Superstability of Approximate Cosine Type Functions on the Monoid $\mathbb{R}^2$

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Abstract: In this paper, we study the superstability problem for the cosine type functional equation

$$f(x_1x_2, x_1y_2 + x_2y_1) + f(x_1x_2, y_1x_2 - x_1y_2) = 2f(x_1, y_1)f(x_2, y_2)$$

on the commutative monoid $(\mathbb{R}^2, \cdot)$. As a result we obtain cosine type functions satisfying the equation approximately.

Key words: Functional equation, cosine function, superstability, multiplicative function.

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1. Introduction

In 1940, S. M. Ulam [17] gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. Among these was the following question concerning the stability of homomorphisms.

**Question 1.1.** Let $(G_1, \ast)$ be a group and let $(G_2, \circ, d)$ be a metric group with the metric $d$. Given $\epsilon > 0$, does there exist $\delta(\epsilon) > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(x \ast y), h(x) \circ h(y)) < \delta$ for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x_1 \in G_1$?

In 1941, Hyers [11] answered this question for the case where $G_1$ and $G_2$ are Banach spaces. In [2] and [15] Aoki and Th. M. Rassias respectively provided a generalization of Hyer’s theorem which allows the Cauchy difference to be unbounded. The interested reader may refer to the book by Hyers, Isac,
Rassias [12] for an in depth account on the subject of stability of functional equations. In 1982, J.M. Rassias [14] solved the Ulam problem by involving a product of powers of norms. Since then, the stability problems of various functional equations has been investigated by many authors (see [9], [10]). In [4] and [7] Baker et al. and Bourgin respectively, introduced the notion that by now is frequently referred to as superstability or Baker’s stability: if a function $f$ satisfies the stability inequality $|E_1(f) - E_2(f)| \leq \varepsilon$, then either $f$ is bounded or $E_1(f) = E_2(f)$. The superstability of d’Alembert’s functional equation $f(x+y) + f(x-y) = 2f(x)f(y)$ was investigated by Baker [5] and Cholewa [8]. Badora and Ger [3] proved its superstability under the condition $|f(x+y) + f(x-y) - 2f(x)f(y)| \leq \varphi(x)$ or $\varphi(y)$. In a previous work, Bouikhalene et al [6] investigated the superstability of the cosine functional equation on the Heisenberg group.

Now, let $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ be the commutative monoid equipped with composition rule
\[(x_1, y_1)(x_2, y_2) := (x_1x_2, x_1y_2 + x_2y_1). \quad (1.1)\]
The map $i : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, given by $i(x, y) = (x, -y)$ for any $(x, y) \in \mathbb{R}^2$, is an involution of $\mathbb{R}^2$, i.e., $i((x_1, y_1)(x_2, y_2)) = i(x_1, y_1)i(x_2, y_2)$ for any $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$ and $i \circ i = id$ (the identity map). Consider the functional equation

\[f(x_1x_2, x_1y_2 + x_2y_1) + f(x_1x_2, y_1x_2 - x_1y_2) = 2f(x_1, y_1)f(x_2, y_2) \quad (1.2)\]
for $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$. By setting $a = (x_1, y_1), b = (x_2, y_2)$ in (1.2) we obtain the cosine type functional equation
\[f(ab) + f(ai(b)) = 2f(a)f(b), \quad a, b \in \mathbb{R}^2 \quad (1.3)\]
on the commutative monoid $\mathbb{R}^2$. This equation has the same form as the cosine functional equation, also called d’Alembert’s functional equation ([1], [13])
\[f(x + y) + f(x - y) = 2f(x)f(y), \quad x, y \in G, \quad (1.4)\]
on an abelian group $G$, except that the group inversion $y \rightarrow -y$ is replaced by the involution $i$. We say that a function $f : \mathbb{R}^2 \rightarrow \mathbb{C}$ is of approximate a cosine type function, if there is $\delta > 0$ such that
\[|f(ab) + f(ai(b)) - 2f(a)f(b)| < \delta, \quad a, b \in \mathbb{R}^2. \quad (1.5)\]
In the case where $\delta = 0$, $f$ satisfies the functional equation $(1.3)$. We call $f$ a cosine type function on $\mathbb{R}^2$. The main purpose of this work is to prove the superstability problem of equation $(1.2)$ in the commutative monoid $\mathbb{R}^2$. 

2. Superstability of equation (1.2)

Proposition 2.1. Let \( \varphi, \psi, \phi, \zeta : \mathbb{R} \rightarrow [0, +\infty] \) be functions and let \( f : \mathbb{R}^2 \rightarrow \mathbb{C} \) satisfies the functional inequality

\[
|f(ab) + f(ai(b)) - 2f(a)f(b)| \leq \min \{\varphi(x_1), \psi(y_1), \phi(x_2), \zeta(y_2)\} \tag{2.1}
\]

for any \( a = (x_1, y_1), b = (x_2, y_2) \in \mathbb{R}^2 \). Then \( m(x) = f(x, 0) \) for any \( x \in \mathbb{R} \), is either bounded or multiplicative function from \( \mathbb{R} \) to \( \mathbb{C} \). Furthermore \( f \) satisfies the following inequality

\[
\left| f(a)^2 - \frac{1}{2} f(a^2) - \frac{1}{2} m(x^2) \right| \leq \frac{1}{2} \min \{\varphi(x), \psi(y), \phi(x), \zeta(y)\} \tag{2.2}
\]

for any \( a = (x, y) \in \mathbb{R}^2 \).

Proof. Setting \( a = (x, 0), b = (y, 0) \) in (2.1), we get

\[
|f(x, 0)f(y, 0) - f(xy, 0)| \leq \frac{1}{2} \min \{\varphi(x), \psi(0), \phi(y), \zeta(0)\}
\]

for any \( x, y \in \mathbb{R} \). According to [16] we get that \( m(x) = f(x, 0) \) for any \( x \in \mathbb{R} \) is either bounded or a multiplicative function from \( \mathbb{R} \) to \( \mathbb{C} \). Once again, putting \( a = (x, y) \) in (2.1) we get that

\[
\left| f(x^2, 2xy) + f(x^2, 0) - 2f(x, y)^2 \right| \leq \min \{\varphi(x), \psi(y), \phi(x), \zeta(y)\}
\]

for any \( x, y \in \mathbb{R} \). So that

\[
\left| f(a)^2 - \frac{1}{2} f(a^2) - \frac{1}{2} m(x^2) \right| \leq \frac{1}{2} \min \{\varphi(x), \psi(y), \phi(x), \zeta(y)\}
\]

for any \( a = (x, y) \in \mathbb{R}^2 \).

Proposition 2.2. Let \( f : \mathbb{R}^2 \rightarrow \mathbb{C} \) satisfies the functional inequality (2.1) and let \( F(y) = f(1, y) \) for any \( y \in \mathbb{R} \). Then

i) \( F \) is either bounded, or

ii) \( F \) satisfies the cosine functional equation

\[
F(x + y) + F(x - y) = 2F(x)F(y), \quad x, y \in \mathbb{R}. \tag{2.3}
\]
Further, in the latter case, there exists an exponential function $\gamma : \mathbb{R} \to \mathbb{C}$ such that

$$F(x) = \frac{1}{2} (\gamma(x) + \gamma(-x))$$

for any $x \in \mathbb{R}$. □

Proof. Let $a = (1, x)$, $b = (1, y)$ for any $x, y \in \mathbb{R}$ in (2.1). By setting $F(y) = f(1, y)$ for any $y \in \mathbb{R}$ we get

$$|F(x + y) + F(x - y) - 2F(x)F(y)| \leq \min \{ \phi(x), \psi(y), \phi(1), \phi(y) \}$$

for any $x, y \in \mathbb{R}$. According to ([3], [5]) it follows that $F$ is either bounded or $F$ is a cosine function. In view of ([1], [5], [13]) we get that there exists an exponential function $\gamma : \mathbb{R} \to \mathbb{C}$ such that $F(x) = \frac{1}{2} (\gamma(x) + \gamma(-x))$ for any $x \in \mathbb{R}$. □

**Proposition 2.3.** Let $f : \mathbb{R}^2 \to \mathbb{C}$ satisfies the functional inequality (2.1). Then $f$ is either bounded or $f \circ i = f$.

Proof. Let $P_f = \frac{f + f \circ i}{2}$. Since $f$ satisfies (2.1), we have

$$|P_f(ab) + P_f(ai(b)) - 2P_f(a)f(b)| \leq \min \{ \phi(x_1), \hat{P}_\phi(y_1), \phi(x_2), \hat{P}_\phi(y_2) \}$$

for any $a = (x_1, y_1), b = (x_2, y_2) \in \mathbb{R}^2$, where $\hat{P}_\phi(x) = \frac{\psi(x)\phi(x) - \phi(-x)\psi(x)}{2}$ for any $x \in \mathbb{R}$. By using the same way as in [3] and [5] we get that $f$ is either bounded or $f$ satisfies the Wilson’s type functional equation

$$P_f(ab) + P_f(ai(b)) = 2P_f(a)f(b), \quad a, b \in \mathbb{R}^2$$

on the commutative monoid $\mathbb{R}^2$. By small computations we get that $f \circ i = f$. □

**Proposition 2.4.** Let $\varphi, \psi, \phi, \xi : \mathbb{R} \to [0, +\infty]$ be functions and let $f : \mathbb{R}^2 \to \mathbb{C}$, with $f(0, 0) \neq 0$, satisfies the functional inequality (2.1). Then $f$ is bounded and we have

$$|f(a) - 1| \leq \frac{1}{2|f(0, 0)|} \min \{ \varphi(x), \psi(y), \phi(0), \xi(0) \}$$

for any $a = (x, y) \in \mathbb{R}^2$. □
Proof. By letting \( b = (0,0) \) in (2.1) we get
\[
|2f(0,0) - 2f(a)f(0,0)| \leq \min \{ \varphi(x), \psi(y), \phi(0), \zeta(0) \}
\]
for any \( a = (x,y) \in \mathbb{R} \). So that we have
\[
|2f(0,0)||f(a) - 1| \leq \min \{ \varphi(x), \psi(y), \phi(0), \zeta(0) \}
\]
for any \( a = (x,y) \in \mathbb{R}^2 \).

**Theorem 2.5.** Let \( \varphi, \psi, \phi, \zeta : \mathbb{R} \rightarrow [0, +\infty[ \) be functions and let \( f : \mathbb{R}^2 \rightarrow \mathbb{C} \) satisfies the functional inequality (2.1). Then

i) \( f \) is either bounded and
\[
|f(0,y)^2 - f(0,0)| \leq \frac{1}{2} \min \{ \varphi(0), \psi(y), \phi(0), \zeta(y) \} \tag{2.5}
\]
for any \( y \in \mathbb{R} \).

ii) \( f \) satisfies the functional inequality
\[
|f(a) - m(x)\frac{\gamma(y) + \gamma(-y)}{2}| \leq \frac{1}{2} \min \{ \varphi(x), \psi(0), \phi(1), \zeta(y) \} \tag{2.6}
\]
for any \( a = (x,y) \in \mathbb{R} \) with \( x \neq 0 \), where \( m : \mathbb{R} \rightarrow \mathbb{C} \) is a multiplicative function and \( \gamma : \mathbb{R} \rightarrow \mathbb{C} \) is an exponential function.

**Proof.** i) Letting \( a = b = (0,y) \) in (2.1), we get
\[
|f(0,y)^2 - f(0,0)| \leq \frac{1}{2} \min \{ \varphi(0), \psi(y), \phi(0), \zeta(y) \}
\]
for any \( y \in \mathbb{R} \).

ii) Let \( f \) be unbounded. Hence by Propositions 2.1 and 2.2 we get that \( f(x,0) = m(x) \) for any \( x \in \mathbb{R} \) is a multiplicative function from \( \mathbb{R} \) to \( \mathbb{C} \) and \( f(1,y) = F(y) \) for any \( y \in \mathbb{R} \) is a solution of the cosine functional equation (1.4). Therefore there exists an exponential function \( \gamma : \mathbb{R} \rightarrow \mathbb{C} \) such that
\[
f(1,y) = F(y) = \frac{\gamma(y) + \gamma(-y)}{2}
\]
for any \( y \in \mathbb{R} \). By letting \( a = (x,0), b = (1, \frac{y}{x}) \), with \( x \neq 0 \), in (2.1) we get the following inequality
\[
|f(x,y) + f(x,-y) - 2f(x,0)f(1, \frac{y}{x})| \leq \min \{ \varphi(x), \psi(0), \phi(1), \zeta(\frac{y}{x}) \} \tag{2.7}
\]
for any $x, y \in \mathbb{R}$ with $x \neq 0$. Therefore by Proposition 2.3 we get that $f(x, y) = f \circ i(x, y) = f(x, -y)$ for any $x, y \in \mathbb{R}$. So that we get from (2.7) that

$$|f(x, y) - m(x)F(\frac{y}{x})| \leq \frac{1}{2} \min \{\varphi(x), \psi(0), \phi(1), \zeta(\frac{y}{x})\}$$

for any $x, y \in \mathbb{R}$ with $x \neq 0$. 

In the next corollary we let $\varphi(x_1) = \psi(y_1) = \varphi(x_2) = \zeta(y_2) = \delta$ for any $x_1, y_1, x_2, y_2 \in \mathbb{R}$.

**Corollary 2.6.** Let $\delta > 0$ and let $f : \mathbb{R}^2 \rightarrow \mathbb{C}$ satisfies the functional inequality

$$|f(ab) + f(ai(b)) - 2f(a)f(b)| \leq \delta$$

for any $a, b \in \mathbb{R}^2$. Then

i) $f$ is bounded and there exists $\eta \in \mathbb{C}^*$ such that $|f(a) - 1| \leq \frac{\delta}{2\eta}$ for any $a = (x, y) \in \mathbb{R}$, with $x \neq 0$. Furthermore $|f(0, y) - \eta| \leq \frac{\delta}{2}$ for any $y \in \mathbb{R}$

ii) $f$ is unbounded and there exist a multiplicative function $m : \mathbb{R} \rightarrow \mathbb{C}$ and an exponential function $\gamma : \mathbb{R} \rightarrow \mathbb{C}$ such that

$$|f(a) - m(x)\gamma(\frac{y}{x}) + \gamma(-\frac{y}{x})| \leq \frac{\delta}{2}$$

for any $a = (x, y) \in \mathbb{R}^2$ with $x \neq 0$.

**Proof.** By using Proposition 2.4 and Theorem 2.5 with $\eta = f(0, 0)$. 

In the next corollary we give the explicit formula of cosine type functions on $\mathbb{R}^2$

**Corollary 2.7.** Let $f : \mathbb{R}^2 \rightarrow \mathbb{C}$ be a cosine type function on $\mathbb{R}^2$. Then

i) $f(x, y) = 1$ for any $x, y \in \mathbb{R}$ or

ii) $f(x, y) = \begin{cases} 0 & \text{if } x = 0, \\ \frac{m(x)}{2}(\gamma(\frac{y}{x}) + \gamma(-\frac{y}{x})) & \text{if } x \neq 0, \end{cases}$

for any $x, y \in \mathbb{R}$.

**Proof.** By letting $\delta = 0$ in Corollary 2.6.
REFERENCES


