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Property (V) Still Fails the Three-Space Property

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Abstract: We construct twisted sums of $C[0,1]$ with itself in which the quotient map does not fix any copy of either $C[0,1]$ or c_0. We moreover show that every twisted sum of $c_0(\Gamma)$ must have Pełczyński's property (V).

Key words: Banach spaces, property (V), twisted sums.

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1. Introduction and preliminaries

In [13], Pełczyński discovered that $C(K)$ spaces enjoy the property that all operators on them are either weakly compact or an isomorphism on a copy of c_0; he called this property (V). Thus, a Banach space X has property (V) if every operator from X into any other Banach space is either weakly compact or an isomorphism on some copy of c_0. Pełczyński's property (V) is not a 3-space property as it was shown in [5] –see also [6]–. The counterexample there presented required a careful inspection of a quite involved example of Ghoussoub and Johnson [8]. In this note we present a rather natural example: there is an exact sequence

$$0 \longrightarrow C[0,1] \longrightarrow \mathfrak{S} \xrightarrow{q_{\mathfrak{S}}} C[0,1] \longrightarrow 0 \tag{1}$$

in which the operator $q_{\mathfrak{S}}$ is not an isomorphism on any copy of c_0, and therefore the space $\mathfrak{S}$ fails to have Pełczyński's property (V). In [3] it was shown that for every separable Banach space X not containing ℓ_1 there exists an exact sequence

$$0 \longrightarrow C[0,1] \longrightarrow \Omega(X) \xrightarrow{q_X} X \longrightarrow 0$$

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in which q_X is a strictly singular operator. Therefore, the choice $X = c_0$ yields an exact sequence

$$0 \longrightarrow C[0,1] \longrightarrow \Omega(c_0) \xrightarrow{q_0} c_0 \longrightarrow 0 \tag{2}$$

in which the space $\Omega(c_0)$ fails property (V), providing in this way another counterexample to the 3-space problem for property (V). Moreover, from (2) it follows the existence of an exact sequence

$$0 \longrightarrow C[0,1] \longrightarrow \Omega(c_0) \oplus C[0,1] \xrightarrow{Q} C[0,1] \longrightarrow 0$$

in which the space $\Omega(c_0) \oplus C[0,1]$ necessarily fails property (V) although the operator Q is an isomorphism on a copy of $C[0,1]$. Let us say that a Banach X space enjoys Rosenthal's property (V^*) when every operator $T : X \to Y$ for which T^*Y^* is nonseparable fixes a copy of $C[0,1]$. Rosenthal proved in [17] that $C[0,1]$ enjoys Rosenthal's property (V^*). Example (1) shows that Rosenthal's property (V^*) also fails to be a 3-space property; we will show in Section 2 another "ad hoc" counterexample. It is worth to mention that Rosenthal's analogue to property (V) –namely, every operator from X into a Banach space Y is either weakly compact or an isomorphism on a copy of ℓ_∞ [16]– was called in [1] Rosenthal's property (V) and shown to be a 3-space property.

An exact sequence $0 \to Y \to X \to Z \to 0$ of Banach spaces is a diagram formed by Banach spaces and linear continuous operators in which the kernel of each operator coincides with the image of the preceding; the middle space X is also called a *twisted sum* of Y and Z. By the open mapping theorem this means that Y is a subspace of X and Z is the corresponding quotient. An exact sequence is said to split if it is equivalent to the trivial sequence $0 \to Y \to Y \oplus Z \to Z \to 0$. There is a correspondence (see [10, 11, 6]) between exact sequences $0 \to Y \to X \to Z \to 0$ of Banach spaces and the so-called z-linear maps which are homogeneous maps $F : Z \curvearrowright Y$ (we use this notation to stress the fact that these are not linear maps) with the property that there exists some constant $C > 0$ such that for all finite sets $x_1, \ldots, x_n \in Z$ one has $\|F(\sum_{n=1}^N x_n) - \sum_{n=1}^N F(x_n)\| \le C \sum_{n=1}^N \|x_n\|$. The infimum of those constants C is called the z-linear constant of F and denoted $Z(F)$. Given a z-linear map $F : Z \curvearrowright Y$ the twisted sum space $Y \oplus_F Z$ is by definition the vector space $Y \times Z$ endowed with the quasi-norm $\|(y,z)\|_F = \|y - Fz\| + \|z\|$. The z-linearity of F makes this quasi-norm $Z(F)$-equivalent to a norm (see [4]). A z-linear map $F : X \curvearrowright Y$ induces an exact sequence of Banach spaces $0 \to Y \to Y \oplus_F X \to X \to 0$. Indeed, the map $y \mapsto (y, 0)$

embeds Y isometrically into $Y \oplus_F X$ and $(y, x) \mapsto x$ extends to a quotient map $q : Y \oplus_F X \to X$ whose kernel is Y. Two quasi-linear maps $F, G : X \to Y$ are said to be equivalent –or that one is a version of the other– when the difference $F - G$ can be written as $B + L$, where $B : X \to Y$ is a homogeneous map bounded on the unit ball and $L : X \to Y$ is linear.

2. Rosenthal's property (V^*) is not a 3-space property

Our starting point is Pełczyński's proof [14] that every subspace of $C[0,1]$ containing a copy C of $C[0,1]$ contains a further copy C' of $C[0,1]$ that is complemented in $C[0,1]$. An examination of the proof shows that for every $\varepsilon > 0$ the copy C' can be chosen $(1+\varepsilon)$-isomorphic to $C[0,1]$ and $(1+\varepsilon)$-complemented. Let thus Γ be the family of all subspaces γ of $C[0,1]$ which are 2-isomorphic to $C[0,1]$ via some isomorphism j_γ and 2-complemented by some projection p_γ. The existence of a nontrivial exact sequence

$$0 \longrightarrow C[0,1] \longrightarrow X \longrightarrow C[0,1] \longrightarrow 0$$

is well known [6, 3]. Let ω be a quasi-linear map inducing this sequence. We define the map $\Omega' : C[0,1] \curvearrowright C[0,1]^\Gamma$ by

$$\Omega'(x) = \big(\omega j_\gamma p_\gamma(x)\big)_{\gamma\in\Gamma}.$$

At each coordinate γ one has $Z(\omega j_\gamma p_\gamma) \leq Z(\omega)\|j_\gamma\|\|p_\gamma\|$. Therefore

$$\sup_{\gamma\in\Gamma}\left\|\omega j_\gamma p_\gamma\Big(\sum x_i\Big) - \sum_i \omega j_\gamma p_\gamma(x_i)\right\| \leq 4Z(\omega)\sum_i \|x_i\|$$

for finite sums. This is therefore enough to define a z-linear map $\Omega : X \curvearrowright \ell_\infty(\Gamma, C[0,1])$ as follows: take $P : C[0,1]^\Gamma \to \ell_\infty(\Gamma, C[0,1])$ a linear projection and set $\Omega = P\Omega'$. It is z-linear with $Z(\Omega) \leq 4Z(\omega)$ since $P\Omega'(\sum x_i) - \sum_i P\Omega'(x_i) = \Omega'(\sum x_i) - \sum_i \Omega'(x_i)$. Thus, it defines an exact sequence

$$0 \longrightarrow \ell_\infty(\Gamma, C[0,1]) \longrightarrow X \xrightarrow{\ q\ } C[0,1] \longrightarrow 0.$$

Let us show that Ω cannot be trivial on any copy of $C[0,1]$. It is enough to show that the restriction of Ω to any copy $\gamma_0 \in \Gamma$ is not trivial. Let $\pi_0 : \ell_\infty(\Gamma, C[0,1]) \to C[0,1]$ be the canonical projection onto the γ_0-th coordinate. For $x \in \gamma_0$ one has

$$\pi_0\Omega(x) = \Omega j_0 p_{\gamma_0} x = \Omega j_0(x).$$

So, $\pi_0\Omega = \omega j_0$ on γ_0. If this map is trivial so must be $\omega = \omega j_0 {j_0}^{-1}$, which is not. Hence Ω cannot be trivial on γ_0.

This shows that the quotient map q cannot be an isomorphism on any copy of $C[0,1]$ (see [7]). Now observe that every z-linear map Ω on a separable space has a version Ω_s having separable range. Since every separable subspace of $\ell_\infty(\Gamma, C[0,1])$ is contained in a copy of $C[0,1]$ inside $\ell_\infty(\Gamma, C[0,1])$, the image of this version actually lies in $C[0,1]$. Moreover, all versions of a map enjoy simultaneously the property of being not trivial on any copy of $C[0,1]$. Thus, this produces an exact sequence

$$0 \longrightarrow C[0,1] \longrightarrow C[0,1] \oplus_{\Omega_s} C[0,1] \stackrel{Q}{\longrightarrow} C[0,1] \longrightarrow 0$$

whose quotient map Q is not an isomorphism on any copy of $C[0,1]$ and consequently the space $C[0,1] \oplus_{\Omega_s} C[0,1]$ cannot have Rosenthal's property (V^*). Moreover, Johnson and Zippin showed in [9] that every separable Lindenstrauss space is a quotient of $C[0,1]$, from which it follows that Lindenstrauss spaces also have Rosenthal's property (V^*). Thus, the twisted sum space $C[0,1] \oplus_{\Omega_s} C[0,1]$ cannot be even a quotient of a Lindenstrauss space.

3. Pełczyński's property (V) is not a 3-space property

We produce now the exact sequence (1) in which the quotient map $q_{\mathfrak{S}}$ is not an isomorphism on any copy of c_0.

We work as before taking as starting point Sobczyk's theorem –every copy of c_0 is 2-complemented in any separable superspace– and the well-known distortion result for c_0 [12]: every Banach space isomorphic to c_0 contains for every $\varepsilon > 0$ a $(1+\varepsilon)$-isometric copy of c_0. Let thus Γ be the family of all subspaces γ of $C[0,1]$ which are 2-isomorphic to c_0 via some isomorphism j_γ and 2-complemented by some projection p_γ.

The existence of a nontrivial exact sequence

$$0 \longrightarrow C[0,1] \longrightarrow \Omega(c_0) \stackrel{q_0}{\longrightarrow} c_0 \longrightarrow 0$$

is well-known [2, 6, 15]. Let ω be a quasi-linear map inducing this sequence. We define the map $\Omega' : C[0,1] \curvearrowright C[0,1]^\Gamma$ by

$$\Omega'(x) = (\omega j_\gamma p_\gamma(x))_{\gamma\in\Gamma}.$$

That this is enough to define a z-linear map $\Omega : C[0,1] \curvearrowright \ell_\infty(\Gamma, C[0,1])$ is as before. Let us show that Ω cannot be trivial on any copy of c_0; equivalently, the restriction of Ω to any copy $\gamma_0 \in \Gamma$ is not trivial. Let $\pi_0 : \ell_\infty(\Gamma, C[0,1]) \to C[0,1]$ be the canonical projection. For $x \in \gamma_0$ one has

$$\pi_0 \Omega(x) = \Omega j_0 p_{\gamma_0} x = \Omega j_0(x).$$

So, $\pi_0 \Omega = \omega j_0$ on γ_0. If this map is trivial so must be $\omega = \omega j_0 {j_0}^{-1}$, which is not. Hence Ω cannot be trivial on γ_0. The reduction of the range to become $C[0,1]$ is as before.

4. Final remarks and conclusions

The previous device can be applied in other more general situations. In particular, we have proved that given Banach spaces X, Y such that:

1. There is some C such that that every copy of Y inside X contains a further C-isomorphic copy of Y that is C-complemented in X.
2. For some compact space S there is a nontrivial exact sequence $0 \to C(S) \to \clubsuit \to Y \to 0$.

Then there is a $C(K)$ space with the same density character as X and an exact sequence

$$0 \longrightarrow C(K) \longrightarrow \diamondsuit \stackrel{q}{\longrightarrow} X \longrightarrow 0$$

such that q is not an isomorphism on any copy of Y. In particular, taking as X a Weakly Compactly Generated space containing c_0 and $Y = c_0$ produces an exact sequence $0 \to C(K) \to \heartsuit \to X \to 0$ in which $\heartsuit$ cannot have Pełczyński's property (V). Further counterexamples can be deduced from [3], where it was shown the existence of an exact sequence

$$0 \longrightarrow C(\omega^\omega) \longrightarrow \Omega_\omega \stackrel{q_0}{\longrightarrow} c_0 \longrightarrow 0$$

with q_0 strictly singular; thus Ω_ω does not have either property (V).

It is an open problem whether every twisted sum of $c_0(\Gamma)$ must be isomorphic to a $C(K)$-space. Related to this is the question of whether every twisted sum of $c_0(\Gamma)$ must have Pełczyński's property (V).

Proposition. *Every twisted sum of $c_0(\Gamma)$ and a space with property (V) has property (V).*

Proof. Let

$$0 \longrightarrow c_0(\Gamma) \longrightarrow X \stackrel{q}{\longrightarrow} Z \longrightarrow 0$$

be an exact sequence in which Z has property (V) and let $\phi : X \to Y$ be an operator. If the restriction $\phi_0 = \phi_{|c_0(\Gamma)}$ is an isomorphism on some copy of c_0 then the same does ϕ. Otherwise it is weakly compact and admits a weakly compact extension $\phi_0^{**} : \ell_\infty(\Gamma) \to Y$. Let $p : X^{**} \to \ell_\infty(\Gamma)$ be a linear continuous projection. This gives a weakly compact extension $\phi_0^{**} p_{|X} : X \to Y$. Moreover, there is an operator $\psi : Z \to Y$ such that $\phi - \phi_0^{**} p_{|X} = \psi q$. If ψ is weakly compact then so is $\phi = \phi_0^{**} p_{|X} + \psi q$. Otherwise, there is a subspace c_0' of Z isomorphic to c_0 on which ψ is an isomorphism. By Sobczyk's theorem, there is a linear continuous projection $P : q^{-1}(c_0') \to c_0(\Gamma)$ and q is thus an isomorphism on $\ker P$, which is isomorphic to c_0'. Finally, since $p_{|\ker P} = 0$ one gets

$$\phi_{|\ker P} = \phi_0^{**} p_{|\ker P} + \psi q_{|\ker P} = \psi_{|c_0'}$$

is an isomorphism. ∎

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