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Integral Operators on Some Classes of Meromorphic Close-to-Convex Multivalent Functions

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Abstract: We introduce some new subclasses of the class of meromorphic multivalent functions, which are defined by subordination and superordination using the close-to-convexity condition. In some particular cases, these new subclasses are the well-known classes of meromorphic close-to-convex functions. We establish the conditions such that when we apply a certain integral operator (similar to Bernardi integral operator) to a function which belongs to one of these subclasses, the image we get belongs to a similar class.

Key words: Meromorphic close-to-convex functions, integral operators, subordination, superordination.

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1. Introduction and preliminaries

For $a \in \mathbb{C}$ and $r > 0$ we consider $U(a,r) = \{z \in \mathbb{C} : |z-a| < r\}$. Let $U = U(0,1) = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disc in the complex plane, $\dot{U} = U \setminus \{0\}$, $H(U) = \{f : U \to \mathbb{C} : f \text{ is holomorphic in } U\}$, $H_u(U) = \{f \in H(U) : f \text{ is univalent in } U\}$, $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$.

For $p \in \mathbb{N}^*$, let Σ_p denote the class of meromorphic functions of the form

$$g(z) = \frac{a_{-p}}{z^p} + a_0 + a_1 z + \cdots + a_n z^n + \cdots,\ z \in \dot{U},\ a_{-p} \neq 0.$$

We will also use the following notations:

$\Sigma_{p,0} = \{g \in \Sigma_p : a_{-p} = 1\}$,

$\Sigma_0 = \{g \in \Sigma_{1,0} : g \text{ is univalent in } \dot{U} \text{ and } g(z) \neq 0,\ z \in \dot{U}\}$,

$\Sigma K_p(\alpha,\delta) = \left\{ g \in \Sigma_p : \alpha < \operatorname{Re}\left[-1 - \frac{zg''(z)}{g'(z)} \right] < \delta,\ z \in U \right\}$, where $\alpha < p < \delta$.

$\Sigma K_{p,0}(\alpha,\delta) = \Sigma K_p(\alpha,\delta) \cap \Sigma_{p,0}$,

$\Sigma\mathcal{C}_{p,0}(\alpha,\delta;\varphi) = \left\{g \in \Sigma_{p,0} : \alpha < \operatorname{Re}\dfrac{g'(z)}{\varphi'(z)} < \delta,\ z \in U\right\}$, where $\alpha < 1 \le p < \delta$ and $\varphi \in \Sigma K_{p,0}(\alpha,\delta)$.

$\Sigma\mathcal{C}_{p,0}(\alpha,\delta) = \left\{g \in \Sigma_{p,0} : (\exists)\varphi \in \Sigma K_{p,0}(\alpha,\delta) \text{ s.t. } \alpha < \operatorname{Re}\dfrac{g'(z)}{\varphi'(z)} < \delta,\ z \in U\right\}$, where $\alpha < 1 \le p < \delta$.

$H[a,n] = \{f \in H(U) : f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \cdots\}$ for $a \in \mathbb{C},\ n \in \mathbb{N}^*$.

$A_n = \{f \in H(U) : f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \cdots\}$, $n \in \mathbb{N}^*$. For $n = 1$ we denote A_1 by A, and this set is called *the class of analytic functions normalized at the origin*.

Definition 1.1. ([4, p. 4]) Let f and F be members of $H(U)$. The function f is said to be subordinate to F, written $f \prec F$ or $f(z) \prec F(z)$, if there exists a function w analytic in U, with $w(0) = 0$ and $|w(z)| < 1$, and such that $f(z) = F(w(z))$.

Definition 1.2. ([4, p. 16]) Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ and let h be univalent in U. If p is analytic in U and satisfies the (second order) differential subordination

$$\psi\big(p(z), zp'(z), z^2p''(z); z\big) \prec h(z), \tag{1}$$

then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination, or more simply, a dominant, if $p \prec q$ for all p satisfying (1). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (1) is said to be the best dominant of (1). (Note that the best dominant is unique up to a rotation of U).

If we require the more restrictive condition $p \in H[a,n]$, then p will be called an (a,n)-solution, q an (a,n)-dominant, and $\tilde{q}$ the best (a,n)-dominant.

Definition 1.3. ([5], [2, p. 98]) Let $\varphi : \mathbb{C}^3 \times U \to \mathbb{C}$ and let h be analytic in U. If p and $\varphi\big(p(z), zp'(z), z^2p''(z); z\big)$ are univalent in U and satisfy the second order differential superordination

$$h(z) \prec \varphi\big(p(z), zp'(z), z^2p''(z); z\big), \tag{2}$$

then p is called a solution of the differential superordination. An analytic function q is called a subordinant of the solutions of the differential superordination, or more simply, a subordinant, if $q \prec p$ for all p satisfying (2). An univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinants q of (2) is said to be the best subordinant. Note that the best subordinant is unique up to a rotation of U.

Definition 1.4. ([2, p. 99]) We denote by Q the set of functions f that are analytic and injective on $\overline{U} \setminus E(f)$, where

$$E(f) = \Big\{\zeta \in \partial U : \lim_{z\to\zeta} f(z) = \infty\Big\},$$

and they are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(f)$. The subclass of Q for which $f(0) = a$, is denoted by $Q(a)$.

Theorem 1.1. ([3]) *Let $\beta, \gamma \in \mathbb{C}$ and let h be a convex function in U, with*

$$\operatorname{Re}[\beta h(z) + \gamma] > 0,\ z \in U.$$

Let q_m and q_k be the univalent solutions of the Briot-Bouquet differential equation

$$q(z) + \frac{nzq'(z)}{\beta q(z) + \gamma} = h(z),\ z \in U, \quad q(0) = h(0),$$

for $n = m$ and $n = k$ respectively. If m/k, then $q_k(z) \prec q_m(z) \prec h(z)$. So, $q_k(z) \prec q_1(z) \prec h(z)$.

Theorem 1.2. ([6]) *Let $p \in \mathbb{N}^*$, $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > p$. If $g \in \Sigma_p$, then $J_{p,\lambda}(g) \in \Sigma_p$, where $J_{p,\lambda}(g)(z) = \frac{\lambda-p}{z^\lambda}\int_0^z g(t)t^{\lambda-1}dt$.*

Theorem 1.3. ([2, p. 102], [5]) *Let $\Omega \subset \mathbb{C}$, $q \in H[a,n], \varphi : \mathbb{C}^2 \times \overline{U} \to \mathbb{C}$, and suppose that*

$$\varphi\big(q(z), tzq'(z); \zeta\big) \in \Omega,$$

for $z \in U$, $\zeta \in \partial U$ and $0 < t \leq \frac{1}{n} \leq 1$. If $p \in Q(a)$ and $\varphi\big(p(z), zp'(z); z\big)$ is univalent in U, then

$$\Omega \subset \big\{\varphi\big(p(z), zp'(z); z\big) : z \in U\big\} \Rightarrow q(z) \prec p(z).$$

Theorem 1.4. ([4, p. 70]) *Let h be convex in U and let $P : U \to \mathbb{C}$ with $\operatorname{Re} P(z) > 0$. If p is analytic in U, then*

$$p(z) + P(z)zp'(z) \prec h(z) \Rightarrow p(z) \prec h(z).$$

Definition 1.5. ([7]) Let $p \in \mathbb{N}^*$ and $h \in H(U)$ with $h(0) = p$. We define:

$$\Sigma K_p(h) = \left\{ g \in \Sigma_p : -\left[1 + \frac{zg''(z)}{g'(z)}\right] \prec h(z) \right\},$$
$$\Sigma K_{p,0}(h) = \Sigma K_p(h) \cap \Sigma_{p,0}.$$

Corollary 1.1. ([7]) *Let $p \in \mathbb{N}^*, \gamma \in \mathbb{C}$ with $\operatorname{Re}\gamma > p$ and $g \in \Sigma K_p(h)$ with h convex in U. If*

$$\operatorname{Re}\left[\gamma - h(z)\right] > 0,\ z \in U,$$

then

$$J_{p,\gamma}(g) \in \Sigma K_p(q),$$

where q is the univalent solution of the Briot-Bouquet differential equation

$$q(z) + \frac{(p+1)zq'(z)}{\gamma - q(z)} = h(z)\,,\ z \in U, \quad q(0) = p.$$

The function q is the best $(p, p+1)$-dominant.

2. Main results

Next we consider some subclasses of $\Sigma_{p,0}$ associated with superordination and subordination, using the close-to-convexity condition and throughout this paper we establish the conditions such that when we apply the integral operator $J_{p,\gamma}$ to a function which belongs to one of these new subclasses, we get an image that belongs to a similar class.

Definition 2.1. Let $p \in \mathbb{N}^*$, $h_1, h_2, h \in H(U)$ with $h_1(0) = h_2(0) = 1$, $h(0) = p$, $h_1 \prec h_2$ and $\varphi \in \Sigma K_{p,0}(h)$. We define:

$$\Sigma \mathcal{C}_{p,0}(h_1, h_2; \varphi, h) = \left\{ g \in \Sigma_{p,0} :\ h_1(z) \prec \frac{g'(z)}{\varphi'(z)} \prec h_2(z) \right\},$$
$$\Sigma \mathcal{C}_{p,0}(h_2; \varphi, h) = \left\{ g \in \Sigma_{p,0} :\ \frac{g'(z)}{\varphi'(z)} \prec h_2(z) \right\}.$$

Definition 2.2. Let $p \in \mathbb{N}^*$ and $h_2, h \in H(U)$ with $h_2(0) = 1$, $h(0) = p$. We define:

$$\Sigma\mathcal{C}_{p,0}(h_2; h) = \left\{ g \in \Sigma_{p,0} : (\exists)\varphi \in \Sigma K_{p,0}(h) \text{ s.t. } \frac{g'(z)}{\varphi'(z)} \prec h_2(z) \right\},$$

$$\Sigma\mathcal{C}_{p,0}(h) = \left\{ g \in \Sigma_{p,0} : (\exists)\varphi \in \Sigma K_{p,0}(h) \text{ s.t. } \frac{g'(z)}{\varphi'(z)} \prec \frac{1}{p} h(z) \right\}.$$

Remark 2.1.

1. If $H \in H(U)$, $H(0) = p$ and $h \prec H$, then $\Sigma C_{p,0}(h_2; h) \subset \Sigma C_{p,0}(h_2; H)$.
2. If $H_2 \in H(U)$, $H_2(0) = 1$ and $h_2 \prec H_2$, then $\Sigma C_{p,0}(h_2; h) \subset \Sigma C_{p,0}(H_2; h)$.
3. If $h_1, h_2, h, H \in H(U)$ with $h_1(0) = h_2(0) = 1$, $h(0) = H(0) = p$, $h_1 \prec h_2$ and $\varphi \in \Sigma K_{p,0}(h) \cap \Sigma K_{p,0}(H)$, then

$$\Sigma\mathcal{C}_{p,0}(h_1, h_2; \varphi, h) = \Sigma\mathcal{C}_{p,0}(h_1, h_2; \varphi, H),$$
$$\Sigma\mathcal{C}_{p,0}(h_2; \varphi, h) = \Sigma\mathcal{C}_{p,0}(h_2; \varphi, H).$$

Next we present some particular cases for the classes defined above.

If $p = 1$ and $h_2(z) = h(z) = \frac{1+z}{1-z}$, $z \in U$, then a function φ is in the class $\Sigma K_{1,0}(h)$ if and only if

$$\operatorname{Re}\left[-1 - \frac{z\varphi''(z)}{\varphi'(z)}\right] > 0,\ z \in U,$$

so, the class of meromorpic close-to-convex functions is included in the class $\Sigma\mathcal{C}_{1,0}\left(\frac{1+z}{1-z}\right)$.

Let $\alpha < 1 \le p < \delta$. We consider $h_2 = h_{1,\alpha,\delta}$ and $h = h_{p,\alpha,\delta}$, where $h_{p,\alpha,\delta} : U \to \mathbb{C}$ is the convex function with $h_{p,\alpha,\delta}(U) = \{z \in \mathbb{C} : \alpha < \operatorname{Re} z < \delta\}$ and $h_{p,\alpha,\delta}(0) = p$. We know that $h_{p,\alpha,\delta}$ exists and it is obtained by composing different well-known elementary functions. It is not difficult to see that

$$\Sigma K_{p,0}(h_{p,\alpha,\delta}) = \Sigma K_{p,0}(\alpha, \delta), \tag{3}$$

$$\Sigma\mathcal{C}_{p,0}(h_{1,\alpha,\delta}; \varphi, h_{p,\alpha,\delta}) = \Sigma\mathcal{C}_{p,0}(\alpha, \delta; \varphi), \text{ where } \varphi \in \Sigma K_{p,0}(\alpha, \delta). \tag{4}$$

We denote the class $\Sigma\mathcal{C}_{p,0}(h_{1,\alpha,\delta}; h_{p,\alpha,\delta})$ by $\Sigma\mathcal{C}_{p,0}(\alpha, \delta)$.

We mention that the class $\Sigma\mathcal{C}_{p,0}(\alpha, \delta; \varphi)$ was introduced and studied in [6]. Also, a class similar with the class $\Sigma\mathcal{C}_{1,0}(\alpha, \delta)$ was defined and studied in [1].

Theorem 2.1. *Let $p \in \mathbb{N}^*$ and $\gamma \in \mathbb{C}$ with $\operatorname{Re}\gamma > p$. Let h_2 and h be convex functions in U with $h_2(0) = 1$, $h(0) = p$ and let $g \in \Sigma\mathcal{C}_{p,0}(h_2; h)$. If we have $\operatorname{Re}[\gamma - h(z)] > 0$, $z \in U$, then*

$$J_{p,\gamma}(g) \in \Sigma\mathcal{C}_{p,0}(h_2; q),$$

where q is the univalent solution of the Briot-Bouquet differential equation

$$q(z) + \frac{(p+1)zq'(z)}{\gamma - q(z)} = h(z)\,,\ z \in U,$$

with $q(0) = p$. The function q is the best $(p, p+1)$-dominant.

Proof. Since $g \in \Sigma\mathcal{C}_{p,0}(h_2; h)$ we know that there is a function $\varphi \in \Sigma K_{p,0}(h)$ such that

$$\frac{g'(z)}{\varphi'(z)} \prec h_2(z). \tag{5}$$

Because $\varphi \in \Sigma K_{p,0}(h)$, where $\Sigma K_{p,0}(h) = \Sigma K_p(h) \cap \Sigma_{p,0}$, and $\operatorname{Re}[\gamma - h(z)] > 0$, $z \in U$, we have from Corollary 1.1 that

$$\Phi = J_{p,\gamma}(\varphi) \in \Sigma K_p(q),$$

where q is the univalent solution of the Briot-Bouquet differential equation

$$q(z) + \frac{(p+1)zq'(z)}{\gamma - q(z)} = h(z)\,,\ z \in U,$$

with $q(0) = p$. Of course, the function q is the best $(p, p+1)$-dominant.

From the definition of the operator $J_{p,\gamma}$ we remark that $\Phi \in \Sigma_{p,0}$, when $\varphi \in \Sigma_{p,0}$, so $\Phi \in \Sigma K_{p,0}(q)$.

Let $G = J_{p,\gamma}(g)$. We know from Theorem 1.2 that $G \in \Sigma_p$ and it is easy to see that $G \in \Sigma_{p,0}$ (since $g \in \Sigma_{p,0}$). Using the definition of the operator $J_{p,\gamma}$ and the fact that $G = J_{p,\gamma}(g)$, $\Phi = J_{p,\gamma}(\varphi)$, we get

$$\gamma G(z) + zG'(z) = (\gamma - p)g(z)$$

and

$$\gamma\Phi(z) + z\Phi'(z) = (\gamma - p)\varphi(z),\ z \in \dot{U},$$

hence

$$(\gamma + 1)G'(z) + zG''(z) = (\gamma - p)g'(z)$$

and

$$(\gamma+1)\Phi'(z)+z\Phi''(z)=(\gamma-p)\varphi'(z).$$

Let us denote

$$P(z)=\frac{G'(z)}{\Phi\,'(z)},\ z\in U.$$

Because $\Phi\in\Sigma K_{p,0}(q)$ we have $z^{p+1}\Phi'(z)\neq 0$, $z\in U$, hence $P\in H(U)$. From $P(z)\Phi'(z)=G'(z)$, we get $G''(z)=P'(z)\Phi'(z)+P(z)\Phi''(z)$, so, the identity

$$(\gamma+1)G'(z)+zG''(z)=(\gamma-p)g'(z),\ z\in\dot{U},$$

can be rewritten as

$$(\gamma+1)P(z)\Phi'(z)+z\big[P'(z)\Phi'(z)+P(z)\Phi''(z)\big]=(\gamma-p)g'(z). \tag{6}$$

Using the identity $(\gamma+1)\Phi'(z)+z\Phi''(z)=(\gamma-p)\varphi'(z)$, we obtain from (6) that

$$P(z)+\frac{zP'(z)}{\gamma+1+\dfrac{z\Phi''(z)}{\Phi'(z)}}=\frac{g'(z)}{\varphi\,'(z)},\ z\in U,$$

which is equivalent to

$$P(z)+\frac{zP'(z)}{R(z)}=\frac{g'(z)}{\varphi\,'(z)},\ \text{ where } R(z)=\gamma+1+\frac{z\Phi''(z)}{\Phi'(z)}. \tag{7}$$

From (5) and (7) we obtain

$$P(z)+\frac{zP'(z)}{R(z)}\prec h_2(z). \tag{8}$$

Next we show that $\operatorname{Re}R(z)>0$, $z\in U$. We know that $\Phi\in\Sigma K_{p,0}(q)$ and $q\prec h$ (see Theorem 1.1), so

$$-1-\frac{z\Phi''(z)}{\Phi'(z)}\prec h(z),$$

which is equivalent to

$$\gamma+1+\frac{z\Phi''(z)}{\Phi'(z)}\prec\gamma-h(z), \tag{9}$$

hence

$$R(z)\prec\gamma-h(z). \tag{10}$$

Since $\operatorname{Re}[\gamma - h(z)] > 0$, $z \in U$, we get from (10) that $\operatorname{Re} R(z) > 0$, $z \in U$.

Because $\operatorname{Re} R(z) > 0$, $z \in U$, we can use Theorem 1.4 for the subordination

$$P(z) + \frac{zP'(z)}{R(z)} \prec h_2(z),$$

and we get $P \prec h_2$, which is equivalent to

$$\frac{G'(z)}{\Phi'(z)} \prec h_2(z). \tag{11}$$

Since $G \in \Sigma_{p,0}$ and $\Phi \in \Sigma K_{p,0}(q)$ we obtain from (11) that $G = J_{p,\gamma}(g) \in \Sigma\mathcal{C}_{p,0}(h_2; q)$. ∎

From the proof of Theorem 2.1 we remark that we also have:

THEOREM 2.2. *Let $p \in \mathbb{N}^*$ and $\gamma \in \mathbb{C}$ with $\operatorname{Re}\gamma > p$. Let h_2 and h be convex functions in U with $h_2(0) = 1$, $h(0) = p$ and $\operatorname{Re}[\gamma - h(z)] > 0$, $z \in U$. If $\varphi \in \Sigma K_{p,0}(h)$ and $g \in \Sigma\mathcal{C}_{p,0}(h_2; \varphi, h)$, then*

$$J_{p,\gamma}(g) \in \Sigma\mathcal{C}_{p,0}\big(h_2; J_{p,\gamma}(\varphi), q\big),$$

where q is the univalent solution of the Briot-Bouquet differential equation

$$q(z) + \frac{(p+1)zq'(z)}{\gamma - q(z)} = h(z)\,,\ z \in U,$$

with $q(0) = p$. The function q is the best $(p, p+1)$-dominant.

If we consider that the conditions from the hypothesis of Theorem 2.1 and Theorem 2.2 respectively, are met, since we know from Theorem 1.1 that $q \prec h$, we obtain the next corollaries:

COROLLARY 2.1. *Let $p \in \mathbb{N}^*$ and $\gamma \in \mathbb{C}$ with $\operatorname{Re}\gamma > p$. Let h_2, h be convex functions in U with $h_2(0) = 1$, $h(0) = p$ and let $g \in \Sigma\mathcal{C}_{p,0}(h_2; h)$. If $\operatorname{Re} h(z) < \operatorname{Re}\gamma$, $z \in U$, then*

$$J_{p,\gamma}(g) \in \Sigma\mathcal{C}_{p,0}(h_2; h).$$

COROLLARY 2.2. *Let $p \in \mathbb{N}^*$ and $\gamma \in \mathbb{C}$ with $\operatorname{Re}\gamma > p$. Let h_2 and h be convex functions in U with $h_2(0) = 1$, $h(0) = p$ and $\operatorname{Re} h(z) < \operatorname{Re}\gamma$, $z \in U$. If $\varphi \in \Sigma K_{p,0}(h)$ and $g \in \Sigma\mathcal{C}_{p,0}(h_2; \varphi, h)$, then*

$$J_{p,\gamma}(g) \in \Sigma\mathcal{C}_{p,0}\big(h_2; J_{p,\gamma}(\varphi), h\big).$$

Next we present two results which concern the particular classes $\Sigma\mathcal{C}_{p,0}(\alpha,\delta)$ and $\Sigma\mathcal{C}_{p,0}(\alpha,\delta;\varphi)$.

THEOREM 2.3. *Let $p\in\mathbb{N}^*$, $\alpha,\delta\in\mathbb{R}$ and $\gamma\in\mathbb{C}$ with $\alpha<1\le p<\delta\le \operatorname{Re}\gamma$. If $g\in\Sigma\mathcal{C}_{p,0}(\alpha,\delta)$, then*

$$J_{p,\gamma}(g)\in\Sigma\mathcal{C}_{p,0}(\alpha,\delta).$$

Proof. We know that the class $\Sigma\mathcal{C}_{p,0}(\alpha,\delta)$ is the class $\Sigma\mathcal{C}_{p,0}(h_{1,\alpha,\delta};h_{p,\alpha,\delta})$. Taking $h_2=h_{1,\alpha,\delta}$, $h=h_{p,\alpha,\delta}$ for Corollary 2.1 we remark that the hypothesis of this corollary is fulfilled, so we get

$$J_{p,\gamma}(g)\in\Sigma\mathcal{C}_{p,0}(h_{1,\alpha,\delta};h_{p,\alpha,\delta})=\Sigma\mathcal{C}_{p,0}(\alpha,\delta).$$

∎

THEOREM 2.4. *Let $p\in\mathbb{N}^*$, $\alpha,\delta\in\mathbb{R}$ and $\gamma\in\mathbb{C}$ with $\alpha<1\le p<\delta\le \operatorname{Re}\gamma$. If $\varphi\in\Sigma K_{p,0}(\alpha,\delta)$ and $g\in\Sigma\mathcal{C}_{p,0}(\alpha,\delta;\varphi)$, then*

$$J_{p,\gamma}(g)\in\Sigma\mathcal{C}_{p,0}(\alpha,\delta;\Phi),$$

where $\Phi=J_{p,\gamma}(\varphi)$.

Proof. From (3) we know that $\Sigma K_{p,0}(h_{p,\alpha,\delta})=\Sigma K_{p,0}(\alpha,\delta)$ and from (4) we have $\Sigma\mathcal{C}_{p,0}(h_{1,\alpha,\delta};\varphi,h_{p,\alpha,\delta})=\Sigma\mathcal{C}_{p,0}(\alpha,\delta;\varphi)$, where $\varphi\in\Sigma K_{p,0}(\alpha,\delta)$. Considering $h_2=h_{1,\alpha,\delta}$ and $h=h_{p,\alpha,\delta}$ for Corollary 2.2, we remark that the hypothesis of this corollary is fulfilled, so we get

$$J_{p,\gamma}(g)\in\Sigma\mathcal{C}_{p,0}\big(h_{1,\alpha,\delta};J_{p,\gamma}(\varphi),h_{p,\alpha,\delta}\big)=\Sigma\mathcal{C}_{p,0}\big(\alpha,\delta;J_{p,\gamma}(\varphi)\big).$$

∎

We remark that a result which is similar to Theorem 2.4 was also obtained in [6] but using a different method. We also remark that in the hypothesis of Theorem 2.4 we do not have the condition $z^{p+1}J'_{p,\gamma}(\varphi)(z)\neq 0$, $z\in U$, which appears in the hypothesis of the result presented in [6].

LEMMA 2.1. *Let $r>0$ and let $\lambda:\overline{U}\to\mathbb{C}$ be an analytic function in U such that $\sup_{z\in\overline{U}}|\lambda(z)|=M<\infty$. If $p\in H[1,1]\cap Q$ and $p(z)+\lambda(z)zp'(z)$ is univalent in U, then*

$$U(1,r)\subset\big\{p(z)+\lambda(z)zp'(z):z\in U\big\}\Rightarrow U\left(1,\frac{r}{1+M}\right)\subset p(U).$$

Proof. To prove this lemma we use Theorem 1.3. Let us consider $\Omega = U(1,r)$, $q(z) = \frac{r}{1+M}z + 1$, $z \in U$, and $\varphi : \mathbb{C}^2 \times \overline{U} \to \mathbb{C}$, $\varphi(u,s;\zeta) = u + \lambda(\zeta)s$. Since we know from the hypothesis that $p \in H[1,1] \cap Q$ and $p(z) + \lambda(z)zp'(z)$ is univalent in U, to apply Theorem 1.3, we need only to verify that

$$\varphi\big(q(z), tzq'(z);\zeta\big) \in \Omega = U(1,r), \text{ when } z \in U,\ \zeta \in \partial U,\ 0 < t \leq 1, \tag{12}$$

which is equivalent to

$$|q(z) + \lambda(\zeta)tzq'(z) - 1| < r, \text{ when } z \in U,\ \zeta \in \partial U,\ 0 < t \leq 1. \tag{13}$$

We have

$$\begin{aligned}\big|q(z) + \lambda(\zeta)tzq'(z) - 1\big| &= \frac{r}{1+M}\big|z[1 + t\lambda(\zeta)]\big| < \frac{r}{1+M}\big|1 + t\lambda(\zeta)\big| \\ &\leq \frac{r}{1+M}\big(1 + t|\lambda(\zeta)|\big) \leq \frac{r}{1+M}(1+M) = r.\end{aligned}$$

Therefore, the condition (13) is satisfied, so we get from Theorem 1.3 that $q \prec p$, which implies

$$U\left(1, \frac{r}{1+M}\right) \subset p(U).$$

∎

Theorem 2.5. *Let $m, r > 0$, $p \in \mathbb{N}^*$ and $\gamma \in \mathbb{C}$ with $\operatorname{Re}\gamma > p$. Let h_2 and h be convex functions in U such that $h_2(0) = 1$, $h(0) = p$ and $\operatorname{Re}[\gamma - h(z)] > m$, $z \in U$. Let $\varphi \in \Sigma K_{p,0}(h)$ and $g \in \Sigma\mathcal{C}_{p,0}(h_1, h_2; \varphi, h)$, where $h_1(z) = rz + 1$, $z \in U$. Suppose that $\frac{g'}{\varphi'}$ is univalent in U and $\frac{J'_{p,\gamma}(g)}{J'_{p,\gamma}(\varphi)} \in Q$. Then*

$$G = J_{p,\gamma}(g) \in \Sigma\mathcal{C}_{p,0}(q_1, h_2; \Phi, q),$$

where

$$\begin{aligned}\Phi &= J_{p,\gamma}(\varphi), \\ q_1(z) &= \frac{rm}{m+1}z + 1,\ z \in U,\end{aligned}$$

and q is the univalent solution of the Briot-Bouquet differential equation

$$q(z) + \frac{(p+1)zq'(z)}{\gamma - q(z)} = h(z),\ z \in U,$$

with $q(0) = p$. The function q is the best $(p, p+1)$-dominant.

Proof. Since $\varphi \in \Sigma K_{p,0}(h) = \Sigma K_p(h) \cap \Sigma_{p,0}$ and $\operatorname{Re}[\gamma - h(z)] > m > 0$, $z \in U$, we have from Corollary 1.1 that

$$\Phi = J_{p,\gamma}(\varphi) \in \Sigma K_p(q),$$

where q is the univalent solution of the Briot-Bouquet differential equation

$$q(z) + \frac{(p+1)zq'(z)}{\gamma - q(z)} = h(z), \; z \in U,$$

with $q(0) = p$. It is easy to see that $\Phi \in \Sigma K_{p,0}(q)$. Of course, the function q is the best $(p, p+1)$-dominant.

We have $G = J_{p,\gamma}(g)$ and $\Phi = J_{p,\gamma}(\varphi)$. Let

$$P(z) = \frac{G'(z)}{\Phi'(z)}, \; z \in U.$$

Since $\Phi \in \Sigma K_{p,0}(q)$ we have $z^{p+1}\Phi'(z) \neq 0$, $z \in U$, so $P \in H(U)$.

Analogously to the proof of Theorem 2.1 we obtain

$$P(z) + \frac{zP'(z)}{R(z)} = \frac{g'(z)}{\varphi'(z)},$$

where

$$R(z) = \gamma + 1 + \frac{z\Phi''(z)}{\Phi'(z)}, \; z \in U.$$

It is obvious that $R \in H(U)$. From $g \in \Sigma\mathcal{C}_{p,0}(h_1, h_2; \varphi, h)$ we have

$$h_1(z) \prec \frac{g'(z)}{\varphi'(z)} \prec h_2(z),$$

hence

$$h_1(z) \prec P(z) + \frac{zP'(z)}{R(z)} \prec h_2(z). \tag{14}$$

Because $\operatorname{Re} R(z) > 0$, $z \in U$, (see the proof of Theorem 2.1), we can use Theorem 1.4 for the subordination of (14), which is

$$P(z) + \frac{zP'(z)}{R(z)} \prec h_2(z),$$

and we get

$$P \prec h_2. \tag{15}$$

Next we consider the superordination of (14), which is

$$h_1(z) \prec P(z) + \frac{zP'(z)}{R(z)}.$$

Since $h_1(z) = rz + 1$, $z \in U$, this superordination implies

$$U(1, r) \subset \left\{ P(z) + \frac{zP'(z)}{R(z)} : z \in U \right\}.$$

Let us denote $\lambda = \frac{1}{R}$. We know from (10) that $R(z) \prec \gamma - h(z)$ and from the hypothesis we have $\operatorname{Re}[\gamma - h(z)] > m > 0$, $z \in U$, hence $\operatorname{Re} R(z) > m$, $z \in U$. We have the function $\lambda : \overline{U} \to \mathbb{C}$ analytic in U and $\sup_{z \in \overline{U}} |\lambda(z)| \leq \frac{1}{m}$. We may apply now Lemma 2.1 and we obtain

$$U\left(1, \frac{rm}{m+1}\right) \subset P(U). \tag{16}$$

Since P is univalent in U and $P(0) = q_1(0)$, we have (16) equivalent to

$$q_1 \prec P, \text{ where } q_1(z) = \frac{rm}{m+1} z + 1, \ z \in U. \tag{17}$$

From (15), (17) and the fact that

$$\Phi = J_{p,\gamma}(\varphi) \in \Sigma K_{p,0}(q),$$

where q is the univalent solution of the Briot-Bouquet differential equation

$$q(z) + \frac{(p+1)zq'(z)}{\gamma - q(z)} = h(z), \ z \in U,$$

with $q(0) = p$, we obtain that

$$G = J_{p,\gamma}(g) \in \Sigma \mathcal{C}_{p,0}(q_1, h_2; \Phi, q).$$

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