) \in (x_0p(\alpha)\backslash x_0q_1) \subset (x_0p(\alpha)\backslash\{x\})$$
$$q(b_1, ..., b_r) \in (x_0p(\beta)\backslash x_0q_1) \subset (x_0p(\beta)\backslash\{x\})$$
$$q(c_1, ..., c_r) \in (x_0p(\gamma)\backslash x_0q_1) \subset (x_0p(\gamma)\backslash\{x\}).$$

But since $(a_1, ..., a_r) = (b_1, ..., b_r) = (c_1, ..., c_r)$ and $\{a_r, b_r, c_r\} \in \{0, 1\}$ then two of them will be equal which is a contradiction since $x_0p(\alpha)\backslash\{x\}$, $x_0p(\beta)\backslash\{x\}$ and $x_0p(\gamma)\backslash\{x\}$ are pairwise disjoint. This ends the proof of the claim. ∎

4. The Periodic Recurrrent Property

In this section we will consider discrete dynamical systems given by the pair (X, f) where X is a dendrite and $f : X \to X$ a continuous map. Generally speaking we are dealing with the knowledge of the behavior of all sequences $(f^n(x))_{n=0}^{\infty}$ where $f^n = f \circ f^{n-1}$ for any $n \in \mathbb{N}$ and $f^0 =$ identity on X.

Definition 7. 1. $x \in X$ is a *periodic point of minimal period* p if $f^p(x) = x$ and $f^i(x) \neq f^j(x)$ for $i \neq j$ and $1 \leq i, j \leq p$. When $p = 1$ we say that p is a fixed point of (X, f).

2. $x \in X$ is a *recurrent point* if for every open neighborhood of $U(x)$ there exists an $n \in \mathbb{N}$ such that $f^n(x) \in U(x)$.

Such type of behaviors means that in strong and weaker way the point x returns close to x by the action of f.

Let X be a continuum and $f \in C(X)$. We recall the following definitions concerning the behavior of continuous and recurrent points.

Definition 8. 1. The discrete dynamical system (X, f) has the *Periodic-Recurrent Property* (*PR Property*) if $\overline{P(f)} = \overline{R(f)}$ (P(f) denotes the set of all periodic points of (X, f) and $R(f)$ the set of its recurrent points. In general is $P(f) \subseteq R(f)$).

2. A continuum X has the *PR Property* if $\overline{P(f)} = \overline{R(f)}$ for all $f \in C(X)$.

3. A family $\mathfrak{X}$ of continua has the *PR Property* if every $X \in \mathfrak{X}$ has the PR Property.

In discrete dynamical systems, the PR Property means that the most interesting dynamical behaviors occur in $\overline{P(f)}$. For example,

- All non-finite minimal sets are contained in it.
- For the topological entropy $h(f)$ it holds $h(f) = h(f|\overline{P(f)})$.
- For every normalized invariant measure μ is $\mu(\overline{P(f)}) = 1$ and no other smaller closed invariant subset holds such property.

The problem we are dealing is: have all dendrites the PR Property?

In general the answer is negative (see [7]), for an example on this using the binary adding machine technique (see section 3). Therefore the most appropriated question is: what are the condition a dendrite has to fulfilled to hold the PR Property? In next result we prove that some type of dendrites have not the PR Property. The proof follows a similar line than in ([6] and [1]).

THEOREM 6. *The Gehman dendrite G^3 has not the PR Property.*

Proof. Let $g : C \to C$ from [7] where C is the Cantor set which verifies $P(g) = \emptyset$ and $R(g) = C$. Since C is homeomorphic to $E(G^3)$ we denote again by $g : E(G^3) \to E(G^3)$ the composition of the first function with the homeomorphism. Now introduce another function $g_1 : \{p\} \cup E(G^3) \to \{p\} \cup E(G^3)$ given by $g_1(p) = p$ and $g_1|E(G^3) = g$. In [10] it is proved that every dendrite is an *absolute retract* (see [10] for this notion) and therefore there is an extension of g_1 (see for example [8]). Let us denote it by $f_1 : G^3 \to G^3$.

Let $h : [0,1] \to [0,1]$ holding $h(0) = 0$, $h(1) = 1$ and $h(s) > s$ for every $s \in (0,1)$ (for example, $h(s) = \sqrt{s}$). Let π_x and π_y the projections in $\mathbb{R}^2$. We observe that for any t, $\pi_y^{-1}([t,1])$ is a sub-dendrite of G^3. Then there exists a monotone retraction $r_t : X \to \pi_y^{-1}([t,1])$.

For every $q \in G^3$, let $t = h(\pi_y(q))$ and define a map $f : G^3 \to G^3$ by $f(q) = h(\pi_y(q))$. It is clear that f is continuous and onto. Moreover, $f|\{p\} \cup E(G^3) = g_1$ and $f|E(G^3) = g$. For every $q \in (G^3 \backslash (\{p\}) \cup E(G^3))$ we have $\pi_y(q) < \pi_y(f(q))$. That is, if $q \in G^3$ is such that $0 < \pi_y(q) < 1$, then f is lifted up and no of such points belong to $R(f)$. We conclude that $R(f) \subset (\{p\} \cup E(G^3))$. By the definition is $R(g) \subset R(f)$ and p is a fixed point. Then $R(f) = \{p\} \cup E(G^3)$. Since $P(f) = \{p\}$, it follows that G^3 has not the PR Property. ∎

THEOREM 7. *Let X be a dendrite. If X contains a topological copy of G^3, then X has not the PR Property.*

Proof. Since X contains a topological copy of G^3, there exists a monotone retraction $r : X \to G^3$. Let $f : X \to X$ the function introduced in the former theorem. If $g = f \circ r$, then it is $R(g) = R(f)$ and $P(g) = P(f)$. Since $\overline{P(f)} \subset R(f)$ then $\overline{P(g)} \subset R(g)$ and X has not the PR Property. ∎

COROLLARY 2. *For every $n \in \mathbb{N}$, the dendrite G^n has not the PR Property.*

Proof. It is a consequence of the fact that all dendrites of the form G^n with $n \geq 3$ contains a topological copy of G^3. ∎

Now we will try to find sufficient conditions for a dendrite to fulfill $\overline{R(f)} = \overline{P(f)}$ for all $f \in C(X)$.

LEMMA 1. *Let X be a dendrite and $A \subset X$ an arc. Let $f : X \to X$ a continuous map and let $p_0 < q_0$ points in A such that $p_0 < r(f(p_0))$ and $r(f(q_0)) < q_0$. Then there exists $z \in A$ such that $f(z) = z$ and $p_0 < z < q_0$.*

Proof. Let $p, q \in p_0q_0$ such that $p_0 < p < q < q_0$, $p < r(f(p))$ and $r(f(q)) < q$. Consider now the map $r \circ f|A : A \to A$. Then by the election of p and q, there exists $y \in pq$ such that $y = r(f(y))$.

Let $B = \{x \in X : f(x)r(f(x))\} \cap r^{-1}(pq)$. Then it is $B \neq \emptyset$ since $y \in B$. To prove that B is closed, take a sequence $(x_n)_{n=1}^{\infty} \subset B$ convergent to a point $x \in X$. We have that $f(x_n) \to f(x)$ and $r(f(x_n)) \to r(f(x))$ by continuity and since X is a dendrite, is $f(x_n)r(f(x_n)) \to f(x)r(f(x))$. Therefore $x \in f(x)r(f(x))$ and $x \in B$ and then B is closed. Let the family $\mathfrak{B} = \{xr(f(x)) \in C(X, X) : x \in B\}$. Let $\{x_nr(f(x_n))\}_{n=1}^{\infty}$ a sequence in $\mathfrak{B}$ convergent to a $D \in C(X)$. Let $x \in B$ such that $x_n \to x$. Since $x_nr(f(x_n)) \to xr(f(x))$, we conclude that $D = xr(f(x))$ which proves that $D \in \mathfrak{B}$ and $\mathfrak{B}$ is closed in the standard topology of $C(X, X)$.

Let $\mu : C(X, X) \to \mathbb{R}$ a Whitney map (see [10]). By the compactness of $\mathfrak{B}$ there exists $z \in B$ such that $\mu(xr(f(x))) \leq \mu(zr(f(z)))$ for every $x \in B$. We claim that $f(z) = z$. To prove it, suppose that on the contrary is $f(Z) \neq z$. Since $z \in f(z)r(f(z))$ we have $f(z) \notin zr(f(z))$ and then $f(z) \notin A$. By previous results, there exists an open and connected by arcs neighborhood of $f(z)$, U, such that $\overline{U} \cap (A \cup zr(f(z))) = \emptyset$. Then there exists $z_1 \in (zf(z) \backslash \{z, f(z)\})$ such that $zz_1 \cap \overline{U} = \emptyset$ and $f(zz_1) \subset U$. Moreover, $f(z_1)r(f(z)) \subset f(z)r(f(z)) \cup \overline{U}$ since the last set is a continuum. This implies that $f(z_1)r(f(z)) \cap A = \{r(f(z))\}$ and therefore $r(f(z_1)) = r(f(z))$.

On other hand, it is easy to see that $z_1 \in B$ and as a consequence of the election of z we have $\mu(z_1r(f(z_1))) \leq \mu(zr(f(z)))$ which is a contradiction since $zr(f(z))$ is contained properly in $z_1r(f(z_1))$. This implies $f(z) = z$. Since $p \leq r(z) \leq q$, we conclude that $p_0 < r(z) < q_0$ and the proof is over. ∎

Theorem 8. *Let X be a dendrite. If X has not the PR Property, then X contains a topological copy of G^3.*

Proof. Let $f : X \to X$ such that $\overline{P(f)} \neq \overline{R(f)}$. Then there exists $p \in \overline{R(f)} \backslash \overline{P(f)}$. Introduce now the set $S = \{f^n(p) : n \in \mathbb{N}\}$. Such set verifies:

$S \subset R(f)$:

Let n be a positive integer and $\epsilon > 0$. Let $\delta > 0$ such that $d(f^n(p), f^n(q)) < \epsilon$. Since $p \in \overline{R(f)}$, there exists $m \in \mathbb{N}$ such that $d(p, f^m) < \epsilon$. Then is $d(f^n(p), f^n(f^m(p))) < \epsilon$. That is, $f^n(p) \in R(f)$ since ϵ is taken arbitrarily.

$S \cap \overline{P(f)} = \emptyset$:

Let $n \in \mathbb{N}$ and $\epsilon > 0$ such that $B(p, \epsilon) \cap P(f) = \emptyset$. Let m be such that $f^m(p) \in B(p, \epsilon)$. We can assume that $m > n$. Let $\rho > 0$ such that

$B(f^m(p),\rho) \subset B(p,\epsilon)$. Then there is $\delta > 0$ such that $d(f^n(p),q) < \delta$ implies $d(f^{m-n}(f^n(p)), f^{m-n}(q)) < \delta$. Suppose now that there is $q \in B(f^n(p),\delta) \cap P(f)$. Then it is

$$f^{m-n}(q) \in (B(p,\epsilon) \cap P(f).$$

This proves that $f^n(p) \notin \overline{P(f)}$.

To finish the proof we will test that S holds the condition of Theorem 5. Firstly $S \neq \emptyset$. Secondly S is dense in itself. To prove it let n a positive integer and $\epsilon > 0$. By the former property (1), we have m such that $f^m(f^n(p)) \in B(f^n(p),\epsilon)$. By the property (2) is $f^n(p) \notin P(f)$. Then $f^{m+n}(p) \neq f^n(p)$ and S is dense in itself.

Thirdly, we are proving that if B is an arc, then $B \cap S$ is a discrete subset of X.

Let us suppose by the contrary that there is $x \in B \cap S$ which is not an isolated point. Let $x = f^n(p)$. By the second property, there is $\epsilon > 0$ such that $B(x,\epsilon) \cap P(f)$. Using the same arguments that in Theorem 5 there is $\delta > 0$ such that $r^{-1}(B \cap (B(x,\delta)\backslash\{x\})) \subset B(x,\epsilon)$. Let $m > n$ such that $f^m(p) \in (B(x,\delta) \cap (B\backslash\{x\}))$ and $xf^m(p) \subset B(x,\delta)$. We assume for example that $x < f^m(p)$ where $<$ is a given ordering in B. If $g = f^{m-n}$, then $x < g(x)$.

We are proving by induction that $y \leq r(g^k(y))$ for every $y \in xg(x)$ and every $k \in \mathbb{N}$. For $k = 1$, suppose that on contrary there is $y \in xg(x)$ such that $r(g(y)) < y$. Since $x < g(x) = r(g^k(y))$ and $x < y$ we apply the former Lemma to find a fixed point z of g such that $x < r(z) < y \geq g(x)$. Then $z \in r^{-1}(B \cap (B(x,\delta)\backslash\{x\})) \subset B(x,\epsilon)$. But $z = f^{m-n}(z)$, that is, $z \in P(f)$ and it contradicts the election of ϵ and proves the first step of the proof.

Now suppose that $y \leq r(g^k(y))$ for all $y \in xg(x)$. In particular, is $g(x) \leq r(g^{k+1}(y))$. Therefore is $x < r(g^{k+1}(x))$. Repeating the argument, but now with the map g^{k+1} we get that $y \leq r(g^{k+1}(y))$ for every $y \in xg(x)$. This completes the induction.

In particular, $g(x) \leq f(g^{k+1}(x))$ for every k and then is $r^{-1}(\{w \in B : w < g(x)\})$ is an open neighborhood of x which does not intersect with $\{g^k(x) : k \in \mathbb{N}\}$. This proves that x is not a recurrent point of g, but $R(g) = R(f^{m-n}) = R(f)$ and therefore, $x \notin R(f)$ which is a contradiction.

The conclusion is that S contains a topological copy of G^3. ∎

We obtain two important consequences of the former results. One is a characterization of dendrites having the PR Property and the other is a different approach more transparent than the results got in the literature ([3] and [11]) to prove that in the setting of compact intervals of the real line, arc and tree maps, the PR Property is held.

THEOREM 9. (ILLANES RESULT) *A dendrite X has the PR Property if and only if does not contain a topological copy of G^3.*

COROLLARY 3. (COVEN-HEDLUND AND YE RESULTS) *Let X be a compact interval $[a, b]$, an arc or a tree, then X has the PR Property.*

5. PR PROPERTY AND CHAOTICITY IN DEVANEY SENSE

The former characterization of dendrites with the PR Property is not easy to test. To our help it comes a sufficient condition obtained in [9]. This is the contain of the following result.

THEOREM 10. *Let X be a dendrite and $f \in C(X)$. If* $\mathrm{Card}(E(X)) < c$ (*where c denotes the cardinal of the continuum*), *then f holds the PR Property.*

Using Theorems 7 and 8 we have that if $\mathrm{Card}(E(X)) < c$ then X does not contain a topological copy of G^3. The PR Property allow us the description of the chaotic character in the Devaney sense of any $f \in C(X)$.

DEFINITION 9. Let (X, f) be any discrete dynamical system where X is a topological space. The system is transitive (or f is transitive) if given two open sets U and V, there exists n such that $f^n(U) \cap V \neq \emptyset$.

DEFINITION 10. The system (X, f) is chaotic in the Devaney sense (respectively, f is chaotic in the Devaney sense) if it is transitive and the set of periodic points are dense in X.

DEFINITION 11. A point $x \in X$ is transitive with respect to f if $\overline{O(f, x)} = \overline{(f^n(x))_{n=0}^{\infty}} = X$ (*forward orbit of* x). The set of transitive points of the system will be denoted by $Tr(f)$.

In the following result, X will be a dendrite and $f \in C(X)$.

THEOREM 11. *Let (X, f) be a discrete dynamical system holding the PR Property. Then f is chaotic in Devaney sense if and only f is transitive.*

Proof. By one hand, it is evident that if f is chaotic, then it is transitive. On other hand, it is clear that $Tr(f) \subseteq R(f)$. Then

$$X = \overline{Tr(f)} \subseteq \overline{R(f)} = \overline{P(f)}.$$

Using now Definition 10 we obtain the result. ∎

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