Compact Hausdorff Pseudoradial Spaces and their Pseudoradial Order*

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Abstract: It is proved that there are compact Hausdorff spaces of any pseudoradial order up to ω₀ included.

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1. Introduction

Given an ordinal γ and a set X, a transfinite sequence in X of length γ is a map $S : γ \rightarrow X$. It is usually denoted $(x_\alpha)_{\alpha < \gamma}$. A transfinite sequence $(x_\alpha)_{\alpha < \gamma}$ in a topological space X converges to a point $x \in X$ (written $x_\alpha \rightarrow x$ or $\lim_{\alpha \rightarrow \gamma} x_\alpha = x$) provided that for each neighborhood $U$ of $x$ there is some $\overline{\sigma} < \gamma$ such that $\{ x_\alpha \mid \overline{\sigma} \leq \alpha < \gamma \} \subseteq U$.

A topological space X is called pseudoradial (see [5], [1] or [3]) provided that for each $A \subseteq X$, if A is not closed, then there are a point $x \in A \setminus A$ and a transfinite sequence $(x_\alpha)_{\alpha < \lambda}$ in A such that $x_\alpha \rightarrow x$.

Following [2] and [6], we define the pseudoradial closure of A in X as the set

$\hat{A} = \{ x \in X \mid \text{there is a transfinite sequence } (x_\alpha)_{\alpha < \lambda} \text{ in } A \text{ converging to } x \}.$

By transfinite recursion define

\begin{align*}
\hat{A}^{(0)} &= A; \\
\hat{A}^{(\alpha+1)} &= \left( \hat{A}^{(\alpha)} \right) \quad \text{for every ordinal } \alpha; \\
\hat{A}^{(\beta)} &= \bigcup_{\alpha < \beta} A^{(\alpha)} \quad \text{if } \beta \text{ is a limit ordinal.}
\end{align*}

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The pseudoradial order of a pseudoradial space $X$ is the least ordinal number $\alpha$ such that for each $A \subseteq X$,

$$\widehat{A}^{(\alpha)} = \overline{A}.$$ 

The pseudoradial order of a pseudoradial space $X$ is denoted by $\text{pro}(X)$.

In a previous article ([6]) we proved that there are normal ($T_4 + T_1$) pseudoradial spaces and compact $T_1$ ones of pseudoradial order given by any ordinal number. Here we exhibit the construction of Hausdorff compact pseudoradial spaces of any pseudoradial order less than or equal to $\omega_0$.

2. The main construction

For each natural number $n \geq 1$, we construct a compact pseudoradial Hausdorff space $G_n$ such that $\text{pro}(G_n) = n$.

For each $j = 0, \ldots, n - 1$, let $x(j), y(j)$ be ordinal numbers. For the sake of convenience $x(j)$ could also assume the value $-1$, so that we can use the notation $(-1, y(j))$ for denoting the segment of ordinals $[0, y(j)]$. Let $x = (x(0), \ldots, x(n-1)), y = (y(0), \ldots, y(n-1))$. We say that $x < y$ if and only if $x(j) < y(j)$ for each $j = 0, \ldots, n - 1$. If $x < y$, let

$$C(x, y) = (x(0), y(0)) \times \cdots \times (x(n-1), y(n-1))$$

be the $n$-dimensional cube with vertexes $x, y$, where each $(x(j), y(j))$ has the order topology and $C(x, y)$ has the product topology. If $x = (-1, \ldots, -1)$, we denote $C(x, y)$ by $C(y)$. If $x \leq x' < y' \leq y$, $C(x', y')$ is both an open and a closed subspace of $C(x, y)$. For each $j = 0, \ldots, n - 1$ we denote by

$$E_j = \{y(0)\} \times \cdots \times \{y(j-1)\} \times (x(j), y(j)) \times \{y(j+1)\} \times \cdots \times \{y(n-1)\}$$

the $j$-th edge of the cube $C(x, y)$ and

$$H_j = (x(0), y(0)) \times \cdots \times (x(j-1), y(j-1)) \times \{y(j)\} \times \cdots \times (x(n-1), y(n-1))$$

the $j$-th hyperface of the cube $C(x, y)$ (we are interested only in the edges and hyperfaces which $y$ belongs to). Finally let us observe that if $z \in C(x, y)$, then $z < y$ if and only if $z \notin H_j$ for each $j = 0, \ldots, n - 1$.

Let $G_n = [0, \omega_0] \times [0, \omega_1] \times \cdots \times [0, \omega_{n-1}]$. $G_n$ is a $T_2$ compact space since it is product of $T_2$ compact spaces. It was proved in [4] that the product of
two pseudoradial $T_2$ compact spaces is pseudoradial if one of them is radial (i.e. its pseudoradial order is 1). Since for each natural number $k$ the segment of ordinals $[0, \omega_k]$ with the order topology is a compact $T_2$ radial space, it is easy to see that $G_n$ is a pseudoradial space.

By the next three lemmas we prove that $\text{pro}(G_n) \leq n$, i.e. that for each subspace $A$ of $G_n$, $\overline{A}^{(n)} = \overline{A}$.

**Lemma 2.1.** As earlier, let $n$ be a natural number, $n \geq 1$ and let $x, y$ be two $n$-tuples of ordinals, $x < y$. Let $A$ be a subspace of $C(x, y)$. Assume that for each $j = 0, \ldots, n-1$, $A \cap H_j = \emptyset$. Then $y \notin \overline{A}$.

**Proof.** If $A = \emptyset$, the proof is trivial. Assume $A \neq \emptyset$. By transfinite recursion we determine an ordinal $\gamma$ and a sequence $(z_\alpha)_{\alpha < \gamma}$ in $A$ of length $\gamma$ in the following way. Let $z_0 \in A$. Assume that we have defined $z_\alpha \in A$. Since for each $j = 0, \ldots, n-1$, $z_\alpha \notin H_j$, $z_\alpha < y$, so we can consider $C(z_\alpha, y)$. If $C(z_\alpha, y) \cap A = \emptyset$, let $\gamma = \alpha + 1$ and break the recursion. If not, choose $z_{\alpha+1} \in C(z_\alpha, y) \cap A$. Assume now that we have defined $z_\alpha$ for each $\alpha < \beta$, a limit ordinal, and for each $j = 0, \ldots, n-1$, let $\bar{z}_\beta(j) = \sup\{z_\alpha(j) \mid \alpha < \beta \}$. Let $\bar{z}_\beta = (\bar{z}_\beta(0), \ldots, \bar{z}_\beta(n-1))$. It is easy to prove that $\bar{z}_\beta = \lim_{\alpha \to \beta} z_\alpha$; then $\bar{z}_\beta \in A$, so $\bar{z}_\beta \notin H_0 \cup \cdots \cup H_{n-1}$, and so $\bar{z}_\beta < y$. Thus we can consider $C(\bar{z}_\beta, y)$. If $C(\bar{z}_\beta, y) \cap A = \emptyset$, let $\gamma = \beta$ and break the recursion. If not, choose $z_\beta \in C(\bar{z}_\beta, y) \cap A$. Then

$$U = \begin{cases} C(z_{\gamma-1}, y) & \text{if } \gamma \text{ is a successor ordinal} \\ C(\bar{z}_\gamma, y) & \text{if } \gamma \text{ is a limit ordinal} \end{cases}$$

is a neighborhood of $y$ in which there are no points of $A$, so $y \notin \overline{A}$. \hfill \blacksquare

**Lemma 2.2.** Let $x, y$ be two $n$-tuples of ordinals. Let $A$ be a subspace of $C(x, y)$. Assume that for each $j = 0, \ldots, n-1$, $A^{(n-1)} \cap E_j = \emptyset$. Then $y \notin \overline{A}$.

**Proof.** By induction on $n$. If $n = 1$ the proof is trivial. If $n = 2$, then $E_0 = H_0$ and $E_1 = H_1$, so by Lemma 2.1 $y \notin \overline{A}$.

Now let $n \geq 3$ and assume that the lemma is proved for $n-1$ and let us prove it for $n$. First let us observe that for each $j = 0, \ldots, n-1$, $H_j$ is homeomorphic to an $(n-1)$-dimensional cube, whose edges are the $E_k$, $k \neq j$. Furthermore $E_j$, $H_j$ are closed subspaces of $C(x, y)$ and so we can use the closure and pseudoradial closure operators in $C(x, y)$, in $E_j$ and in $H_j$ without ambiguity.
Now, for each \( j = 0, \ldots, n - 1 \), let \( B_j = \hat{A} \cap H_j \). First we prove that for each \( j = 0, \ldots, n - 1 \) and for each \( k \neq j \), \( \overline{B}_j^{(n-2)} \cap E_k = \emptyset \). If not, \( \emptyset \neq \overline{B}_j^{(n-2)} \cap E_k \subseteq \widehat{A}^{(n-1)} \cap \widehat{H}_j^{(n-2)} \cap E_k = \hat{A}^{(n-1)} \cap H_j \cap E_k \), but this contradicts the hypothesis. So for each \( j = 0, \ldots, n - 1 \) the hyperface \( H_j \) of \( C(x, y) \) is homeomorphic to an \((n-1)\)-dimensional hypercube such that in each of its edges there are no points of \( B_j^{(n-2)} \). So by inductive assumption, \( y \in \hat{A} \). Thus for each \( j = 0, \ldots, n - 1 \), and for each \( k \neq j \), there is an ordinal \( w_j(k) < y(k) \) such that in 

\[
(w_j(0), y(0)) \times \cdots \times (w_j(j - 1), y(j - 1)) \times \{y(j)\} \times \\
\times (w_j(j + 1), y(j + 1)) \times \cdots \times (w_j(n - 1), y(n - 1))
\]

there are no points of \( B_j = \hat{A} \cap H_j \). Let 

\[
w(0) = \max\{w_j(0) \mid j = 0, \ldots, n - 1\} < y(0) \\
\ldots \\
w(n - 1) = \max\{w_j(n - 1) \mid j = 0, \ldots, n - 1\} < y(n - 1)
\]

and let \( w = (w(0), \ldots, w(n - 1)) \). Thus \( C(w, y) \) is an \( n \)-dimensional hypercube such that in each of its hyperfaces there are no points of \( \hat{A} \). So by Lemma 2.1 \( y \notin \overline{A} \). 

\[\text{Łemma 2.3. Let } y \text{ be an } n\text{-tuple of ordinals. Let } A \text{ be a subspace of } C(y) \text{ and } y \in \overline{A}. \text{ Then } y \in \hat{A}.\]

\[\text{Proof. By contradiction assume that } y \notin \hat{A}. \text{ Then there is } x = (x(0), \ldots, x(n - 1)) \text{ such that in each edge } E_j \text{ of the cube } C(x, y) \text{ there are no points of } \hat{A}^{(n-1)}. \text{ By Lemma 2.2, } y \notin A \cap C(x, y) \text{ and so } y \notin \overline{A}.\]

By the next lemma we prove that \( \text{pro}(G_n) \geq n \), i.e. that there is a subspace \( A \) of \( G_n \) such that \( \hat{A}(k) \subseteq \overline{A} \) for each \( k = 0, \ldots, n - 1 \).

\[\text{Lemema 2.4. Let } A = [0, \omega_0) \times \cdots \times [0, \omega_{n-1}) \subseteq G_n. \text{ Then for each } k = 0, \ldots, n,
\]

\[\hat{A}(k) = \{(x(0), \ldots, x(n - 1)) \mid x(j) = \omega_j \text{ for at most } k \text{ indices}\}.
\]

\[\text{Proof. By induction on } k. \text{ For } k = 0 \text{ the proof is trivial. Assume that the lemma is proved for } k - 1 \text{ and let us prove it for } k.\]
Let $x \in \widehat{A}(k)$. Assume $x(j) = \omega_j$ for more than $k$ indices. We can assume without restriction $x = (\omega_0, \ldots, \omega_{k-1}, \omega_k, x(k+1), \ldots, x(n-1))$. Since $x \in \widehat{A}(k)$, there is a sequence $(x_\alpha)_{\alpha < \lambda}$ of length $\lambda$ in $\widehat{A}(k-1)$ such that $x_\alpha \to x$.

First assume $\lambda \leq \omega_{k-1}$. Let $\overline{\gamma} = \sup\{x_\alpha(k) \mid \alpha < \lambda\}$. Since $\lambda \leq \omega_{k-1}$, then $\overline{\gamma}$ is strictly less than $\omega_k$ and so $x_\alpha$ cannot converge to $x$. Now assume $\lambda \geq \omega_k$. Let $h \in \{0, \ldots, k-1\}$. Since $x_\alpha \to x$, for each $\gamma < \omega_k$ there is $\alpha(h, \gamma) < \lambda$ such that for each $\alpha > \alpha(h, \gamma)$, $x_\alpha(h) > \gamma$. Let $\overline{\alpha}_h = \sup\{\alpha(h, \gamma) \mid \gamma < \omega_h\}$ and $\overline{\alpha} = \max\{\alpha_h \mid h = 0, \ldots, k-1\}$. Since $\lambda \geq \omega_k$, $\overline{\alpha}_h < \omega_k$ for each $h$ and so $\overline{\alpha} < \omega_k$. Then for each $\alpha > \overline{\alpha}$, $x_\alpha(h) = \omega_h$ for each $h = 0, \ldots, k-1$. Then by inductive assumption $x_\alpha \notin \widehat{A}(k-1)$, a contradiction.

Let $x = (x(0), \ldots, x(n-1))$ such that $x(j) = \omega_j$ for at most $k$ indices.

If $x(j) = \omega_j$ for at most $k - 1$ indices, by inductive assumption $x \in \widehat{A}(k-1)$.

So assume $x(j) = \omega_j$ for exactly $k$ indices. We can assume without restriction that $x = (\omega_0, \ldots, \omega_{k-1}, x(k), \ldots, x(n-1))$ and $x(k) = \omega_k, \ldots, x(n-1) = \omega_{n-1}$.

For each $\alpha < \omega_{k-1}$, let $x_\alpha = (\omega_0, \ldots, \omega_{k-1}, \alpha, x(k), \ldots, x(n-1))$. By inductive assumption $x_\alpha \in \widehat{A}(k-1)$. Clearly $x_\alpha \to x$ and so $x \in \widehat{A}(k)$.

**Theorem 2.5.** $G_n = [0, \omega_0] \times [0, \omega_1] \times \cdots \times [0, \omega_{n-1}]$ is a compact pseudoradial Hausdorff space and pro($G_n$) = $n$.

**Proof.** Clearly $G_n$ is a $T_2$ compact space since it is product of $T_2$ compact spaces. We have already observed that $G_n$ is a pseudoradial space. In order to prove that pro($G_n$) = $n$ it suffices to prove that:

(i) for each $A \subseteq G_n$, $\widehat{A}(n) = \overline{A}$;

(ii) there exists $A \subseteq G_n$ such that for each $k < n$, $\widehat{A}(k) \subseteq \overline{A}$.

Let us prove the first claim. Let $A \subseteq G_n$. Let $y \in \overline{A}$. Since $C(y)$ is both an open and a closed subspace of $G_n$, $x \in \overline{A} \cap C(y)$. Thus, by Lemma 2.3, $x \in A \cap C(y)$. Therefore, $x \in A \cap C(y)$ and so $x \in \widehat{A}(n)$.

Let us prove the second claim. Let $A$ be as in Lemma 2.4 and let $x = (\omega_0, \ldots, \omega_{n-1})$. Clearly $x \in \overline{A}$, but by Lemma 2.4, $x \notin \widehat{A}(k)$, for each $k = 0, \ldots, n-1$.

3. A space of order $\omega_0$

Let $X$ be the disjoint topological sum of the spaces $G_n$, $n < \omega_0$, constructed in the previous section. Let $G_\omega$ be the one-point compactification of $X$, i.e. $G_\omega = X \cup \{\infty\}$.
Remark 3.1. Let us observe that:

(i) \( \infty \notin X \);
(ii) a basic neighborhood of \( \infty \) has the form \( G_\omega \setminus K \), where \( K \) is a compact subspace of \( X \);
(iii) if \( K \) is a compact subspace of \( X \), then there is \( n < \omega_0 \) such that \( K \subseteq \bigcup_{1 \leq k \leq n} G_k \).

Theorem 3.2. \( G_\omega \) is a compact Hausdorff pseudoradial space and its pseudoradial order is \( \omega_0 \).

Proof. Clearly \( G_\omega \) is a compact Hausdorff space. In order to prove that \( G_\omega \) is pseudoradial and \( \text{pro}(G_\omega) = \omega_0 \) it suffices to prove that:

(i) for each \( A \subseteq G_\omega \), \( \widehat{A}(\omega_0) = \overline{A} \);
(ii) for each \( n < \omega_0 \), there exists \( A \subseteq G_\omega \) such that \( \widehat{A}(n) \not\subseteq \overline{A} \).

Let us prove the first claim. Let \( A \subseteq G_\omega \) and let \( x \in \overline{A} \setminus A \). If \( x = \infty \), then for each \( n < \omega_0 \),

\[
U_n = (G_\omega \setminus \bigcup_{1 \leq k \leq n} G_k)
\]

is a neighborhood of \( \infty \) and so there is \( x_n \in A \cap U_n \). It follows immediately from Remark 3.1 that \( x_n \to \infty \). So \( \infty \in \overline{A} \subseteq \widehat{A}(\omega_0) \). If \( x \neq \infty \), then there is \( n < \omega_0 \) such that \( x \in G_n \). Since \( G_n \) is a compact open subspace of \( G_\omega \) and \( \text{pro}(G_n) = n \), then \( x \in \widehat{A}(n) \subseteq \widehat{A}(\omega_0) \).

The second claim is an easy consequence of the fact that for each \( n < \omega_0 \) the space \( G_n \) is a compact open subspace of \( G_\omega \) and its pseudoradial order is \( n \).

References
