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Invariance of the Schechter Essential Spectrum under Polynomially Compact Operators Perturbation

Boulbeba Abdelmoumen, Aref Jeribi, Maher Mnif

Département de Mathématiques, Université de Sfax, Faculté des Sciences de Sfax, Route de Soukra Km 3.5, B.P. 1171, 3000, Sfax, Tunisie

Boulbeba.Abdelmoumen@ipeis.rnu.tn, aref.jeribi@fss.rnu.tn, maher.mnif@ipeis.rnu.tn

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Abstract: In this work, we use the notion of the measure of noncompactness in order to establish some results concerning the class of semi-Fredholm and Fredholm operators. Further, we apply the results we obtained to prove the invariance of the Schechter essential spectrum on Banach spaces by means of polynomially compact perturbations.

Key words: Measures of noncompactness in Banach spaces, Fredholm operators, Schechter essential spectrum.

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1. Introduction

Let $(X, \|.\|)$ be an infinite-dimensional Banach space and let $\mathcal{C}(X)$ be the set of all closed densely defined linear operators on X. We denote by $\mathcal{L}(X)$ the space of all bounded linear operators on X and by $\mathcal{K}(X)$ the ideal of all compact operators of $\mathcal{L}(X)$. For $T \in \mathcal{C}(X)$ we use the following notations: $\mathcal{D}(T)$ is the domain, $N(T)$ is the kernel and $R(T)$ is the range of T. The nullity, $\alpha(T)$, of T is defined as the dimension of $N(T)$ and the deficiency, $\beta(T)$, of T is defined as the codimension of $R(T)$ in X. We use $\sigma(T)$ and $\rho(T)$ to denote the spectrum and the resolvent set of T. The next sets of upper and lower semi-Fredholm operators are well-known:

$$\Phi_+(X) = \{T \in \mathcal{C}(X) : \alpha(T) < \infty \text{ and } \mathcal{R}(T) \text{ is closed in } X\}$$

and

$$\Phi_-(X) = \{T \in \mathcal{C}(X) : \beta(T) < \infty \text{ (then } \mathcal{R}(T) \text{ is closed in } X)\}.$$

$\Phi(X) := \Phi_+(X) \cap \Phi_-(X)$ is the set of Fredholm operators in $\mathcal{C}(X)$, while $\Phi_\pm(X) := \Phi_+(X) \cup \Phi_-(X)$ is the set of semi-Fredholm operators in $\mathcal{C}(X)$.

For $T \in \Phi_{\pm}(X)$, the index is defined as $i(T) := \alpha(T) - \beta(T)$. A complex number λ is in Φ_{+T}, Φ_{-T}, $\Phi_{\pm T}$ or Φ_T if $\lambda - T$ is in $\Phi_+(X)$, $\Phi_-(X)$, $\Phi_{\pm}(X)$ or $\Phi(X)$ respectively. For the properties of these sets we refer to [6, 7, 9, 18].

The next proposition is well known in [8, 19].

Proposition 1.1. ([8, 19]) *For any $T \in \mathcal{C}(X)$, we have*

(i) Φ_{+T}, Φ_{-T}, *and* Φ_T *are open,*

(ii) $i(\lambda - T)$ *is constant on any component of* Φ_T.

There are many ways to define the essential spectrum of $T \in \mathcal{C}(X)$ (see, for example, [8, 17] or the comments in [18, p. 283]). Throughout this paper, we are concerned with the so-called Schechter essential spectrum.

Definition 1.1. Let $T \in \mathcal{C}(X)$. We define the essential spectrum of the operator T by:

$$\sigma_{ess}(T) = \bigcap_{K \in \mathcal{K}(X)} \sigma(T + K).$$

The following proposition, owing to Schechter [19, Theorem 5.4, p. 180], gives a characterization of the Schechter essential spectrum by means of Fredholm operators.

Proposition 1.2. ([19, Theorem 5.4, p. 180]) *Let $T \in \mathcal{C}(X)$. Then*

$$\lambda \notin \sigma_{ess}(T) \quad \text{if and only if} \quad \lambda \in \Phi_T^0,$$

where $\Phi_T^0 := \{\lambda \in \Phi_T \text{ such that } i(\lambda - T) = 0\}$.

Throughout this paper, we use the concept of measure of noncompactness, we do not recall the general definition, it can be found in [2, 3, 5]. But we deal only with two specific measures of noncompactness defined as follow [16]:

For any bounded set A of the Banach space X,

- Kuratowski measure of noncompactness,

$$\gamma(A) = \inf\{\varepsilon > 0 : A \text{ may be covered by finitely many sets of diameter} \leq \varepsilon\}.$$

- Hausdorff measure of noncompactness,

$$\overline{\gamma}(A) = \inf\{\varepsilon > 0 : A \text{ may be covered by finitely many open balls of radius} \leq \varepsilon\}.$$

The relations between these measures are given by the following inequalities, which are obtained by Danes [4]:

$$\overline{\gamma}(A) \leq \gamma(A) \leq 2\overline{\gamma}(A), \text{ for any bounded set } A \text{ of } X.$$

Let $T \in \mathcal{L}(X)$. We say that T is k-set-contraction if for every bounded set A in X, we have $\gamma(T(A)) \leq k\gamma(A)$. T is called k-ball-contraction if $\overline{\gamma}(T(A)) \leq k\overline{\gamma}(A)$ for every bounded set A in X. We define $\gamma(T)$ and $\overline{\gamma}(T)$, respectively, by

$$\gamma(T) := \inf\{k : T \text{ is } k\text{-set-contraction}\}$$

and

$$\overline{\gamma}(T) := \inf\{k : T \text{ is } k\text{-ball-contraction}\}.$$

In the following lemma, we give some important properties of $\gamma(T)$ and $\overline{\gamma}(T)$.

LEMMA 1.1. ([1]) *Let X be a Banach space and $T \in \mathcal{L}(X)$. We have the following:*

(i) $\frac{1}{2}\gamma(T) \leq \overline{\gamma}(T) \leq 2\gamma(T)$.

(ii) $\gamma(T) = 0$ *if and only if* $\overline{\gamma}(T) = 0$ *if and only if* T *is compact.*

(iii) *If* $T, S \in \mathcal{L}(X)$ *then* $\gamma(ST) \leq \gamma(S)\gamma(T)$ *and* $\overline{\gamma}(ST) \leq \overline{\gamma}(S)\overline{\gamma}(T)$. *We say that* γ *(resp.* $\overline{\gamma}$*) has the property algebraic semi-multiplicative.*

(iv) *If* $K \in \mathcal{K}(X)$ *then* $\gamma(T+K) = \gamma(T)$ *and* $\overline{\gamma}(T+K) = \overline{\gamma}(T)$.

(v) $\gamma(T^*) \leq \overline{\gamma}(T)$ *and* $\gamma(T) \leq \overline{\gamma}(T^*)$, *where* T^* *denotes the dual operator of* T.

For further facts concerning measures of noncompactness and its properties we refer to [2, 3, 5].

The purpose of this work is to pursue the analysis started in [1, 10, 11, 12, 14]. From [1], for $T \in \mathcal{C}(X)$, the Schechter essential spectrum is characterized by:

$$\sigma_{ess}(T) = \bigcap_{K \in \mathcal{G}_T^n(X)} \sigma(T+K),$$

where

$$\mathcal{G}_T^n(X) = \left\{K \in \mathcal{L}(X) : \gamma\big(\big[(\lambda - T - K)^{-1}K\big]^n \big) < \frac{1}{2} \text{ for all } \lambda \in \varrho(T+K)\right\}$$

and γ is the Kuratowski measure of noncompactness. Moreover, if X has the property Dunford-Pettis, then $\sigma_{ess}(T+K)=\sigma_{ess}(T)$, $\forall K\in\mathcal{L}(X)$ such that, $\forall p\in\mathbb{Z}$, $\rho_{ess}(T+pK)$ is a connected set of $\mathbb{C}$ and $K(\lambda-T)^{-1}K$ is weakly compact (see [1, Corollary 4.1]).

In this work, using the concept of measure of noncompactness, we show (see Section 2, Theorem 2.1) that, for $a\in\mathbb{C}^*$ and P,Q two complex polynomials satisfying Q divides $P-a$, if $\gamma(P(T))<|a|$, then $Q(T)\in\Phi(X)$. In Section 3, we will apply the results obtained in Section 2 to give some new characterization of the Schechter essential spectrum (see Proposition 3.1) by means of the measure of noncompactness and we will show that, under some sufficient conditions on the perturbed operator, we get the invariance of the Schechter essential spectrum on Banach spaces (see Theorem 3.1). Our work provides an extension to the work in [1], indeed, to establish some characterization of Schechter essential spectrum in a Banach space X, the set $\mathcal{G}_T^n(X)$ will be replaced respectively by certain sets $\mathcal{S}_T(X)$ and $\mathcal{Q}_T(X)$ (see Proposition 3.1). To establish the invariance of the Schechter essential spectrum, the condition

$$K\in\mathcal{L}(X);\ K(\lambda-T)^{-1}K \text{ is weakly compact},$$

where X is a Dunford-Pettis space, given in [1, Corollary 4.1], can be replaced respectively by $K(\lambda-T-K)^{-1}\in\mathcal{P}_1\mathcal{K}(X)$, $K(\lambda-T)^{-1}\in\mathcal{P}_1\mathcal{K}(X)$, (see Theorem 3.1 and Corollary 3.1) where X is a Banach space and

$$\mathcal{P}_1\mathcal{K}(X)=\Big\{A\in\mathcal{L}(X) \text{ such that there exists a nonzero complex polynomial } P(z)=\sum_{k=0}^{p}a_kz^k \text{ satisfying } P(\tfrac{1}{n})\neq 0,\ \forall n\in\mathbb{Z}^* \text{ and } P(A)\in\mathcal{K}(X)\Big\}.$$

2. Preliminaries results

The goal of this section is to establish all auxiliary results which are needed in the next section. We begin by the following theorem which is a generalization of Theorem 3.1 in [1]:

Theorem 2.1. *Let X be a Banach space, $T\in\mathcal{L}(X)$, $a\in\mathbb{C}^*$ and let P,Q two complex polynomials such that Q divides $P-a$. Then,*

$$\gamma(P(T))<|a|, \text{ implies } Q(T)\in\Phi(X).$$

Proof. Since Q divides $\frac{P}{a}-1$ and $\gamma(\frac{P(T)}{a})<1$, then from [1, Theorem 3.1], we get $Q(T)\in\Phi_+(X)$. Furthermore, there exists $k_0\in\mathbb{N}^*$ such that

$\big(\gamma(\frac{P(T)}{a})\big)^{k_0} < \frac{1}{2}$. According to Lemma 1.1 (iii), we deduce $\gamma\big((\frac{P(T)}{a})^{k_0}\big) < \frac{1}{2}$. Combining the assertions (i) and (v) of Lemma 1.1 one has $\gamma\big((\frac{P(T)}{a}^{*})^{k_0}\big) \leq 2\gamma\big((\frac{P(T)}{a})^{k_0}\big) < 1$. Hence $\gamma\big((\frac{P(T)}{a}^{*})^{k_0}\big) < 1$. Since Q divides $(\frac{P}{a})^{k_0} - 1$, then $\alpha(Q(T)^*) = \beta(Q(T)) < \infty$ and $Q(T) \in \Phi(X)$ which completes the proof of theorem. ∎

Corollary 2.1. *Let X be a Banach space, $T \in \mathcal{L}(X)$, $\lambda \in \mathbb{C}^*$, $P(z) = \sum_{k=0}^{n} a_k z^k$ a nonzero complex polynomial satisfying $P(\lambda) \neq 0$ and let $|P|(z) := \sum_{k=0}^{n} |a_k| z^k$.*

(i) *If $\gamma(P(T)) < |P(\lambda)|$, then $\lambda - T$ is a Fredholm operator.*

(ii) *If $|P|\,\gamma(T) < |P(\lambda)|$, then $\lambda - T$ is a Fredholm operator with index zero.*

Proof. (i) Let $\lambda \in \mathbb{C}^*$ with $P(\lambda) \neq 0$. We have the following

$$\begin{aligned} P(\lambda) - P(z) &= \sum_{k=1}^{n} a_k(\lambda^k - z^k) \\ &= (\lambda - z)\left(\sum_{k=1}^{n} a_k \sum_{r=0}^{k-1} \lambda^r z^{k-r-1}\right). \end{aligned}$$

Then $(\lambda - z)$ divides $P(\lambda) - P(z)$. Applying Theorem 2.1 with $a := P(\lambda)$, we get $(\lambda - T) \in \Phi(X)$.
(ii) Let $t \in [0, 1]$. We have the following:

$$\begin{aligned} \gamma\big(P(tT)\big) &\leq \sum_{k=0}^{n} |a_k| t^k \gamma(T^k) \\ &\leq |P|\gamma(T). \end{aligned}$$

If $|P|\,\gamma(T) < |P(\lambda)|$ then $\gamma(P(tT)) \leq |P(\lambda)|$. Applying (i), we get $(\lambda - tT) \in \Phi(X)$ for all $t \in [0, 1]$. On the other hand, from Proposition 1.1, the index is constant on any component of $\Phi(X)$, since $[0, 1]$ is connect, we infer that $i(\lambda - tT) = i(I) = 0$. ∎

Remark. Corollary 2.1 is an amelioration of [13, Theorem 2.1]. Indeed, in the particular case where $P(T) \in \mathcal{K}(X)$ and $P(\lambda) \neq 0$, the operator $\lambda - T$ is Fredholm of index zero.

3. Main results

Let $T \in \mathcal{C}(X)$. Since T is closed, then $\mathcal{D}(T)$ (the domain of T) endowed with the graph norm $\|.\|_T$ (i.e $\|x\|_T := \|x\| + \|Tx\|$) is a Banach space. In this new space, denoted by X_T, the operator T satisfies $\|Tx\| \leq \|x\|_T$ and consequently, $T \in \mathcal{L}(X_T, X)$. If J is a linear operator with $\mathcal{D}(T) \subset \mathcal{D}(J)$, then J will be called T-defined. If J is T-defined, we will denote by $\widehat{J}$ its restriction to $\mathcal{D}(T)$. Moreover, if $\widehat{J} \in \mathcal{L}(X_T, X)$, we say that J is T-bounded.

In this section, we will establish some characterization and invariance of the Schechter essential spectrum in a Banach space X. For this, let $T \in \mathcal{C}(X)$, S a T-bounded operator on X and let $\lambda \in \rho(T+S)$. Since S is T-bounded, according to Lemma 2.1 in [17], $S(\lambda - T - S)^{-1}$ (resp., $S(\lambda - T)^{-1}$) is a closed linear operator defined on all of X and therefore bounded by the closed graph theorem. In the first part, we begin by giving theorem which is a generalization of Theorem 3.2 in [1]. For this, let $T \in \mathcal{C}(X)$. We define the following sets:

$$\mathcal{S}_T(X) = \Big\{ S \in \mathcal{C}(X) : S \text{ is } T\text{-bounded and } S(\lambda - T - S)^{-1} \in \mathcal{P}_1\mathcal{K}(X), \text{ for all } \lambda \in \rho(T+S) \Big\},$$

$$\mathcal{Q}_T(X) = \Big\{ S \in \mathcal{C}(X) : S \text{ is } T\text{-bounded and there exists a nonzero complex polynomial } P(z) = \sum_{k=0}^{p} a_k z^k \text{ satisfying } P(-1) \neq 0 \text{ and } |P|\big(\gamma\big(S(\lambda - T - S)^{-1}\big)\big) < |P(-1)|, \text{ for all } \lambda \in \rho(T+S) \Big\},$$

where $|P|(z) = \sum_{k=0}^{p} |a_k| z^k$ and γ is the Kuratowski measure of noncompactness.

Remark. Observe that, for $T \in \mathcal{C}(X)$, we have $\mathcal{K}(X) \subset \mathcal{S}_T(X) \cap \mathcal{Q}_T(X)$. Indeed, let $K \in \mathcal{K}(X)$. If we take $P(z) = z$, then, for all $\lambda \in \rho(T+K)$, $P(K(\lambda - T - K)^{-1}) \in \mathcal{K}(X)$, and $|P|\left(\gamma\left(K(\lambda - T - K)^{-1}\right)\right) = 0 < |P(-1)|$.

In what follows, we say that an operator $T \in \mathcal{C}(X)$ satisfies the hypothesis $(\mathcal{A})$ if

$$(\mathcal{A}) \begin{cases} \text{(i) For all } R \in \mathcal{C}(X), \text{ such that } R \text{ is } T\text{-bounded, there exists } \lambda \in \mathbb{R}; \\ \quad]\lambda, +\infty[\subset \rho(T+R). \\ \text{(ii) } \rho_{ess}(T) \text{ is a connected set of } \mathbb{C}. \end{cases}$$

We begin with the following proposition which is fundamental for our purpose. It provides a new characterization of the Schechter essential spectrum in a Banach space X and extends that established in [1, Theorem 3.2].

Proposition 3.1. *Let $T \in \mathcal{C}(X)$. Then*

$$\sigma_{ess}(T) = \bigcap_{S \in \mathcal{Q}_T(X) \cup \mathcal{S}_T(X)} \sigma(T+S).$$

Proof. Since $\mathcal{K}(X) \subset Q_T(X) \cup \mathcal{S}_T(X)$, we infer that

$$\bigcap_{S \in \mathcal{Q}_T(X) \cup \mathcal{S}_T(X)} \sigma(T+S) \subset \sigma_{ess}(T).$$

On the other hand, we claim the opposite inclusion holds. Indeed, suppose that

$$\lambda \notin \bigcap_{S \in \mathcal{Q}_T(X) \cup \mathcal{S}_T(X)} \sigma(T+S),$$

then there exists $S \in \mathcal{Q}_T(X) \cup \mathcal{S}_T(X)$, such that $\lambda \in \rho(T+S)$. We have two possibilities:

(i) If $S \in \mathcal{Q}_T(X)$, then there exists a nonzero complex polynomial P satisfying $P(-1) \neq 0$ and $|P|\left(\gamma\left(S(\lambda - T - S)^{-1}\right)\right) < |P(-1)|$. Applying Corollary 2.1, we get $I + S(\lambda - T - S)^{-1} \in \Phi(X)$ and $i\left(I + S(\lambda - T - S)^{-1}\right) = 0$.

(ii) If $S \in \mathcal{S}_T(X)$, then there exists a nonzero complex polynomial P satisfying $P(\frac{1}{n}) \neq 0$, $\forall\, n \in \mathbb{N}^*$ and $P(S(\lambda - T - S)^{-1})$ is a compact operator on X. Since $P(-1) \neq 0$, then from [13, Theorem 2.1], $I + S(\lambda - T - S)^{-1} \in \Phi(X)$ and $i(I + S(\lambda - T - S)^{-1}) = 0$.

For each above case, using the equality

$$\lambda - T = [I + S(\lambda - T - S)^{-1}](\lambda - T - S),$$

together with Atkinson's theorem [15, Proposition 2.c.7(ii), p. 77] one gets $\lambda - T \in \Phi(X)$ and $i(\lambda - T) = 0$. Finally, the use of Proposition 1.2, shows that $\lambda \notin \sigma_{ess}(T)$, and so, $\sigma_{ess}(T) \subset \bigcap_{S \in \mathcal{Q}_T(X) \cup \mathcal{S}_T(X)} \sigma(T+S)$. ∎

Arguing as the proof of Corollary 4.2 in [1], we show:

Corollary 3.1. *Let $T \in \mathcal{C}(X)$. Let us consider $\mathcal{H}$ including in $\mathcal{S}_T(X)$, containing the subspace of all compact operators $\mathcal{K}(X)$ and satisfying that for all $K, K' \in \mathcal{H}$, $K \pm K' \in \mathcal{H}$. Then, for each $K \in \mathcal{H}$, we have*

$$\sigma_{ess}(T) = \sigma_{ess}(T+K).$$

The next result prove the invariance of the Schechter essential spectrum on Banach spaces by means of polynomially compact perturbations.

Theorem 3.1. *Let* $T \in \mathcal{C}(X)$ *satisfying the hypothesis* $(\mathcal{A})$ *and let* $S \in \mathcal{S}_T(X)$ *such that, for all* $p \in \mathbb{Z}$, $\rho_{ess}(T+pS)$ *is a connected set of* $\mathbb{C}$. *Then*

$$\sigma_{ess}(T) = \sigma_{ess}(T+S).$$

To prove Theorem 3.1, we shall state some lemmas we need.

Lemma 3.1. ([1, Lemma 4.2]) *Let* Ω *be an open connected set of* $\mathbb{C}$, Y *be a Banach space and* $f : \Omega \longrightarrow \mathcal{L}(Y)$ *be an analytic operator. We define* $K(f) = \{\lambda \in \Omega$ *such that* $f(\lambda)$ *is compact*$\}$. *Then one of the two possibilities must holds:*

(a) $K(f) = \Omega$,

(b) $K(f)$ *does not have accumulation point in* Ω.

Lemma 3.2. *Let* $T \in \mathcal{C}(X)$ *satisfying hypothesis* $(\mathcal{A})$ *and let* S *be a* T*-bounded operator on* X.

(i) *If* $\rho_{ess}(T+S)$ *is a connected set of* $\mathbb{C}$, *then* $\rho(T+S+K)$ *is a connected set of* $\mathbb{C}$ *for all* K *compact operator.*

(ii) *For each compact operator* K, $\rho(T+S+K) \cap \rho(T+S) \cap \rho(T)$ *has a point of accumulation.*

Proof. The proof is similar to the proof of Proposition 4.1(i)-(ii) in [1]. ∎

Lemma 3.3. *Let* $T \in \mathcal{C}(X)$ *satisfying hypothesis* $(\mathcal{A})$ *and let* S *be a* T*-bounded operator on* X *such that* $\rho_{ess}(T+S)$ *is a connected set. Then we have:*

(i) *If* $S \in \mathcal{S}_T(X)$, *then* $S+K \in \mathcal{S}_T(X)$ *for all compact operator* K.

(ii) *If* $S \in \mathcal{S}_T(X)$, *then* $pS \in \mathcal{S}_T(X)$ *for all* $p \in \mathbb{Z}^*$.

Proof. (i) Let $K \in \mathcal{K}(X)$ and let $\lambda \in \rho(T+S+K) \cap \rho(T+S)$. We have the following identity:

$$S(\lambda-T-S-K)^{-1} = S(\lambda-T-S)^{-1} + S(\lambda-T-S-K)^{-1}K(\lambda-T-S)^{-1}.$$

Since $S(\lambda-T-S-K)^{-1}K(\lambda-T-S)^{-1}$ is a compact operator, then for all $k\in\mathbb{N}^*$ we have

$$\left[S(\lambda-T-S-K)^{-1}\right]^k=\left[S(\lambda-T-S)^{-1}\right]^k+K_k, \tag{3.1}$$

where K_k is a compact operator. According to (3.1), for all nonzero complex polynomial $P(z)=\sum_{k=1}^{p}a_kz^k$,

$$P\big(S(\lambda-T-S-K)^{-1}\big)=P\big(S(\lambda-T-S)^{-1}\big)+K',$$

where $K'=\sum_{k=1}^{p}a_kK_k$ is a compact operator. Then $P(S(\lambda-T-S-K)^{-1})\in\mathcal{K}(X)$ if and only if $P\left(S(\lambda-T-S)^{-1}\right)\in\mathcal{K}(X)$. Hence, for all $\lambda\in\rho(T+S+K)\cap\rho(T+S)$, we have $S(\lambda-T-S-K)^{-1}\in\mathcal{P}_1\mathcal{K}(X)$ if and only if $S(\lambda-T-S)^{-1}\in\mathcal{P}_1\mathcal{K}(X)$. Let $S\in\mathcal{S}_T(X)$ and $K\in\mathcal{K}(X)$. We will prove that $S+K\in\mathcal{S}_T(X)$. To do this, let

$$G=\left\{\lambda\in\rho(T+S+K)\ \text{ such that } S(\lambda-T-S-K)^{-1}\in\mathcal{P}_1\mathcal{K}(X)\right\}.$$

We have $\rho(T+S+K)\cap\rho(T+S)\subset G$. According to Lemma 3.2(ii), G has a point of accumulation and $\rho(T+S+K)$ is a connected set of $\mathbb{C}$. Applying Lemma 3.1, we get $G=\rho(T+S+K)$. So, $S+K\in\mathcal{S}_T(X)$.

(ii) Let $P(z)=\sum_{i=0}^{p}a_iz^i$ be a nonzero complex polynomial satisfying $P(\frac{1}{n})\neq 0$ for all $n\in\mathbb{Z}^*$, and $P(S(\lambda-T-S)^{-1})\in\mathcal{K}(X)$ for all $\lambda\in\rho(T+S)$. We consider $Q(z):=(1-(1-p)z)^pP((1-(1-p)z)^{-1}z)$. Then $Q(\frac{1}{n})\neq 0$ for all $n\in\mathbb{Z}^*$. In what follows we claim that $Q(S(\lambda-T-pS)^{-1})\in\mathcal{K}(X)$ for all $\lambda\in\rho(T+pS)$. From the hypothesis $(\mathcal{A})$, there exist $\alpha_1,\beta_1\in\mathbb{R}$ such that $]\alpha_1,+\infty[\subset\rho(T+S)$ and $]\beta_1,+\infty[\subset\rho(T+pS)$. If we take $\gamma_1=\max\{\alpha_1,\beta_1\}$, we have necessarily $]\gamma_1,+\infty[\subset\rho(T+S)\cap\rho(T+pS)$. Let $\omega_1>\gamma_1$. Then, for all $\lambda\in\mathbb{R}$ satisfying $\lambda\geq\omega_1$,

$$\begin{aligned}&\left[I-(1-p)S(\lambda-T-pS)^{-1}\right]\left[I+(1-p)S(\lambda-T-S)^{-1}\right]\\=&\left[I+(1-p)S(\lambda-T-S)^{-1}\right]\left[I-(1-p)S(\lambda-T-pS)^{-1}\right]=I.\end{aligned}$$

Hence $\Big[I-(1-p)S(\lambda-T-pS)^{-1}\Big]^{-1}$ is invertible in $\mathcal{L}(X)$. Let $\lambda\in[\omega_1,+\infty[$. We have the following identity:

$$S(\lambda-T-S)^{-1}=S(\lambda-T-pS)^{-1}\Big[I-(1-p)S(\lambda-T-pS)^{-1}\Big]^{-1}.$$

Since

$$\Big[I-(1-p)S(\lambda-T-pS)^{-1}\Big]S(\lambda-T-pS)^{-1} = S(\lambda-T-pS)^{-1}\Big[I-(1-p)S(\lambda-T-pS)^{-1}\Big],$$

then, for all $k \in \mathbb{N}$, we have

$$\Big[S(\lambda-T-S)^{-1}\Big]^k = \Big[S(\lambda-T-pS)^{-1}\Big]^k\Big[I-(1-p)S(\lambda-T-pS)^{-1}\Big]^{-k}.$$

Since, for all $\lambda \in \rho(T+S)$, $P(S(\lambda-T-S)^{-1}) \in \mathcal{K}(X)$, then, for all $\lambda \in [\omega_1,+\infty[$, $Q(S(\lambda-T-pS)^{-1}) \in \mathcal{K}(X)$. From the fact that $\rho(T+pS)$ is a connected set of $\mathbb{C}$, and $[\omega_1,+\infty[\subset \rho(T+pS)$, we infer from Lemma 3.1 that $Q(S(\lambda-T-pS)^{-1}) \in \mathcal{K}(X)$ for all $\lambda \in \rho(T+pS)$. This proves the claim and completes the proof of lemma. ∎

Proof of Theorem 3.1. Let $S \in \mathcal{S}_T(X)$ be such that, for all $p \in \mathbb{Z}$, $\rho_{ess}(T+pS)$ is a connected set of $\mathbb{C}$. Define the following set:

$$\mathcal{I}_{T,S}(X) = \{K + pS : K \in \mathcal{K}(X)\}.$$

It is obvious that $\mathcal{K}(X) \subset \mathcal{I}_{T,S}(X)$. By Lemma 3.3, $\mathcal{I}_{T,S}(X) \subset \mathcal{S}_T(X)$. It is easy to show that, for all $K \in \mathcal{I}_{T,S}(X)$, $\mathcal{I}_{T,S}(X) \pm K = \mathcal{I}_{T,S}(X)$. For $S \in \mathcal{I}_{T,S}(X)$, we have

$$\sigma_{ess}(T+S) = \bigcap_{K\in\mathcal{I}_{T,S}(X)} \sigma(T+S+K) = \bigcap_{K-S\in\mathcal{I}_{T,S}(X)} \sigma(T+K) = \bigcap_{K\in\mathcal{I}_{T,S}(X)} \sigma(T+K) = \sigma_{ess}(T).$$

Which completes the proof. ∎

For $T \in \mathcal{C}(X)$ satisfying the hypothesis $(\mathcal{A})$, we define the following set

$$\mathcal{M}_T(X) = \{S \in \mathcal{C}(X) : S \text{ is } T\text{-bounded and } S(\lambda-T)^{-1} \in \mathcal{P}_1\mathcal{K}(X), \text{ for all } \lambda \in \rho(T)\}.$$

Remark. Observe that if we take $P(z) = z$, then $P(K(\lambda-T)^{-1}) \in \mathcal{K}(X)$ for all $\lambda \in \rho(T)$, and therefore $\mathcal{K}(X) \subset \mathcal{M}_T(X)$.

Applying Theorem 3.1, we have:

Corollary 3.2. *Let $S \in \mathcal{M}_T(X)$ be such that, for all $p \in \mathbb{Z}$, $\rho_{ess}(T+pS)$ is a connected set of $\mathbb{C}$. Then*

$$\sigma_{ess}(T) = \sigma_{ess}(T+S).$$

Proof. It suffices to prove that $\mathcal{M}_T(X) \subset \mathcal{S}_T(X)$. For this, let $S \in \mathcal{M}_T(X)$. Since T satisfies the hypothesis $(\mathcal{A})$, there exist $\lambda_1, \lambda_2 \in \mathbb{R}$ such that $]\lambda_1, +\infty[\subset \rho(T)$ and $]\lambda_2, +\infty[\subset \rho(T+S)$. If we take $\overline{\lambda} = \max\{\lambda_1, \lambda_2\}$, then $]\overline{\lambda}, +\infty[\subset \rho(T+S) \cap \rho(T)$. Let $\omega > \overline{\lambda}$. Then, for all $\lambda \in \mathbb{R}$ satisfying $\lambda \geq \omega$,

$$\Big[I + S(\lambda - T - S)^{-1}\Big]\Big[I - S(\lambda - T)^{-1}\Big] = \Big[I - S(\lambda - T)^{-1}\Big]\Big[I + S(\lambda - T - S)^{-1}\Big] = I.$$

Hence, $\big[I + S(\lambda - T - S)^{-1}\big]$ is invertible in the algebra of bounded linear operators on X. Let $\Omega' = \{\lambda \in \rho(T+S) \cap \rho(T) : \lambda \geq \omega\}$ and let $\lambda \in \Omega'$. We have the following identity:

$$S(\lambda - T)^{-1} = \Big[I + S(\lambda - T - S)^{-1}\Big]^{-1} S(\lambda - T - S)^{-1}.$$

Since

$$\Big[I + S(\lambda - T - S)^{-1}\Big] S(\lambda - T - S)^{-1} = S(\lambda - T - S)^{-1}\Big[I + S(\lambda - T - S)^{-1}\Big],$$

then, for all $k \in \mathbb{N}$, we have

$$\Big[S(\lambda - T)^{-1}\Big]^k = \Big[I + S(\lambda - T - S)^{-1}\Big]^{-k}\Big[S(\lambda - T - S)^{-1}\Big]^k.$$

Let $P(z) = \sum_{k=0}^{p} a_k z^k$, satisfying $P(\frac{1}{n}) \neq 0$ for all $n \in \mathbb{Z}^*$ and $a_p \neq 0$, be such that $P(S(\lambda - T)^{-1}) \in \mathcal{K}(X)$, and let

$$Q(z) := \sum_{k=0}^{p} a_{p-k}(1+z)^k z^{p-k} = (1+z)^p P\left(\frac{z}{1+z}\right).$$

Then, for all $n \in \mathbb{Z}$, $Q(\frac{1}{n}) \neq 0$ and $Q(S(\lambda - T - S)^{-1})$ is a compact operator. According to the hypothesis $(\mathcal{A})(i)$, Ω' has a point of accumulation. Applying Lemma 3.1, we get that $Q(S(\lambda - T - S)^{-1}) \in \mathcal{K}(X)$ for all $\lambda \in \rho(T+S)$, which completes the proof. ∎

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