

Trees and Point of Continuity Property in Banach Spaces[†]

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Abstract: We introduce the notion of topologically weakly null tree and, as a consequence, we get a characterization of the point of continuity property in general Banach spaces by extending to the general case some known results.

Key words: Point of continuity property, trees, boundedly complete sequences.

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1. INTRODUCTION

We recall that a bounded subset C of a Banach space satisfies the point of continuity property (PCP) if every nonempty closed subset of C admits a point of continuity of the identity map from the weak to norm topologies. A Banach space is said to verify the point of continuity property whenever its closed unit ball satisfies this property. It is well known that Banach spaces with Radon-Nikodym property, including separable dual spaces, satisfy PCP, but the converse is false (see [3]). The PCP has been characterized for separable Banach spaces in [3] and [6], and this characterization implies that Banach spaces with PCP have many boundedly complete basic sequences, and as many subspaces which are separable dual spaces. As PCP is separably determined [1], that is, a Banach space satisfies PCP if every separable subspace has PCP, it is natural to look for a characterization of PCP in terms of boundedly complete basic sequences. In this sense, it has been proved in [11] that every semi-normalized basic sequence in a Banach space with PCP has a boundedly complete subsequence. The converse of the above result is false in general, (see Remark 2 in [11]), even for Banach spaces not containing ℓ_1 [7]. Also, it

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was proved in [5] that a Banach space X with a separable dual satisfies PCP if, and only if, every weakly null tree in the unit sphere of X has some boundedly complete branch. In [8] it is proved that the last characterization of PCP holds for Banach spaces not containing isomorphic subspaces to ℓ_1 . It seems then natural to look for a characterization of PCP for general Banach spaces in terms of trees with boundedly complete branches. For this, we introduce the concept of topologically weakly null tree, which is a weaker condition than the weakly null tree condition, and we characterize in terms of trees the PCP for general bounded subsets of Banach spaces in theorem 2.2. As a consequence, we get in theorem 2.3 that a general Banach space X has PCP if, and only if, every seminormalized and topologically weakly null tree in the unit sphere of X has some boundedly complete branch. This result extends then to the general case the aforementioned statement in [5].

We begin with some notation and preliminaries. Let X be a Banach space and let B_X , respectively S_X , be the closed unit ball, respectively sphere, of X . The weak topology in X , will be denoted by w . If A is a subset in X , \overline{A}^w stands for the weak closure of A in X . Recall that for every subset A in a Banach space X and for every $a \in \overline{A}^w$ there is a countable subset F of A such that $a \in \overline{F}^w$ (see [2]). Given $\{e_n\}$ a basic sequence in X , $\{e_n\}$ is said to be *semi-normalized* if $0 < \inf_n \|e_n\| \leq \sup_n \|e_n\| < \infty$ and the closed linear span of $\{e_n\}$ is denoted by $[e_n]$. $\{e_n\}$ is called *boundedly complete* provided whenever scalars $\{\lambda_i\}$ satisfy $\sup_n \|\sum_{i=1}^n \lambda_i e_i\| < \infty$, then $\sum_n \lambda_n e_n$ converges. $\{e_n\}$ is called *shrinking* if $[e_n]^* = [e_n^*]$, where $\{e_n^*\}$ denotes the sequence of biorthogonal functionals associated to $\{e_n\}$.

A boundedly complete basic sequence $\{e_n\}$ in a Banach space X spans a dual space. In fact, $[e_n^*]^* = [e_n]$, where $\{e_n^*\}$ denotes the sequence of biorthogonal functionals in the dual space X^* [9]. Following the notation in [12], it is said that a sequence $\{e_n\}$ in a Banach space is *type P* if the set $\{\sum_{k=1}^n e_k : n \in \mathbb{N}\}$ is bounded. Observe, from the definitions, that type P seminormalized basic sequences always fail to be boundedly complete basic sequences.

$\mathbb{N}^{<\omega}$ stands for the set of all ordered finite sequences of natural numbers joint to the empty sequence denoted by \emptyset . We consider the natural order in $\mathbb{N}^{<\omega}$, that is, given $\alpha = (\alpha_1, \dots, \alpha_p)$, $\beta = (\beta_1, \dots, \beta_q) \in \mathbb{N}^{<\omega}$, one has $\alpha \leq \beta$ if $p \leq q$ and $\alpha_i = \beta_i \forall 1 \leq i \leq p$. Also $|\alpha|$ denotes the *length* of sequence α , and \emptyset is the minimum of $\mathbb{N}^{<\omega}$ with this partial order. A *tree* in a Banach space X is a family $\{x_A\}_{A \in \mathbb{N}^{<\omega}}$ of vectors in X indexed on $\mathbb{N}^{<\omega}$. The tree will be said *seminormalized* if $0 < \inf_A \|x_A\| \leq \sup_A \|x_A\| < \infty$. We will say

that the tree $\{x_A\}_{A \in \mathbb{N}^{<\omega}}$ is *weakly null* if the sequence $\{x_{(A,n)}\}_n$ is weakly null for every $A \in \mathbb{N}^{<\omega}$. The tree $\{x_A\}_{A \in \mathbb{N}^{<\omega}}$ is *topologically weakly null* if $0 \in \overline{\{x_{(A,n)} : n \in \mathbb{N}\}}^w$ for every $A \in \mathbb{N}^{<\omega}$. A sequence $\{x_{A_n}\}_{n \geq 0}$ is called a *branch* if $\{A_n\}$ is a maximal totally ordered subset of $\mathbb{N}^{<\omega}$, that is, there exists a sequence $\{\alpha_n\}$ of natural numbers such that $A_n = (\alpha_1, \dots, \alpha_n)$ for every $n \in \mathbb{N}$ and $A_0 = \emptyset$. The tree $\{x_A\}_{A \in \mathbb{N}^{<\omega}}$ is said to be *uniformly type P* if every branch of the tree is type P and the partial sums of every branch are uniformly bounded.

Finally, we recall that a *boundedly complete skipped blocking finite dimensional decomposition* (BCSBFDD) in a separable Banach space X is a sequence $\{F_j\}$ of finite dimensional subspaces in X such that:

1. $X = [F_j : j \in \mathbb{N}]$.
2. $F_k \cap [F_j : j \neq k] = \{0\}$ for every $k \in \mathbb{N}$.
3. For every sequence $\{n_j\}$ of non-negative integers with $n_j + 1 < n_{j+1}$ for all $j \in \mathbb{N}$ and for every $f \in [F_{(n_j, n_{j+1})} : j \in \mathbb{N}]$ there exists a unique sequence $\{f_j\}$ with $f_j \in F_{(n_j, n_{j+1})}$ for all $j \in \mathbb{N}$ such that $f = \sum_{j=1}^{\infty} f_j$.
4. Whenever $f_j \in F_{(n_j, n_{j+1})}$ for all $j \in \mathbb{N}$ and $\sup_n \|\sum_{j=1}^n f_j\| < \infty$ then $\sum_{j=1}^{\infty} f_j$ converges.

Here, $[A]$ denotes the closed linear span in X of the set A and, for some nonempty interval of non-negative integers I , we denote the linear span of the F'_j s for $j \in I$ by F_I .

If $\{F_j\}$ is a BCSBFDD in a separable Banach space X and $\{x_j\}$ is a sequence in X such that $x_j \in F_{(n_j, n_{j+1})}$ for some sequence $\{n_j\}$ of non-negative integers with $n_j + 1 < n_{j+1}$ for all $j \in \mathbb{N}$, we say that $\{x_j\}$ is a *skipped block sequence* of $\{F_n\}$. It is standard to prove that there is a positive constant K such that every skipped block sequence $\{x_j\}$ of $\{F_n\}$ with $x_j \neq 0$ for every j is a boundedly complete basic sequence with constant at most K .

From [6], we know that the family of separable Banach spaces with PCP is exactly the family of separable Banach spaces with a BCSBFDD.

2. MAIN RESULTS

As we have said in the introduction, we know that every basic sequence in a Banach space with PCP has a boundedly complete basic subsequence [11] and that every weakly null tree in the unit sphere of a Banach space with a separable dual and PCP has a boundedly complete branch [5]. The converse

of the first result is false, even for Banach spaces not containing ℓ_1 -copies [7] and the second one fails to be true in general Banach spaces since, for example in Banach spaces with the Schur property (spaces where the weak sequential convergence implies the norm convergence), there are no seminormalized weakly null trees and there are Banach spaces with the Schur property failing PCP (see [4]). So it is natural to look for a general characterization of PCP in terms of trees and boundedly complete basic sequences. Our main goal is getting such a characterization. For this we begin with an easy lemma which shows that PCP is separably determined from a local point of view.

LEMMA 2.1. *Let K be a bounded subset of a Banach space X . If K fails PCP, then there is a closed and separable subset $B \subset K$ failing PCP.*

Proof. As K fails PCP, then there is $B \subset K$ and $\delta > 0$ such that every relatively weak open subset of B has diameter at least 2δ . Thus $b \in \overline{B \setminus B(b, \delta)}^w$ for every $b \in B$, where $B(b, \delta)$ stands for the open ball with center b and radius δ . Now, from [2], for every $b \in B$ there is a countable subset $B_b \subset B \setminus B(b, \delta)$ such that $b \in \overline{B_b}^w$. Choose $b_1 \in B$ and define $B_1 = B_{b_1} \cup \{b_1\}$ and $B_{n+1} = B_n \cup \cup_{b \in B_n} B_b$ for every $n \in \mathbb{N}$. Put $A = \cup_{n \in \mathbb{N}} B_n$ and let us see that A fails PCP. For this we show that every relatively weak open subset of A has diameter at least δ . Indeed, let U be a relatively weakly open subset of A . Then there is $n \in \mathbb{N}$ such that $B_n \cap U \neq \emptyset$. Pick $b \in B_n \cap U$. Now $b \in \overline{B_b}^w$ and so there is some $x \in B_b \cap U$. Therefore $x \in A \cap U$ and $\|x - b\| > \delta$ which shows that U has diameter at least δ . It is now clear that $B = \overline{A}$ is a closed and separable subset of K failing PCP. ■

The following result is a local characterization of PCP for general bounded subsets of Banach spaces.

THEOREM 2.2. *Let X be a Banach space and let K be a bounded subset of X . Then the following assertions are equivalent:*

- i) K fails PCP.
- ii) *There is a seminormalized topologically weakly null tree $\{x_A\}_{A \in \mathbb{N}^{<\omega}}$ in X such that $\{\sum_{C \leq A} x_C : A \in \mathbb{N}^{<\omega}\} \subset K$.*

Proof. i) \Rightarrow ii) Assume that K fails PCP. Then, from lemma 2.1, there is B a closed and separable subset of K failing PCP and then there exists $\delta > 0$ such that every relatively weak open subset of B has diameter greater than 2δ . So $b \in \overline{B \setminus B(b, \delta)}^w$ for every $b \in A$, where $B(b, \delta)$ stands for the open ball

with center b and radius δ . Now, from [4], for every $b \in B$ there is a countable set $F_b \subset B \setminus B(b, \delta)$ such that $b \in \overline{F_b}^w$.

First, we construct a tree $\{y_A\}_{A \in \mathbb{N}^{<\omega}}$ in B satisfying:

- a) $y_A \in \overline{B \setminus B(y_A, \delta)}^w$ for every $A \in \mathbb{N}^{<\omega}$.
- b) $\|y_A - y_{(A,i)}\| > \delta$ for every $A \in \mathbb{N}^{<\omega}$.
- c) $y_A \in \overline{\{y_{(A,i)} : i \in \mathbb{N}\}}^w$ for every $A \in \mathbb{N}^{<\omega}$.

We do the construction by induction on $n = |A|$, the length of $A \in \mathbb{N}^{<\omega}$. For $n = 0$, $A = \emptyset$ and we pick $y_\emptyset \in B \setminus \{0\}$. For $n = 1$, as $y_\emptyset \in \overline{B \setminus B(y_\emptyset, \delta)}^w$ then there is a countable set $F_{y_\emptyset} = \{y_{(i)} : i \in \mathbb{N}\} \subset B \setminus B(y_\emptyset, \delta)$ such that $y_\emptyset \in \overline{F_{y_\emptyset}}^w$. Then a), b) and c) are verified.

Assume that y_A is constructed whenever $A \in \mathbb{N}^{<\omega}$ and $|A| = n$. Pick $A \in \mathbb{N}^{<\omega}$. From a), $y_A \in \overline{B \setminus B(y_A, \delta)}^w$ and so, from [4], there is a countable set $F_{y_A} = \{y_{(A,i)} : i \in \mathbb{N}\} \subset B \setminus B(y_A, \delta)$ such that $y_A \in \overline{F_{y_A}}^w$. This finishes the construction of the tree $\{y_A\}_{A \in \mathbb{N}^{<\omega}}$ satisfying a), b) and c).

Now we define a new tree $\{x_A\}_{A \in \mathbb{N}^{<\omega}}$ by $x_\emptyset = y_\emptyset$ and $x_{(A,i)} = y_{(A,i)} - y_A$ for every $i \in \mathbb{N}$ and $A \in \mathbb{N}^{<\omega}$. From b) we get that $\{x_A\}_{A \in \mathbb{N}^{<\omega}}$ is a seminormalized tree, since B is bounded. From c), we deduce that $\{x_A\}_{A \in \mathbb{N}^{<\omega}}$ is topologically weakly null. Furthermore, if $A \in \mathbb{N}^{<\omega}$ then $\sum_{C \leq A} x_C = y_A$, from the definition of the tree $\{x_A\}_{A \in \mathbb{N}^{<\omega}}$. So $\{x_A\}_{A \in \mathbb{N}^{<\omega}}$ is a uniformly type P tree, since B is bounded and $y_A \in B$ for every $A \in \mathbb{N}^{<\omega}$. This finishes the proof of i) \Rightarrow ii).

ii) \Rightarrow i) Let $\{x_A\}$ be a seminormalized weakly null tree such that $B = \{\sum_{C \leq A} x_C : A \in \mathbb{N}^{<\omega}\} \subset K$ and let $\delta > 0$ such that $\|x_A\| > \delta$ for every $A \in \mathbb{N}^{<\omega}$. For every $A \in \mathbb{N}^{<\omega}$ and for every $n \in \mathbb{N}$ we have that $\sum_{C \leq (A,n)} x_C = \sum_{C \leq A} x_C + x_{(A,n)}$, but $0 \in \overline{\{x_{(A,n)} : n \in \mathbb{N}\}}^w$, since the tree $\{x_A\}$ is topologically weakly null. So $\sum_{C \leq A} x_C \in \overline{\{\sum_{C \leq (A,n)} x_C : n \in \mathbb{N}\}}^w$ and $\|\sum_{C \leq (A,n)} x_C - \sum_{C \leq A} x_C\| > \delta$. This proves that B has no points where the identity map is continuous from the weak to the norm topologies. In fact, we have proved that every relatively weak open subset of B has diameter greater than δ . Now, \overline{B}^w is a closed and bounded subset of K such that every relatively weak open subset of \overline{B}^w has diameter greater than δ , since B is weakly dense in \overline{B}^w , and so K fails PCP. ■

We show now our characterization of PCP in terms of boundedly complete basic sequences in a general setting.

THEOREM 2.3. *Let X be a Banach space. Then the following assertions are equivalent:*

- i) X has PCP.
- ii) Every topologically weakly null tree in S_X is not uniformly type P .
- iii) Every topologically weakly null tree in S_X does not have type P branches.
- iv) Every topologically weakly null tree in S_X has a boundedly complete branch.

We need the following easy

LEMMA 2.4. *Let X be a Banach space and let $\{x_i\}$ a bounded net in X . Then the following assertion are equivalent:*

- i) $\{x_i\}$ is weakly null.
- ii) The scalars net $\{\text{dist}(x_i, Y)\}$ is null for every Y subspace of X with finite codimension.
- iii) The scalars net $\{\text{dist}(x_i, Y)\}$ is null for every Y subspace of X with codimension one.

Proof. i) \Rightarrow ii). Assume that ii) is false. Then there is a subspace Y of X with finite codimension such that the scalars sequence $\{\text{dist}(x_i, Y)\}$ is not null. We can assume, passing to a subnet if it is necessary, that there is $\delta > 0$ such that $\text{dist}(x_i, Y) > \delta$ for every i . By a separation argument, for every i there is $f_i \in S_{X^*}$ such that $f_i(x_i) = \text{dist}(x_i, Y) > \delta$ and $f_i \in Y^\circ := \{f \in X^* : f(y) = 0 \forall y \in Y\}$. As Y has finite codimension, $(X/Y)^* = Y^\circ$ is finite dimensional and so we can assume, passing to a subnet if it is necessary, that $\{f_i\}$ converges to some $f \in S_{X^*}$ in the norm topology, since $\{f_i\}$ is a bounded sequence by hypothesis. Now one has that $f(x_i) = (f - f_i)(x_i) + f_i(x_i)$ for every i . But $\{(f - f_i)(x_i)\}$ is null and $f_i(x_i) > \delta$ for every i , thus $\{f(x_i)\}$ is not null, which is a contradiction with i).

ii) \Rightarrow iii) is trivial and for iii) \Rightarrow i) pick $f \in X^*$, $f \neq 0$ and do $Y = \text{Ker } f$. Then Y is a subspace of X with codimension one and so $\{\text{dist}(x_i, Y)\}$ is null. As $\text{dist}(x_i, Y) = \frac{|f(x_i)|}{\|f\|}$ for every i we deduce that $\{f(x_i)\}$ is a null net and thus $\{x_i\}$ is weakly null. ■

Proof of theorem 2.3. iv) \Rightarrow iii) is a consequence of the fact that every boundedly complete basic sequence is not type P , commented in the introduction and iii) \Rightarrow ii) is trivial.

For ii)⇒i) it is enough to apply theorem 2.2 for $K = B_X$ by assuming that X fails PCP and normalizing.

i)⇒iv) Assume that X has PCP. As it was said in the introduction we can assume that X has a BCSBFDD from [6]. Then there is a sequence $\{F_j\}$ of finite dimensional subspaces in X such that:

1. $X = [F_j : j \in \mathbb{N}]$.
2. $F_k \cap [F_j : j \neq k] = \{0\}$ for every $k \in \mathbb{N}$.
3. For every sequence $\{n_j\}$ of non-negative integers with $n_j + 1 < n_{j+1}$ for all $j \in \mathbb{N}$ and for every $f \in [F_{(n_j, n_{j+1})} : j \in \mathbb{N}]$ there exists a unique sequence $\{f_j\}$ with $f_j \in F_{(n_j, n_{j+1})}$ for all $j \in \mathbb{N}$ such that $f = \sum_{j=1}^{\infty} f_j$.
4. Whenever $f_j \in F_{(n_j, n_{j+1})}$ for all $j \in \mathbb{N}$ and $\sup_n \|\sum_{j=1}^n f_j\| < \infty$ then $\sum_{j=1}^{\infty} f_j$ converges.

Here, $[A]$ denotes the closed linear span in X of the set A and, for some nonempty interval of non-negative integers I , we denote the linear span of the F_j 's for $j \in I$ by F_I .

Take a topologically weakly null tree $\{x_A\}$ in S_X . We have to construct a boundedly complete branch of the tree $\{x_A\}$. For this, fix a sequence $\{\varepsilon_j\}_{j \geq 0}$ of positive real numbers with $\sum_{j=0}^{\infty} \varepsilon_j < 1/2K$, where $K > 0$ is such that every skipped block sequence of $\{F_n\}_n$ has basic constant at most K , as we said in the introduction. Now we construct a sequence $\{n_j\}_{j \geq 0}$ of integers positive numbers with $n_{j-1} + 1 < n_j$ for all $j \geq 0$, doing $n_{-1} = 0$, a sequence $\{y_j\}_{j \geq 0}$ in X with $y_j \in F_{(n_{j-1}, n_j)}$ for all $j \geq 0$, that is, a skipped block sequence of $\{F_n\}_n$, and a branch $\{x_{A_j}\}$ of the tree such that $\|x_{A_j} - y_j\| < \varepsilon_j$ for all $j \geq 0$.

Define $A_0 = \emptyset$. By i), there exists $n_0 > 2$ and $y_0 \in F_{(0, n_0)}$ such that $\|x_{A_0} - y_0\| < \varepsilon_0$. Now, assume that $n_0, \dots, n_j, y_0, \dots, y_j$ and A_0, \dots, A_j have been constructed such that $A_0 \leq A_1 \leq \dots \leq A_j$ and $|A_k| = k$ for $1 \leq k \leq j$. Then $A_k = (p_1, p_2, \dots, p_k)$ for all $1 \leq k \leq j$ for some positive integers p_1, p_2, \dots, p_j . As the tree is topologically weakly null we have that $0 \in \overline{\{x_{(A_j, p)} : p \in \mathbb{N}\}}^w$ and so there is a net $\{x_{(A_j, p_\lambda)}\}_{\lambda \in \Lambda}$ converging weakly to 0. Then, by lemma 2.4, we deduce that $\{\text{dist}(x_{(A_j, p_\lambda)}, F_{[n_j+1, \infty)})\}_\lambda$ converges to 0, since $F_{[n_j+1, \infty)}$ is a finite codimensional subspace in X . Then there exist $p_{j+1} \in \mathbb{N}$, $n_{j+1} > n_j + 1$ and $y_{j+1} \in F_{(n_j, n_{j+1})}$ such that $\|x_{(A_j, p_{j+1})} - y_{j+1}\| < \varepsilon_{j+1}$. Now we define $A_{j+1} = (A_j, p_{j+1})$ and we have that $A_j \leq A_{j+1}$ and $|A_{j+1}| = j + 1$. This finishes the inductive construction of the branch $\{x_{A_j}\}$ satisfying that $\|x_{A_j} - y_j\| < \varepsilon_j$ for all j . Finally we get that $\sum_{j=1}^{\infty} \|x_{A_j} - y_j\| < 1/2K$, being $\{y_j\}$ a skipped block sequence of $\{F_j\}$. Then $\{x_{A_j}\}$ is a branch

of the tree $\{x_A\}_{A \in \mathbb{N}^{<\omega}}$ which is a basic sequence equivalent to $\{y_j\}$, hence boundedly complete and the proof of theorem 2.3 is finished. ■

Remark 1. If X is a Banach space with a separable dual, the result in [5] stating that X has PCP if, and only if, every weakly null tree in S_X has a boundedly complete branch is a consequence of the above theorem, since in this case the weak topology in X has a metrizable behavior, and then we can change the topologically weakly null condition by the weakly null condition. Furthermore, in this setting, the fact that every weakly null tree in S_X has a boundedly complete branch can be seen in terms of the following game G : player I chooses finite codimensional subspaces X_1, X_2 , etc of X and player II chooses normalized vectors $x_1 \in X_1, x_2 \in X_2$, etc. Following [10], if X is a Banach space with a separable dual, then the fact that every weakly null tree in S_X has a boundedly complete branch, equivalently X has PCP, is equivalent to the fact that the player I in the game G can always force player II to come up with a boundedly complete basis. We don't know if PCP can be characterized in this way for general Banach spaces. In other words, we don't know if the fact that every topologically weakly null tree in a Banach space X has a boundedly complete branch is equivalent to the fact that the player I in the game G can always force player II to come up with a boundedly complete basis. In the case of a Banach space X with a separable dual the above equivalence is true because we can take a dense sequence $\{x_n^*\}$ in X^* and consider in X the subspaces $\{X_n = [x_1, \dots, x_n]^\perp\}$. Then player I only needs to choose among the spaces $\{X_n\}$ and thus player II, who tries to avoid boundedly complete sequences, has a winner strategy if, and only if, there is a tree $\{x_A\}_{A \in \mathbb{N}^{<\omega}}$ in X with $x_A \in X_{|A|}$ for every $A \in \mathbb{N}^{<\omega}$ and without boundedly complete branches. This tree can then be seen as a winner strategy for player II.

Remark 2. It is unknown if PCP is basically determined, that is, it is a well known open problem if every Banach space failing PCP has a subspace with a Schauder basis and failing PCP. In relation with this problem we consider interesting to know if every sequence $\{x_n\}$ in a Banach space X such that $0 \in \overline{\{x_n : n \in \mathbb{N}\}}^w$ and $0 \notin \overline{\{x_n : n \in \mathbb{N}\}}^{\|\cdot\|}$ has a basic subsequence $\{y_n\}$ such that $0 \in \overline{\{y_n : n \in \mathbb{N}\}}^w$. The existence of $\{y_n\}$ without the requirement $0 \in \overline{\{y_n : n \in \mathbb{N}\}}^w$ is a consequence of the well known Mazur's procedure to extract basic sequences. Observe that if a Banach space X fails PCP, from theorem 2.2 we get that there is a topologically weakly null tree in S_X without boundedly complete branches. The question then is if it is possible to

construct a basic subtree from this tree. If this is the case, again from theorem 2.2 we get a subspace with a Schauder basis failing PCP. On the other hand, it is proved in [8] that a Banach space failing PCP contains a seminormalized basic tree without boundedly complete branches.

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