Integral Operators on Some Classes of Meromorphic Close-to-Convex Multivalent Functions

ALINA TOTOI

Department of Mathematics, Faculty of Science, University “Lucian Blaga” of Sibiu, Str. Dr. Ion Rațiu, nr. 5-7, 550012-Sibiu, Romania
totoialina@yahoo.com

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Abstract: We introduce some new subclasses of the class of meromorphic multivalent functions, which are defined by subordination and superordination using the close-to-convexity condition. In some particular cases, these new subclasses are the well-known classes of meromorphic close-to-convex functions. We establish the conditions such that when we apply a certain integral operator (similar to Bernardi integral operator) to a function which belongs to one of these subclasses, the image we get belongs to a similar class.

Key words: Meromorphic close-to-convex functions, integral operators, subordination, superordination.

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1. Introduction and preliminaries

For $a \in \mathbb{C}$ and $r > 0$ we consider $U(a, r) = \{z \in \mathbb{C} : |z - a| < r\}$. Let $U = U(0, 1) = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disc in the complex plane, $\dot{U} = U \setminus \{0\}$, $H(U) = \{f : U \to \mathbb{C} : f$ is holomorphic in $U\}$, $H_n(U) = \{f \in H(U) : f$ is univalent in $U\}$, $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$.

For $p \in \mathbb{N}^*$, let $\Sigma_p$ denote the class of meromorphic functions of the form

$$g(z) = \frac{a_{-p}}{z^p} + a_0 + a_1 z + \cdots + a_n z^n + \cdots, \ z \in \dot{U}, \ a_{-p} \neq 0.$$ 

We will also use the following notations:

$\Sigma_{p,0} = \{g \in \Sigma_p : a_{-p} = 1\}$,

$\Sigma_0 = \{g \in \Sigma_{1,0} : g$ is univalent in $\dot{U}$ and $g(z) \neq 0, \ z \in \dot{U}\}$,

$\Sigma K_p(\alpha, \delta) = \left\{g \in \Sigma_p : \alpha < \text{Re} \left[-1 - \frac{zg''(z)}{g'(z)}\right] < \delta, \ z \in U\right\}$, where $\alpha < p < \delta$.

$\Sigma K_{p,0}(\alpha, \delta) = \Sigma K_p(\alpha, \delta) \cap \Sigma_{p,0}$. 

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Let \( f \) be a function, or more simply, a subordinant, if \( p \) and \( q \) are analytic in \( U \). If \( p \) and \( q \) are analytic in \( U \) and satisfy the second order differential superordination

\[
\text{Re} \left( \frac{q'(z)}{p'(z)} \right) < \delta, \quad z \in U
\]

then \( q \) is said to be subordinate to \( p \) of (1). (Note that the best dominant is unique up to a rotation of \( U \)).

**Definition 1.1.** ([4, p. 4]) Let \( f \) and \( F \) be members of \( H(U) \). The function \( f \) is said to be subordinate to \( F \), written \( f \prec F \) or \( f(z) \prec F(z) \), if there exists a function \( w \) analytic in \( U \), with \( w(0) = 0 \) and \( |w(z)| < 1 \), and such that \( f(z) = F(w(z)) \).

**Definition 1.2.** ([4, p. 16]) Let \( \psi : \mathbb{C}^3 \times U \to \mathbb{C} \) and let \( h \) be univalent in \( U \). If \( p \) is analytic in \( U \) and satisfies the (second order) differential subordination

\[
\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z), \quad (1)
\]

then \( p \) is called a solution of the differential subordination. The univalent function \( q \) is called a dominant of the solutions of the differential subordination, or more simply, a dominant, if \( p \prec q \) for all \( p \) satisfying (1). A dominant \( \tilde{q} \) that satisfies \( \tilde{q} \prec q \) for all dominants \( q \) of (1) is said to be the best dominant of (1). (Note that the best dominant is unique up to a rotation of \( U \)).

If we require the more restrictive condition \( p \in H[a, n] \), then \( p \) will be called an \((a, n)\)-solution, \( q \) an \((a, n)\)-dominant, and \( \tilde{q} \) the best \((a, n)\)-dominant.

**Definition 1.3.** ([5], [2, p. 98]) Let \( \varphi : \mathbb{C}^3 \times U \to \mathbb{C} \) and let \( h \) be analytic in \( U \). If \( p \) and \( \varphi(p(z), zp'(z), z^2p''(z); z) \) are univalent in \( U \) and satisfy the second order differential superordination

\[
h(z) \prec \varphi(p(z), zp'(z), z^2p''(z); z), \quad (2)
\]

then \( p \) is called a solution of the differential superordination. An analytic function \( q \) is called a subordinant of the solutions of the differential superordination, or more simply, a subordinant, if \( q \prec p \) for all \( p \) satisfying (2). An univalent subordinant \( \tilde{q} \) that satisfies \( q \prec \tilde{q} \) for all subordinants \( q \) of (2) is said
to be the best subordinant. Note that the best subordinant is unique up to a rotation of $U$.

**Definition 1.4.** ([2, p. 99]) We denote by $Q$ the set of functions $f$ that are analytic and injective on $\overline{U} \setminus E(f)$, where

$$E(f) = \left\{ \zeta \in \partial U : \lim_{z \to \zeta} f(z) = \infty \right\},$$

and they are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(f)$. The subclass of $Q$ for which $f(0) = a$, is denoted by $Q(a)$.

**Theorem 1.1.** ([3]) Let $\beta, \gamma \in \mathbb{C}$ and let $h$ be a convex function in $U$, with

$$\text{Re} [\beta h(z) + \gamma] > 0, \ z \in U.$$ 

Let $q_m$ and $q_k$ be the univalent solutions of the Briot-Bouquet differential equation

$$q(z) + \frac{nzq'(z)}{\beta q(z) + \gamma} = h(z), \ z \in U, \ q(0) = h(0),$$

for $n = m$ and $n = k$ respectively. If $m/k$, then $q_k(z) \prec q_m(z) \prec h(z)$. So, $q_k(z) \prec q_1(z) \prec h(z)$.

**Theorem 1.2.** ([6]) Let $p \in \mathbb{N}^*, \lambda \in \mathbb{C}$ with $\text{Re} \lambda > p$. If $g \in \Sigma_p$, then $J_{p,\lambda}(g) \in \Sigma_p$, where $J_{p,\lambda}(g)(z) = \frac{\lambda - p}{z^p} \int_0^z g(t)t^{\lambda-1}dt$.

**Theorem 1.3.** ([2, p. 102], [5]) Let $\Omega \subset \mathbb{C}$, $q \in H[a,n], \varphi : \mathbb{C} \times \overline{U} \to \mathbb{C}$, and suppose that

$$\varphi(q(z), tzq'(z); \zeta) \in \Omega,$$

for $z \in U$, $\zeta \in \partial U$ and $0 < t \leq \frac{1}{n} \leq 1$. If $p \in Q(a)$ and $\varphi(p(z), zp'(z); z)$ is univalent in $U$, then

$$\Omega \subset \left\{ \varphi(p(z), zp'(z); z) : z \in U \right\} \Rightarrow q(z) \prec p(z).$$

**Theorem 1.4.** ([4, p. 70]) Let $h$ be convex in $U$ and let $P : U \to \mathbb{C}$ with $\text{Re} P(z) > 0$. If $p$ is analytic in $U$, then

$$p(z) + P(z)zp'(z) \prec h(z) \Rightarrow p(z) \prec h(z).$$
Definition 1.5. ([7]) Let \( p \in \mathbb{N}^* \) and \( h \in H(U) \) with \( h(0) = p \). We define:

\[
\Sigma K_p(h) = \left\{ g \in \Sigma_p : -\left[ 1 + \frac{zg''(z)}{g'(z)} \right] \prec h(z) \right\},
\]

\[
\Sigma K_{p,0}(h) = \Sigma K_p(h) \cap \Sigma_{p,0}.
\]

Corollary 1.1. ([7]) Let \( p \in \mathbb{N}^* \), \( \gamma \in \mathbb{C} \) with \( \text{Re } \gamma > p \) and \( g \in \Sigma K_p(h) \) with \( h \) convex in \( U \). If

\[
\text{Re } [\gamma - h(z)] > 0, \ z \in U,
\]

then

\[
J_{p,\gamma}(g) \in \Sigma K_p(q),
\]

where \( q \) is the univalent solution of the Briot-Bouquet differential equation

\[
q(z) + \frac{(p + 1)zq'(z)}{\gamma - q(z)} = h(z), \ z \in U, \ q(0) = p.
\]

The function \( q \) is the best \((p, p+1)\)-dominant.

2. Main results

Next we consider some subclasses of \( \Sigma_{p,0} \) associated with superordination and subordination, using the close-to-convexity condition and throughout this paper we establish the conditions such that when we apply the integral operator \( J_{p,\gamma} \) to a function which belongs to one of these new subclasses, we get an image that belongs to a similar class.

Definition 2.1. Let \( p \in \mathbb{N}^* \), \( h_1, h_2, h \in H(U) \) with \( h_1(0) = h_2(0) = 1 \), \( h(0) = p \), \( h_1 \prec h_2 \) and \( \varphi \in \Sigma K_{p,0}(h) \). We define:

\[
\Sigma C_{p,0}(h_1, h_2; \varphi, h) = \left\{ g \in \Sigma_{p,0} : h_1(z) \prec \frac{g'(z)}{\varphi'(z)} \prec h_2(z) \right\},
\]

\[
\Sigma C_{p,0}(h_2; \varphi, h) = \left\{ g \in \Sigma_{p,0} : \frac{g'(z)}{\varphi'(z)} \prec h_2(z) \right\}.
\]
DEFINITION 2.2. Let \( p \in \mathbb{N}^* \) and \( h_2, h \in H(U) \) with \( h_2(0) = 1, h(0) = p \). We define:

\[
\Sigma C_{p,0}(h_2; h) = \left\{ g \in \Sigma_{p,0} : (\exists) \varphi \in \Sigma K_{p,0}(h) \text{ s.t. } \frac{g'(z)}{\varphi'(z)} < h_2(z) \right\},
\]

\[
\Sigma C_{p,0}(h) = \left\{ g \in \Sigma_{p,0} : (\exists) \varphi \in \Sigma K_{p,0}(h) \text{ s.t. } \frac{g'(z)}{\varphi'(z)} < \frac{1}{p} h(z) \right\}.
\]

Remark 2.1.

1. If \( H \in H(U), H(0) = p \) and \( h \prec H \), then \( \Sigma C_{p,0}(h_2; h) \subset \Sigma C_{p,0}(h_2; H) \).
2. If \( H_2 \in H(U), H_2(0) = 1 \) and \( h_2 \prec H_2 \), then \( \Sigma C_{p,0}(h_2; h) \subset \Sigma C_{p,0}(H_2; h) \).
3. If \( h_1, h_2, h, H \in H(U) \) with \( h_1(0) = h_2(0) = 1, h(0) = H(0) = p, h_1 \prec h_2 \) and \( \varphi \in \Sigma K_{p,0}(h) \cap \Sigma K_{p,0}(H) \), then

\[
\Sigma C_{p,0}(h_1, h_2; \varphi, h) = \Sigma C_{p,0}(h_1, h_2; \varphi, H),
\]

\[
\Sigma C_{p,0}(h_2; \varphi, h) = \Sigma C_{p,0}(h_2; \varphi, H).
\]

Next we present some particular cases for the classes defined above.

If \( p = 1 \) and \( h_2(z) = h(z) = \frac{1+z}{1-z}, z \in U \), then a function \( \varphi \) is in the class \( \Sigma K_{1,0}(h) \) if and only if

\[
\text{Re} \left[ -1 - \frac{z\varphi''(z)}{\varphi'(z)} \right] > 0, \ z \in U,
\]

so, the class of meromorphic close-to-convex functions is included in the class \( \Sigma C_{1,0}\left(\frac{1+z}{1-z}\right) \).

Let \( \alpha < 1 \leq p < \delta \). We consider \( h_2 = h_{1,0,\delta} \) and \( h = h_{p,0,\delta} \), where \( h_{p,0,\delta} : U \to \mathbb{C} \) is the convex function with \( h_{p,0,\delta}(U) = \{ z \in \mathbb{C} : \alpha < \text{Re} z < \delta \} \) and \( h_{p,0,\delta}(0) = p \). We know that \( h_{p,0,\delta} \) exists and it is obtained by composing different well-known elementary functions. It is not difficult to see that

\[
\Sigma K_{p,0}(h_{p,0,\delta}) = \Sigma K_{p,0}(\alpha, \delta),
\]

(3)

\[
\Sigma C_{p,0}(h_{1,0,\delta}; \varphi, h_{p,0,\delta}) = \Sigma C_{p,0}(\alpha, \delta; \varphi), \text{ where } \varphi \in \Sigma K_{p,0}(\alpha, \delta).
\]

(4)

We denote the class \( \Sigma C_{p,0}(h_{1,0,\delta}; h_{p,0,\delta}) \) by \( \Sigma C_{p,0}(\alpha, \delta) \).

We mention that the class \( \Sigma C_{p,0}(\alpha, \delta; \varphi) \) was introduced and studied in [6]. Also, a class similar with the class \( \Sigma C_{1,0}(\alpha, \delta) \) was defined and studied in [1].
Theorem 2.1. Let $p \in \mathbb{N}^*$ and $\gamma \in \mathbb{C}$ with $\Re \gamma > p$. Let $h_2$ and $h$ be convex functions in $U$ with $h_2(0) = 1$, $h(0) = p$ and let $g \in \Sigma C_{p,0}(h_2; h)$. If we have $\Re [\gamma - h(z)] > 0$, $z \in U$, then
\[ J_{p,\gamma}(g) \in \Sigma C_{p,0}(h_2; q), \]
where $q$ is the univalent solution of the Briot-Bouquet differential equation
\[ q(z) + \frac{(p+1)zq'(z)}{\gamma - q(z)} = h(z), \quad z \in U, \]
with $q(0) = p$. The function $q$ is the best $(p, p+1)$-dominant.

Proof. Since $g \in \Sigma C_{p,0}(h_2; h)$ we know that there is a function $\varphi \in \Sigma K_{p,0}(h)$ such that
\[ \frac{g'(z)}{\varphi'(z)} < h_2(z). \] (5)
Because $\varphi \in \Sigma K_{p,0}(h)$, where $\Sigma K_{p,0}(h) = \Sigma K_{p}(h) \cap \Sigma_{p,0}$, and $\Re [\gamma - h(z)] > 0$, $z \in U$, we have from Corollary 1.1 that
\[ \Phi = J_{p,\gamma}(\varphi) \in \Sigma K_{p}(q), \]
where $q$ is the univalent solution of the Briot-Bouquet differential equation
\[ q(z) + \frac{(p+1)zq'(z)}{\gamma - q(z)} = h(z), \quad z \in U, \]
with $q(0) = p$. Of course, the function $q$ is the best $(p, p+1)$-dominant.

From the definition of the operator $J_{p,\gamma}$, we remark that $\Phi \in \Sigma_{p,0}$, when $\varphi \in \Sigma_{p,0}$, so $\Phi \in \Sigma K_{p,0}(q)$.

Let $G = J_{p,\gamma}(g)$. We know from Theorem 1.2 that $G \in \Sigma p$ and it is easy to see that $G \in \Sigma_{p,0}$ (since $g \in \Sigma_{p,0}$). Using the definition of the operator $J_{p,\gamma}$ and the fact that $G = J_{p,\gamma}(g)$, $\Phi = J_{p,\gamma}(\varphi)$, we get
\[ \gamma G(z) + zG'(z) = (\gamma - p)g(z) \]
and
\[ \gamma \Phi(z) + z\Phi'(z) = (\gamma - p)\varphi(z), \quad z \in U, \]
therefore
\[ (\gamma + 1)G'(z) + zG''(z) = (\gamma - p)g'(z) \]
and

$$(\gamma + 1)\Phi'(z) + z\Phi''(z) = (\gamma - p)\varphi'(z).$$

Let us denote

$$P(z) = \frac{G'(z)}{\Phi'(z)}, \ z \in U.$$

Because $\Phi \in \Sigma_{K_p,0}(q)$ we have $z^{p+1}\Phi'(z) \neq 0$, $z \in U$, hence $P \in H(U)$. From $P(z)\Phi'(z) = G'(z)$, we get $G''(z) = P'(z)\Phi'(z) + P(z)\Phi''(z)$, so, the identity

$$(\gamma + 1)G'(z) + zG''(z) = (\gamma - p)\varphi'(z), \ z \in \hat{U},$$

can be rewritten as

$$(\gamma + 1)P(z)\Phi'(z) + z[P'(z)\Phi'(z) + P(z)\Phi''(z)] = (\gamma - p)\varphi'(z). \quad (6)$$

Using the identity $(\gamma + 1)\Phi'(z) + z\Phi''(z) = (\gamma - p)\varphi'(z)$, we obtain from (6) that

$$P(z) + \frac{zP'(z)}{\gamma + 1 + \frac{z\Phi''(z)}{\Phi'(z)}} = \frac{g'(z)}{\varphi'(z)}, \ z \in U,$$

which is equivalent to

$$P(z) + \frac{zP'(z)}{R(z)} = \frac{g'(z)}{\varphi'(z)}, \ \text{where} \ R(z) = \gamma + 1 + \frac{z\Phi''(z)}{\Phi'(z)}. \quad (7)$$

From (5) and (7) we obtain

$$P(z) + \frac{zP'(z)}{R(z)} < h_2(z). \quad (8)$$

Next we show that $\Re R(z) > 0$, $z \in U$. We know that $\Phi \in \Sigma_{K_p,0}(q)$ and $q \prec h$ (see Theorem 1.1), so

$$-1 - \frac{z\Phi''(z)}{\Phi'(z)} < h(z),$$

which is equivalent to

$$\gamma + 1 + \frac{z\Phi''(z)}{\Phi'(z)} < \gamma - h(z), \quad (9)$$

hence

$$R(z) < \gamma - h(z). \quad (10)$$
Since $\text{Re} \left[ \gamma - h(z) \right] > 0$, $z \in U$, we get from (10) that $\text{Re} \, R(z) > 0$, $z \in U$.

Because $\text{Re} \, R(z) > 0$, $z \in U$, we can use Theorem 1.4 for the subordination

$$P(z) + \frac{zP'(z)}{R(z)} \prec h_2(z),$$

and we get $P \prec h_2$, which is equivalent to

$$\frac{G'(z)}{\Phi'(z)} < h_2(z). \quad (11)$$

Since $G \in \Sigma_{p,0}$ and $\Phi \in \Sigma_{K,p,0}(q)$ we obtain from (11) that

$$G = J_{p,\gamma}(g) \in \Sigma_{C,p,0}(h_2;q).$$

From the proof of Theorem 2.1 we remark that we also have:

**Theorem 2.2.** Let $p \in \mathbb{N}^*$ and $\gamma \in \mathbb{C}$ with $\text{Re} \, \gamma > p$. Let $h_2$ and $h$ be convex functions in $U$ with $h_2(0) = 1$, $h(0) = p$ and $\text{Re} \left[ \gamma - h(z) \right] > 0$, $z \in U$.

If $\varphi \in \Sigma_{K,p,0}(h)$ and $g \in \Sigma_{C,p,0}(h_2;\varphi,h)$, then

$$J_{p,\gamma}(g) \in \Sigma_{C,p,0}(h_2;J_{p,\gamma}(\varphi),q),$$

where $q$ is the univalent solution of the Briot-Bouquet differential equation

$$q(z) + \frac{(p+1)q'(z)}{\gamma - q(z)} = h(z), \ z \in U,$$

with $q(0) = p$. The function $q$ is the best $(p,p+1)$-dominant.

If we consider that the conditions from the hypothesis of Theorem 2.1 and Theorem 2.2 respectively, are met, since we know from Theorem 1.1 that $q \prec h$, we obtain the next corollaries:

**Corollary 2.1.** Let $p \in \mathbb{N}^*$ and $\gamma \in \mathbb{C}$ with $\text{Re} \, \gamma > p$. Let $h_2$, $h$ be convex functions in $U$ with $h_2(0) = 1$, $h(0) = p$ and let $g \in \Sigma_{C,p,0}(h_2;h)$. If $\text{Re} \, h(z) < \text{Re} \, \gamma$, $z \in U$, then

$$J_{p,\gamma}(g) \in \Sigma_{C,p,0}(h_2;h).$$

**Corollary 2.2.** Let $p \in \mathbb{N}^*$ and $\gamma \in \mathbb{C}$ with $\text{Re} \, \gamma > p$. Let $h_2$ and $h$ be convex functions in $U$ with $h_2(0) = 1$, $h(0) = p$ and $\text{Re} \, h(z) < \text{Re} \, \gamma$, $z \in U$. If $\varphi \in \Sigma_{K,p,0}(h)$ and $g \in \Sigma_{C,p,0}(h_2;\varphi,h)$, then

$$J_{p,\gamma}(g) \in \Sigma_{C,p,0}(h_2;J_{p,\gamma}(\varphi),h).$$
Next we present two results which concern the particular classes $\Sigma_{C,p,0}^\alpha, \delta$ and $\Sigma_{C,p,0}^\alpha, \delta; \varphi$.

**Theorem 2.3.** Let $p \in \mathbb{N}^*$, $\alpha, \delta \in \mathbb{R}$ and $\gamma \in \mathbb{C}$ with $\alpha < 1 \leq p < \delta \leq \Re \gamma$. If $g \in \Sigma_{C,p,0}^\alpha, \delta$, then

$$J_{p,\gamma}(g) \in \Sigma_{C,p,0}^\alpha, \delta.$$

**Proof.** We know that the class $\Sigma_{C,p,0}^\alpha, \delta$ is the class $\Sigma_{C,p,0}^{h_1,\alpha, \delta; h_p,\alpha, \delta}$. Taking $h_2 = h_1, h = h_p, \alpha, \delta$ for Corollary 2.1 we remark that the hypothesis of this corollary is fulfilled, so we get

$$J_{p,\gamma}(g) \in \Sigma_{C,p,0}^{h_1,\alpha, \delta; h_p,\alpha, \delta} = \Sigma_{C,p,0}^\alpha, \delta.$$

**Theorem 2.4.** Let $p \in \mathbb{N}^*$, $\alpha, \delta \in \mathbb{R}$ and $\gamma \in \mathbb{C}$ with $\alpha < 1 \leq p < \delta \leq \Re \gamma$. If $\varphi \in \Sigma_{K,p,0}^\alpha, \delta$ and $g \in \Sigma_{C,p,0}^\alpha, \delta; \varphi$, then

$$J_{p,\gamma}(g) \in \Sigma_{C,p,0}^\alpha, \delta; \Phi,$$

where $\Phi = J_{p,\gamma}(\varphi)$.

**Proof.** From (3) we know that $\Sigma_{K,p,0}^{h_p,\alpha, \delta} = \Sigma_{K,p,0}^\alpha, \delta$ and from (4) we have $\Sigma_{C,p,0}^{h_1,\alpha, \delta; \varphi, h_p,\alpha, \delta} = \Sigma_{C,p,0}^\alpha, \delta; \varphi$, where $\varphi \in \Sigma_{K,p,0}^\alpha, \delta$. Considering $h_2 = h_1, \alpha, \delta$ and $h = h_p, \alpha, \delta$ for Corollary 2.2, we remark that the hypothesis of this corollary is fulfilled, so we get

$$J_{p,\gamma}(g) \in \Sigma_{C,p,0}^{h_1,\alpha, \delta; J_{p,\gamma}(\varphi), h_p,\alpha, \delta} = \Sigma_{C,p,0}^\alpha, \delta; J_{p,\gamma}(\varphi).$$

We remark that a result which is similar to Theorem 2.4 was also obtained in [6] but using a different method. We also remark that in the hypothesis of Theorem 2.4 we do not have the condition $z^{p+1}J_{p,\gamma}'(\varphi)(z) \neq 0, z \in U$, which appears in the hypothesis of the result presented in [6].

**Lemma 2.1.** Let $r > 0$ and let $\lambda : \overline{U} \to \mathbb{C}$ be an analytic function in $U$ such that $\sup_{z \in \overline{U}}|\lambda(z)| = M < \infty$. If $p \in H[1, 1] \cap Q$ and $p(z) + \lambda(z)zp'(z)$ is univalent in $U$, then

$$U(1, r) \subset \{p(z) + \lambda(z)zp'(z) : z \in U\} \Rightarrow U\left(1, \frac{r}{1+M}\right) \subset p(U).$$
Proof. To prove this lemma we use Theorem 1.3. Let us consider \( \Omega = U(1, r), q(z) = \frac{r}{1 + M} z + 1, z \in U, \) and \( \varphi : \mathbb{C}^2 \times \overline{U} \to \mathbb{C}, \varphi(u, s; \zeta) = u + \lambda(\zeta)s. \) Since we know from the hypothesis that \( p \in H[1, 1] \cap Q \) and \( p(z) + \lambda(z)zp'(z) \) is univalent in \( U, \) to apply Theorem 1.3, we need only to verify that

\[
\varphi(q(z), tzq'(z); \zeta) \in \Omega = U(1, r), \text{ when } z \in U, \zeta \in \partial U, 0 < t \leq 1,
\]

which is equivalent to

\[
|q(z) + \lambda(\zeta)tzq'(z) - 1| < r, \text{ when } z \in U, \zeta \in \partial U, 0 < t \leq 1.
\]

We have

\[
|q(z) + \lambda(\zeta)tzq'(z) - 1| = \frac{r}{1 + M}|z[1 + t\lambda(\zeta)]| < \frac{r}{1 + M}|1 + t\lambda(\zeta)|
\]

\[
\leq \frac{r}{1 + M}(1 + t|\lambda(\zeta)|) \leq \frac{r}{1 + M}(1 + M) = r.
\]

Therefore, the condition (13) is satisfied, so we get from Theorem 1.3 that \( q \prec p, \) which implies

\[
U \left( 1, \frac{r}{1 + M} \right) \subset p(U).
\]

Theorem 2.5. Let \( m, r > 0, p \in \mathbb{N}^* \) and \( \gamma \in \mathbb{C} \) with \( \text{Re } \gamma > p. \) Let \( h_2 \) and \( h \) be convex functions in \( U \) such that \( h_2(0) = 1, h(0) = p \) and \( \text{Re } [\gamma - h(z)] > m, z \in U. \) Let \( \varphi \in \Sigma K_{p,0}(h) \) and \( g \in \Sigma C_{p,0}(h_1, h_2; \varphi, h), \) where \( h_1(z) = rz + 1, z \in U. \) Suppose that \( \frac{g'}{g} \) is univalent in \( U \) and \( \frac{J_p,\gamma(g)}{J_p,\gamma(\varphi)} \in Q. \) Then

\[
G = J_{p,\gamma}(g) \in \Sigma C_{p,0}(q_1, h_2; \Phi, q),
\]

where

\[
\Phi = J_{p,\gamma}(\varphi),
\]

\[
q_1(z) = \frac{rm}{m + 1} z + 1, z \in U,
\]

and \( q \) is the univalent solution of the Briot-Bouquet differential equation

\[
q(z) + \frac{(p + 1)zq'(z)}{\gamma - q(z)} = h(z), z \in U,
\]

with \( q(0) = p. \) The function \( q \) is the best \((p, p + 1)\)-dominant.
**Proof.** Since $\varphi \in \Sigma K_{p,0}(h) = \Sigma K_p(h) \cap \Sigma_{p,0}$ and $\text{Re} [\gamma - h(z)] > m > 0$, $z \in U$, we have from Corollary 1.1 that

$$\Phi = J_{p,\gamma}(\varphi) \in \Sigma K_p(q),$$

where $q$ is the univalent solution of the Briot-Bouquet differential equation

$$q(z) + \frac{(p + 1)zq'(z)}{\gamma - q(z)} = h(z), \; z \in U,$$

with $q(0) = p$. It is easy to see that $\Phi \in \Sigma K_{p,0}(q)$. Of course, the function $q$ is the best $(p, p+1)$-dominant.

We have $G = J_{p,\gamma}(g)$ and $\Phi = J_{p,\gamma}(\varphi)$. Let

$$P(z) = \frac{G'(z)}{\Phi'(z)}, \; z \in U.$$

Since $\Phi \in \Sigma K_{p,0}(q)$ we have $z^{p+1}\Phi'(z) \neq 0$, $z \in U$, so $P \in H(U)$.

Analogously to the proof of Theorem 2.1 we obtain

$$P(z) + \frac{zP'(z)}{R(z)} = \frac{g'(z)}{\varphi'(z)},$$

where

$$R(z) = \gamma + 1 + \frac{z\Phi''(z)}{\Phi'(z)}, \; z \in U.$$  

It is obvious that $R \in H(U)$. From $g \in \Sigma C_{p,0}(h_1, h_2; \varphi, h)$ we have

$$h_1(z) \prec \frac{g'(z)}{\varphi'(z)} \prec h_2(z),$$

hence

$$h_1(z) \prec P(z) + \frac{zP'(z)}{R(z)} \prec h_2(z). \quad (14)$$

Because $\text{Re} R(z) > 0$, $z \in U$, (see the proof of Theorem 2.1), we can use Theorem 1.4 for the subordination of (14), which is

$$P(z) + \frac{zP'(z)}{R(z)} \prec h_2(z),$$

and we get

$$P \prec h_2. \quad (15)$$
Next we consider the superordination of (14), which is

\[ h_1(z) \prec P(z) + \frac{zP'(z)}{R(z)}. \]

Since \( h_1(z) = rz + 1, z \in U \), this superordination implies

\[ U(1, r) \subset \left\{ P(z) + \frac{zP'(z)}{R(z)} : z \in U \right\}. \]

Let us denote \( \lambda = \frac{1}{r} \). We know from (10) that \( R(z) \prec \gamma - h(z) \) and from the hypothesis we have \( \text{Re} [\gamma - h(z)] > m > 0 \), \( z \in U \), hence \( \text{Re} R(z) > m, z \in U \). We have the function \( \lambda : \mathcal{U} \to \mathbb{C} \) analytic in \( U \) and \( \sup_{z \in \mathcal{U}} |\lambda(z)| \leq \frac{1}{m} \). We may apply now Lemma 2.1 and we obtain

\[ U \left( 1, \frac{rm}{m+1} \right) \subset P(U). \] (16)

Since \( P \) is univalent in \( U \) and \( P(0) = q_1(0) \), we have (16) equivalent to

\[ q_1 \prec P, \text{ where } q_1(z) = \frac{rm}{m+1}z + 1, z \in U. \] (17)

From (15), (17) and the fact that

\[ \Phi = J_{p, \gamma}(\varphi) \in \Sigma K_{p,0}(q), \]

where \( q \) is the univalent solution of the Briot-Bouquet differential equation

\[ q(z) + \frac{(p+1)zq'(z)}{\gamma - q(z)} = h(z), \ z \in U, \]

with \( q(0) = p \), we obtain that

\[ G = J_{p, \gamma}(g) \in \Sigma \mathcal{C}_{p,0}(q_1, h_2; \Phi, q). \]

References


