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Derivations of Generalized B*-algebras

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Abstract: It is well known that a commutative C*-algebra has no nonzero derivations. In this article, we extend this result to complete commutative GB*-algebras having jointly continuous multiplication. We also give some results about derivations of GB*-algebras, with their underlying C*-algebras being W*-algebras.

Key words: GB*-algebra, topological algebra, derivation, locally convex bimodule.

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1. Introduction

GB*-algebras (i.e., generalized B*-algebras) are locally convex $*$-algebras which are generalizations of C*-algebras. They were introduced in 1967 by G.R. Allan in [2], and later, the concept was extended by P.G. Dixon in [16] to include non-locally convex algebras. GB*-algebras are also abstract algebras of unbounded operators on Hilbert spaces, i.e., O*-algebras. The latter algebras were introduced by G. Lassner in [26] and play an important role in the theory of unbounded operators and their physical applications. To be more precise, the observables of a quantum mechanical system can be realized as unbounded self-adjoint operators on a Hilbert space, and one considers these operators to be elements of an algebra of unbounded operators (O*-algebra). The time-evolution of the quantum mechanical system can be modeled by one-parameter automorphism groups of the latter algebras, and derivations are the generators of these groups.

If A is an algebra, and X is an A-bimodule, then a linear map $\delta : A \to X$ is called a *derivation* if $\delta(ab) = a\delta(b) + \delta(a)b$ for all $a, b \in A$. We say that δ is *inner* if there exists $x \in X$ such that $\delta(a) = ax - xa$ for all $a \in A$. The theory of derivations of C*-algebras is well developed and, as mentioned above, is of importance to the algebraic formalism of quantum mechanics ([11], [32]). For instance, it is well known that all derivations of a C*-algebra are continuous [32, Theorem 2.3.1], and that all derivations of a von Neumann algebra are inner [32, Theorem 2.5.3]. Also, the zero derivation is the only derivation of a commutative C*-algebra [15]. A wealth of automatic continuity results for derivations and homomorphisms of Banach algebras are given in [15].

The first article about derivations of unbounded operator algebras to appear in the literature is the article of C. Brödel and G. Lassner [12]. In this article, they proved that every derivation of a complete O*-algebra A of type R is spatial, and is the generator of a one-parameter automorphism group of A. A special type of GB*-algebra is a pro-C*-algebra, i.e., a complete topological $*$-algebra $A[\tau_\Gamma]$ for which there exists a directed family of C*-seminorms $\Gamma = \{p_\lambda : \lambda \in \Gamma\}$ defining the topology τ_Γ [18, Definition 7.1]. If $A[\tau_\Gamma]$ is a pro-C*-algebra, R. Becker proved in 1992 that all derivations $\delta : A \to A$ are continuous [7, Proposition 2]. He also proved that the zero derivation is the only derivation of a commutative pro-C*-algebra [7, Corollary 3]. Other results concerning derivations of non-normed topological $*$-algebras and unbounded operator algebras can be found in [30], [23], [5], [6], [8], [4], [34] and [35]. For a more detailed survey of derivations of locally convex $*$-algebras, see [20].

All of the above, together with [20, discussion after Theorem 5.2], provides good motivation for a general investigation of derivations of GB*-algebras. We prove in Section 3 that the zero derivation is the only derivation of a complete commutative GB*-algebra having jointly continuous multiplication. This is an extension to GB*-algebras of the well known fact that every commutative C*-algebra (more, generally, a pro-C*-algebra) has no nonzero derivations. In Section 4, we give an example of a commutative O*-algebra admitting a nonzero derivation.

A GB*-algebra $A[\tau]$ has the property that there is a C*-algebra $A[B_0]$ dense in A (Proposition 2.3), which plays an important role for its study. In Section 5, we give some results about derivations of GB*-algebras, with $A[B_0]$ being a W*-algebra. Examples of GB*-algebras having $A[B_0]$ as a W*-algebra are given. Section 2 consists of all the necessary background for understanding and proving the main results of this paper.

2. Preliminaries

All vector spaces in this paper are over the field $\mathbb{C}$ of complex numbers and all topological spaces are assumed to be Hausdorff. Moreover, all algebras are assumed to have an identity element denoted by 1.

A *topological algebra* is an algebra, which is also a topological vector space such that the multiplication is separately continuous in both variables [18]. A *topological $*$-algebra* is a topological algebra endowed with a continuous involution. A topological $*$-algebra which is also a locally convex space is called a *locally convex $*$-algebra*. The symbol $A[\tau]$ will stand for a topological $*$-algebra A endowed with given topology τ.

Definitions 2.1. ([2]) Let $A[\tau]$ be a topological $*$-algebra and $\mathcal{B}^*$ a collection of subsets B of A with the following properties:

(i) B is absolutely convex, closed and bounded,

(ii) $1 \in B$, $B^2 \subset B$ and $B^* = B$.

For every $B \in \mathcal{B}^*$, denote by $A[B]$ the linear span of B, which is a normed algebra under the gauge function $\|\cdot\|_B$ of B. If $A[B]$ is complete for every $B \in \mathcal{B}^*$, then $A[\tau]$ is called *pseudo-complete*.

An element $x \in A$ is called (Allan) *bounded* if for some nonzero complex number λ, the set $\{(\lambda x)^n : n = 1, 2, 3, \dots\}$ is bounded in A. We denote by A_0 the set of all bounded elements in A.

A topological $*$-algebra $A[\tau]$ is called *symmetric* if, for every $x \in A$, the element $(1 + x^*x)^{-1}$ exists and belongs to A_0.

In [16], the collection $\mathcal{B}^*$ in the definition above is defined to be the same as above, except that $B \in \mathcal{B}^*$ is no longer assumed to be absolutely convex. The notion of a bounded element is a generalization of the concept of bounded operator on a Banach space, and was introduced by G.R. Allan in [1] in order to develop a spectral theory for general locally convex $*$-algebras.

Definition 2.2. ([2]) A symmetric pseudo-complete locally convex $*$-algebra $A[\tau]$ such that the collection $\mathcal{B}^*$ has a greatest member denoted by B_0, is called a *GB^*-algebra* over B_0.

Every sequentially complete locally convex algebra is pseudo-complete [1, Proposition 2.6]. In [16], P.G. Dixon extended the notion of GB*-algebras to include topological $*$-algebras which are not locally convex. In this definition, GB*-algebras are not assumed to be pseudo-complete, B_0 is the only element in $\mathcal{B}^*$ which is necessarily absolutely convex (see the paragraph before Definition 2.2), and only $A[B_0]$ is assumed to be complete with respect to the gauge function $\|\cdot\|_{B_0}$. For a survey on GB*-algebras, see [19].

Proposition 2.3. ([2, Theorem 2.6], [10, Theorem 2]) *If $A[\tau]$ is a GB*-algebra, then the Banach $*$-algebra $A[B_0]$ is a C*-algebra sequentially dense in A, and $(1+x^*x)^{-1} \in A[B_0]$ for every $x \in A$. Furthermore, B_0 is the unit ball of $A[B_0]$.*

The C*-algebra $A[B_0]$ of Proposition 2.3 is also called the *bounded part* of the GB*-algebra A. If A is commutative, then $A_0 = A[B_0]$ [2, p. 94]. In general, A_0 is not a $*$-subalgebra of A, and $A[B_0]$ contains all normal elements of A_0 [2, p. 94].

It is well known that every commutative C*-algebra is topologically and algebraically $*$-isomorphic to $C(X)$ for some compact Hausdorff space (in fact, X is the maximal ideal space of A). More generally, any commutative GB*-algebra is algebraically $*$-isomorphic to an algebra of functions on a compact Hausdorff space X, which are allowed to take the value infinity on at most a nowhere dense subset of X [2, Theorem 3.9]. This algebraic $*$-isomorphism extends the Gelfand isomorphism of $A[B_0]$ onto the corresponding $C(X)$.

Recall that every C*-algebra is topologically-algebraically $*$-isomorphic to a norm closed $*$-subalgebra of $B(H)$ for some Hilbert space H. In general, every GB*-algebra is algebraically $*$-isomorphic to an algebra of unbounded operators on a Hilbert space [16, Theorem 7.6 and Theorem 7.11]. Therefore, in light of Proposition 2.3, one can think of a GB*-algebra as a C*-algebra with "unbounded elements" adjoined to it.

A *pro-C*-algebra* is a complete locally convex $*$-algebra $A[\tau]$, whose topology τ is defined by a directed family of C^*-seminorms [18, Definition 7.1]. Every pro-C*-algebra is topologically $*$-isomorphic to an inverse limit of C*-algebras [18], and every pro-C*-algebra is a *GB*-algebra [2, p. 95].

Suppose now that $A[\tau]$ is a locally convex $*$-algebra, where τ is defined by a directed family $\{p_\nu\}_{\nu\in\Lambda}$ of seminorms with the following properties: for every $\nu \in \Lambda$, there is $\nu' \in \Lambda$ such that $p_\nu(xy) \leq p_{\nu'}(x)p_{\nu'}(y)$, $p_\nu(x^*) \leq p_{\nu'}(x)$ and $p_\nu(x)^2 \leq p_{\nu'}(x^*x)$ for all $x, y \in A$. Such a family of seminorms is called *C^*-like*. A complete locally convex $*$-algebra $A[\tau]$ for which τ is defined by a family of C^*-like seminorms is called a *C*-like locally convex $*$-algebra* if

$$A_b := \left\{ x \in A \,:\, \sup_\nu p_\nu(x) < \infty \right\}$$

is τ-dense in A [22]. Every C*-like locally convex $*$-algebra is a GB*-algebra over $B_0 = \{x \in A : \sup_\nu p_\nu(x) \leq 1\}$ [22, Theorem 2.1]. Clearly, every pro-C*-algebra is a C*-like locally convex $*$-algebra. Examples of GB*-algebras, including pro-C*-algebras and C*-like locally convex $*$-algebras, can be found in [2], [16], [18] and [22]. We give the following example, which we will need in Section 3.

EXAMPLE 2.4. ([22, Example 3.3]) Let M be a von Neumann algebra with a faithful finite normal trace τ. Let $LS(M)$ denote the $*$-algebra of all locally measurable operators affiliated with M (see Definition 2.6 below), and let $L^p(M,\tau) = \{x \in LS(M) : \tau(|x|^p) < \infty\}$ for all $p \geq 1$, where $|x| = (x^*x)^{\frac{1}{2}}$. Then $L^p(M,\tau)$ is a Banach space with respect to the norm

$$\|x\|_p = \big(\tau(|x|^p)\big)^{\frac{1}{p}}$$

for every $p \geq 1$. Let $L^\omega(M,\tau) = \bigcap_{p\geq 1} L^p(M,\tau)$. Then $L^\omega(M,\tau)$ is a C*-like locally convex $*$-algebra, and hence a GB*-algebra, with respect to the seminorms $\|\cdot\|_p$, where $p \geq 1$.

If $\mathcal{D}$ denotes an inner product space, then $\mathcal{L}^\dagger(\mathcal{D})$ denotes the set of all closable linear operators a such that $a\mathcal{D} \subset \mathcal{D}$, the domain of a^* contains $\mathcal{D}$ and $a^*\mathcal{D} \subset \mathcal{D}$. We define an involution on $\mathcal{L}^\dagger(\mathcal{D})$ by $a^\dagger = a^*|_\mathcal{D}$ for all $a \in \mathcal{L}^\dagger(\mathcal{D})$. Then $\mathcal{L}^\dagger(\mathcal{D})$ is a $*$-algebra with respect to this involution, and with multiplication being defined by the usual composition of operators [26]. A $*$-subalgebra of $\mathcal{L}^\dagger(\mathcal{D})$ containing the identity operator on $\mathcal{D}$ is called an *O^*-algebra* on $\mathcal{D}$ [26].

DEFINITION 2.5. Let x and y be closed operators on a Hilbert space $\mathcal{H}$. If $x + y$ is closable, then its closure $\overline{x+y}$ is called the strong sum of x and y, and is denoted by $x\,\dot{+}\,y$. The strong product of x and y is defined similarly by $\overline{xy}$, and is denoted by $x \cdot y$. If $0 \neq \lambda \in \mathbb{C}$, then we define $\lambda \cdot x$ to be λx, and if $\lambda = 0$, then $\lambda \cdot x$ is defined to be the zero operator defined on the whole of $\mathcal{H}$.

The following concepts of locally measurable operator and EW*-algebra will be needed in Section 5.

DEFINITION 2.6. ([36, Theorem 2.1 and Definition 2.2]) Let M be a von Neumann algebra on a Hilbert space H and x a closed operator affiliated with M.

(i) The operator x is called measurable if the domain of x is dense in H and $1 - E_\lambda$ is finite for some $\lambda > 0$, where $|x| = \int_0^\infty \lambda \, dE_\lambda$ is the spectral decomposition of $|x|$.

(ii) If there exist projections q_n in the centre of M such that $q_n \uparrow 1$ and xq_n is measurable for each n, then x is called locally measurable.

We denote the set of *all locally measurable operators affiliated with a von Neumann algebra* M by $LS(M)$. This is a $*-$algebra with respect to the usual adjoint, the strong sum and strong product [36, p. 260].

Definition 2.7. ([17, Definition 1.2]) Let A be a set of closed, densely defined operators on a Hilbert space $\mathcal{H}$ which is a $*$-algebra under strong sum, strong product, scalar multiplication (it is understood that $\lambda x = 0$, the zero operator on the whole of $\mathcal{H}$, if $\lambda = 0$) and the usual adjoint of operators. We call A an EW*-algebra if the following conditions are satisfied:

(i) $(1 + x^* x)^{-1}$ exists in A for every $x \in A$,

(ii) the subalgebra A_e of bounded operators in A is a W*-algebra.

We sometimes say that *A is an EW*-algebra over the von Neumann algebra A_e.*

Proposition 2.8. ([29, Proposition 3.4]) *If $A[\tau_\Gamma]$ is a pro-C*-algebra and $X[\tau]$ is a complete locally convex A-bimodule having $\tau_\Gamma \times \tau - \tau$ jointly continuous module actions, then the topology τ on X can be defined by a directed family of seminorms Γ' such that for every $q \in \Gamma'$, there is a C*-seminorm $p \in \Gamma$ satisfying $q(ax) \leq p(a)q(x)$ and $q(xa) \leq p(a)q(x)$ for all $a \in A$ and $x \in X$.*

If, in particular, $A[\| \cdot \|]$ is a C*-algebra and $X[\tau]$ is a complete locally convex A-bimodule having $\| \cdot \| \times \tau - \tau$ jointly continuous module actions, then the topology τ on X can be defined by a family of seminorms Γ' such that for every $q \in \Gamma'$, $q(ax) \leq \|a\| q(x)$ and $q(xa) \leq \|a\| q(x)$ for all $a \in A$ and $x \in X$.

3. Derivations of commutative GB*-algebras

The main result of this section is that a complete commutative GB*-algebra having jointly continuous multiplication has no nonzero derivations.

This result is a partial answer to the question in [20, discussion after Theorem 5.2], concerning the structure of derivations of GB*-algebras.

The strategy of the proof is as follows: given a complete commutative GB*-algebra $A[\tau]$ with jointly continuous multiplication, and a derivation $\delta : A \to A$, we prove that $\delta|_{A[B_0]} = 0$. The result then follows from the following proposition.

PROPOSITION 3.1. *If $\delta : A \to A$ is a derivation of a GB*$-$algebra $A[\tau]$ such that there is an $a \in A$ satisfying $\delta(x) = ax - xa$ for all $x \in A[B_0]$, then $\delta(x) = ax - xa$ for all $x \in A$.*

Proof. Let $x \in A$ such that $x \geq 0$. Then $(1+x)^{-1} \in A[B_0]$ ([16, Proposition 5.1] and [2, Theorem 2.6]). Also, we have that

$$\begin{aligned} 0 = \delta(1) &= \delta((1+x)(1+x)^{-1}) \\ &= \delta((1+x)^{-1} + x(1+x)^{-1}) \\ &= \delta((1+x)^{-1}) + x\delta((1+x)^{-1}) + \delta(x)(1+x)^{-1}. \end{aligned}$$

Therefore

$$\begin{aligned} \delta(x) &= -\delta((1+x)^{-1})(1+x) - x\delta((1+x)^{-1})(1+x) \\ &= -(a(1+x)^{-1} - (1+x)^{-1}a)(1+x) \\ &\quad - x(a(1+x)^{-1} - (1+x)^{-1}a)(1+x) \\ &= \ ax - xa\,. \end{aligned}$$

Now let $x \in A$ be arbitrary. By the proof of [16, Theorem 6.5], there exist positive elements $x_i \in A, 1 \leq i \leq 4$, such that $x = x_1 - x_2 + \mathrm{i}\, x_3 - \mathrm{i}\, x_4$. Therefore, from the above, $\delta(x) = ax - xa$. ∎

If A is a commutative amenable Banach algebra, X a commutative Banach A-bimodule, and $\delta : A \to X$ a continuous derivation, then $\delta = 0$ [24, Proposition 8.2]. Also, every derivations of a C*-algebra A into any Banach A-bimodule is continuous [31, Theorem 2]. These facts are needed in the proof of the following theorem, which is the key for proving that the zero derivation is the only derivation of a commutative Fréchet GB*-algebra.

THEOREM 3.2. *Let A be a commutative C^*-algebra and $X[\tau]$ a commutative complete locally convex $A-$bimodule with jointly continuous module actions. Then every derivation $\delta : A \to X$ is inner and thus the zero derivation.*

Proof. From Proposition 2.8, we have that the topology τ of X is determined by a family $(q_i)_{i\in I}$ of seminorms such that $q_i(ax) \leq ||a||q_i(x)$ and $q_i(xa) \leq ||a||q_i(x)$ for all $x \in X$ and $a \in A$. Then, for all $i \in I$, it follows that $X_i \equiv X/\ker q_i$ is a normed A-bimodule with respect to the following (well defined) module actions:

$$a \cdot (x + N_i) = ax + N_i \qquad \text{and} \qquad (x + N_i) \cdot a = xa + N_i\,,$$

where $N_i = \{x \in X \,:\, q_i(x) = 0\}$ for each $i \in I$. Therefore $X = \varprojlim \overline{X}_i$, up to isomorphism of locally convex spaces, where $\overline{X}_i$ is the completion of X_i with respect the norm $\overline{q}_i$, where $\overline{q}_i(x + \ker q_i) = q_i(x)$ for every $x \in X$ and $i \in I$. Therefore $\overline{X}_i$ is a commutative Banach A-bimodule for every $i \in I$. We now consider the map

$$\delta_i : A \longrightarrow \overline{X}_i\,, \qquad \delta_i = \pi_i \circ \delta\,,$$

where $\pi_i : X \to \overline{X}_i$ is the i^{th} projection (module) map of X into $\overline{X}_i$. It is easily verified that δ_i is a derivation for every $i \in I$. By [31, Theorem 2], δ_i is $\|\cdot\| - \overline{q}_i$ continuous for every $i \in I$. Since A is a commutative C*-algebra, A is an amenable Banach algebra, and therefore, by [24, Proposition 8.2], $\delta_i = 0$ for all $i \in I$. Hence $\delta = 0$. ▮

THEOREM 3.3. *Let $A[\tau]$ be a commutative complete GB*$-$algebra with jointly continuous multiplication. Then the zero derivation is the only derivation of A.*

Proof. Let $\delta : A \to A$ be a derivation of A. Then $\delta_{|A[B_0]} : A[B_0] \to A$ is a derivation from the commutative C^*-algebra $A[B_0]$ into A, which is a complete locally convex $A[B_0]-$bimodule with $\|\cdot\| \times \tau - \tau$ jointly continuous module actions (the module actions being the multiplication on A). The latter comes from the fact that the multiplication in A is jointly continuous and that $\tau \preceq \|\cdot\|$ on $A[B_0]$. Therefore, from Theorem 3.2, we have that $\delta_{|A[B_0]} = 0$. Hence, by Proposition 3.1, $\delta = 0$. ▮

Every Fréchet topological algebra has the property that multiplication is jointly continuous [18], and therefore the following result is an immediate consequence of Theorem 3.3.

COROLLARY 3.4. *If $A[\tau]$ is a commutative Fréchet GB*$-$algebra, then the zero derivation is the only derivation of A.*

Since C^*-like locally convex $*-$algebras are complete GB*$-$algebras having jointly continuous multiplication, we get the following corollary.

COROLLARY 3.5. *If $A[\tau]$ is a commutative C^*-like locally convex $*-$algebra, then the zero derivation is the only derivation of A.*

Since $L^\omega(M,\tau)$ is a C*-like locally convex $*$-algebra, as in Example 2.4, one can deduce the following result from Corollary 3.5, which is a special case of [5, Corollary 3.5].

COROLLARY 3.6. *If M is a commutative von Neumann algebra with a faithful finite normal trace τ, then the zero derivation is the only derivation of $L^\omega(M,\tau)$.*

Remark. If A is a pro-C*-algebra and X is a complete locally convex A-bimodule with jointly continuous module actions, then every derivation $\delta : A \to X$ is continuous (this follows from Proposition 2.8 and [35, Theorem 3.8]).

4. AN EXAMPLE OF A COMMUTATIVE O*-ALGEBRA WITH A NONZERO DERIVATION

Consider the inner product space $\mathcal{D} = S(\mathbb{R})$ of all infinitely differentiable functions on $\mathbb{R}$ which are rapidly decreasing. The completion of $\mathcal{D}$ is the Hilbert space $\mathcal{H} = L_2(\mathbb{R})$. Recall the position and momentum operators q and p from quantum mechanics.

Let A be the commutative $*$-subalgebra of $\mathcal{L}^\dagger(\mathcal{D})$ generated by q and 1. Then A is a commutative O*-algebra. For each $a \in A$, let $\delta(a) = pa - ap$. Observe that δ is nonzero since $q \in A$ and $\delta(q) = pq - qp = -\mathrm{i}\,\hbar 1 \neq 0$, where $\hbar$ is Planck's constant. We prove that $\delta(a) \in A$ for every $a \in A$, implying that δ is a nonzero derivation of A.

In proving that $\delta(A) \subset A$, we require the following observation.

LEMMA 4.1. *$q^n p - pq^n \in A$ for all $n \in \mathbb{N}$.*

Proof. We will use mathematical induction. Firstly, $qp - pq = \mathrm{i}\,\hbar 1 \in A$. Now assume that $q^m p - pq^m \in A$ for some $m \in \mathbb{N}$. For any $k \in \mathbb{N}$, it follows from the identity $qp-pq = \mathrm{i}\,\hbar 1$ that $q^k p - pq^k = q^{k-1}(pq) - (pq)q^{k-1} + \mathrm{i}\,\hbar q^{k-1}$. Then

$$\begin{aligned} q^{m+1}p - pq^{m+1} &= q^m(pq) - (pq)q^m + \mathrm{i}\,\hbar q^m \\ &= (q^m p)q - (pq^m)q + \mathrm{i}\,\hbar q^m \\ &= (q^m p - pq^m)q + \mathrm{i}\,\hbar q^m \ \in \ A \end{aligned}$$

by assumption. By induction, $q^n p - pq^n \in A$ for all $n \in \mathbb{N}$. ∎

Coming back to our claim, let $a \in A$. Then, by the very definition of A, it follows that $a = \alpha_n q^n + \alpha_{n-1} q^{n-1} + \cdots + \alpha_1 q + \alpha_0 1$, for some $n \in \mathbb{N}$, $\alpha_i \in \mathbb{C}$, $i = 0, \ldots, n$. Therefore

$$\begin{aligned}\delta(a) = pa - ap &= p(\alpha_n q^n + \alpha_{n-1} q^{n-1} + \cdots + \alpha_1 q + \alpha_0 1) \\ &\quad - (\alpha_n q^n + \alpha_{n-1} q^{n-1} + \cdots + \alpha_1 q + \alpha_0 1)p \\ &= \alpha_n (pq^n - q^n p) + \alpha_{n-1}(pq^{n-1} - q^{n-1}p) + \cdots + \alpha_1 (pq - qp) \ \in \ A\end{aligned}$$

by Lemma 4.1. Consequently, the commutative O*-algebra A, defined as above, admits at least one nonzero derivation.

The *graph topology* [26] t_B on $\mathcal{D}$ induced by an O*-algebra B on $\mathcal{D}$ is defined by the family of seminorms $\|\phi\|_a = \|a\phi\|$ for all $\phi \in \mathcal{D}$, where $a \in B$.

We equip an O*-algebra B on $\mathcal{D}$ with the *uniform topology* [26], which is defined by the following family of seminorms:

$$p_{\mathcal{M}}(a) = \sup_{\phi, \psi \in \mathcal{M}} |\langle a\phi, \psi \rangle|,$$

for all t_B-bounded subsets $\mathcal{M}$ of $\mathcal{D}$. The uniform topology of $\mathcal{L}^\dagger(\mathcal{D})$ is a direct generalization of the norm topology of the algebra of bounded linear operators on a Hilbert space, although the preceding seminorms are not C*-seminorms. This motivates the following example.

Example 4.2. Consider the $*$-algebra A from above, and let $\overline{A}$ denote the closure of A in $\mathcal{L}^\dagger(\mathcal{D})$ with respect to the uniform topology on $\mathcal{L}^\dagger(\mathcal{D})$. We remark that our derivation δ can be defined, with the same formula, on the whole $\mathcal{L}^\dagger(\mathcal{D})$, for which we retain the same symbol. Then $\delta(\overline{A}) \subset \overline{A}$, so that δ is a nonzero derivation of the commutative $*$-subalgebra $\overline{A}$ of $\mathcal{L}^\dagger(\mathcal{D})$.

In contrast to this fact, recall that commutative C*-algebras have no nonzero derivations.

If there is a $*$-subalgebra B of $\mathcal{L}^\dagger(\mathcal{D})$ which is also a GB*-algebra in some topology τ, and it contains A, then $\delta(\overline{A}^\tau) \subset \overline{A}^\tau$, where $\overline{A}^\tau$ denotes the τ-closure of A in B. Furthermore, $\overline{A}^\tau$ is a (commutative) GB*-algebra [2, Proposition 2.9], implying that there is a commutative GB*-algebra having a nonzero derivation. The authors currently do not know if such a GB*-algebra B exists.

5. Derivations of GB*-algebras with $A[B_0]$ a W*-algebra

In this section, we give some results about derivations of GB*-algebras whose bounded part is a W*-algebra. We first give some examples of such GB*-algebras below. The motivation for this section comes mainly from [3], [4], [9] and [13].

Example 5.1. ([22, Example 3.3], [5, p. 292]) If M is a von Neumann algebra with a faithful semifinite normal trace τ, then the algebra $A = L^\omega(M, \tau)$ of Example 2.4 is a GB*-algebra with $A[B_0] = M$. Therefore $A[B_0]$ is a W*-algebra.

Example 5.2. If M is a von Neumann algebra with a faithful finite normal trace τ, then the algebra $A = LS(M)$ (see Section 2), equipped with the topology of convergence in measure τ_{cm}, is a (not necessarily locally convex) GB*-algebra with $A[B_0] = M$ [33, Theorem 1.5.29]. Under reasonable conditions, the topology τ_{cm} above is a locally convex topology [14, Section 1.5], implying that A is a (locally convex) GB*-algebra with $A[B_0]$ a W*-algebra.

Example 5.3. If M is a finite von Neumann algebra, we denote by $\mathcal{F}$ the set of all faithful finite normal traces on M. Let $M_f = \bigcap_{\mu \in \mathcal{F}} L^\omega(M, \mu)$ (we refer to Example 2.4 for the latter notation). By [6, Theorem 3.1] and the remark thereafter, $A = M_f$ is a GB*-algebra with $A[B_0] = M$.

If A is an algebra, we will, from here on, use the notation $Z(A)$ to denote the center of A.

Proposition 5.4. *Let $A[\tau]$ be a GB*$-$algebra with $A[B_0]$ a W*$-$algebra. If $\delta : A \to A$ is a continuous derivation of A, then $\delta(xz) = \delta(x)z$, for all $x \in A$ and $z \in Z(A[B_0])$ (such a derivation is called $Z-$linear, with $Z = Z(A[B_0])$).*

Proof. Since $A[B_0]$ is $\tau-$dense in A [10, Theorem 2], we have that $Z(A[B_0]) \subset Z(A)$. Therefore, for a projection $p \in Z(A[B_0])$, we get that

$$\delta(p) = \delta(p^2) = \delta(p)p + p\delta(p) = 2p\delta(p) .$$

Therefore $p\delta(p) = 2p\delta(p)$, implying that $p\delta(p) = 0$, and hence $\delta(p) = 0$.

Let $z \in Z(A[B_0])$. Since $Z(A[B_0])$ is a W*$-$algebra, then z is the norm limit of the sequence $\left(\sum_{k=1}^{n} \lambda_{i_k} p_{i_k}\right)$, where $\lambda_{i_k} \in \mathbb{C}$ and p_{i_k} are projections in $Z(A[B_0])$ for all $i_k \in \mathbb{N}$. So for $x \in A$ and $z \in Z(A[B_0])$, it follows from the continuity of δ, and the fact that τ is weaker than the norm topology on $A[B_0]$, that

$$\begin{aligned}\delta(xz) &= \delta(x)z + x\delta(z) \\ &= \delta(x)z + x\delta\left(\lim_{n\to\infty}\sum_{k=1}^{n}\lambda_{i_k}p_{i_k}\right) \\ &= \delta(x)z + x\lim_{n\to\infty}\sum_{k=1}^{n}\lambda_{i_k}\delta(p_{i_k}) = \delta(x)z\,.\end{aligned}$$

∎

At this point, we remark that if $A[B_0]$ in Proposition 5.4 is a properly infinite W*-algebra, then the derivation $\delta : A \to A$ is automatically Z-linear without the assumption of continuity [9, Proposition 6.22] and [13, Theorem 1]. Results involving Z-linearity of derivations of locally measurable operators can be found in [3].

THEOREM 5.5. ([3] and [4]) *Let M be a type I von Neumann algebra with center Z, and let A be an arbitrary $*-$subalgebra of the $*-$algebra $LS(M)$ of locally measurable operators affiliated with M, such that A contains M. If δ is a $Z-$linear derivation of A, then δ is spatial, i.e., there exists $a \in LS(M)$ such that $\delta(x) = ax - xa$ for all $x \in A$.*

Any GB*-algebra, whose bounded part is a W*-algebra, is $*-$isomorphic to an EW$^*-$algebra [13, Corollary 2]. Moreover, every EW$^*-$algebra B over the von Neumann algebra M is a full $*-$subalgebra of $LS(M)$ [13, Theorem 1] (see Section 2 for the definition of $LS(M)$). The term full means that $1 \in B$ and if $y \in B$, $x \in LS(M)$ and $0 \leq x \leq y$, then $x \in B$.

Using these facts, Corollary 5.6, Corollary 5.10, Proposition 5.12 and Proposition 5.13, given below, are analogues of the corresponding results for measurable and locally measurable operators given in [4], [3] and [9]. We give the proofs for sake of completeness.

COROLLARY 5.6. *Let $A[\tau]$ be a GB*-algebra with $A[B_0]$ a type I von Neumann algebra, such that all derivations on A are continuous. Then A is identifiable with an EW$^*-$algebra B over the von Neumann algebra $M \cong A[B_0]$, such that all derivations of B are spatial and implemented by an element of $LS(M)$.*

Proof. From [13, Corollary 2], there exists an algebra $*$-isomorphism $\phi : A \to B$ of A onto B, where B is an EW$^*-$algebra over the von Neumann algebra M, say. Therefore B admits a GB$^*-$topology τ' such that $B[\tau']$ is a GB$^*-$algebra topologically $*-$isomorphic to A, with bounded part B_{bd}, say (see discussion immediately after Proposition 2.3): Let $(p_i)_{i\in I}$ denote a family of seminorms defining the GB*-topology on A, and let $q_i(\phi(x)) = p_i(x)$ for every $x \in A$. Then the family of seminorms $(q_i)_{i\in I}$ defines a locally convex topology τ on B, such that $\phi : A \to B$ is a topological-algebraic $*$-isomorphism. It now follows easily that $B[\tau']$ is a GB*-algebra.

By [13, Corollary 2], $M = B_{bd}$. Therefore, since $A \cong B$ and thus $A[B_0] \cong B_{bd}$ [16, Theorem 7.14], we get that $A[B_0] \cong M$. This last isomorphism implements the isomorphism $Z(A[B_0])$ with $Z(M)$.

Let now $\delta : B \to B$ be a derivation of B. We then have that the map $\delta_\phi : A \to A : \delta_\phi(a) = \phi^{-1}(\delta(\phi(a)))$, for all $a \in A$, is a derivation of A, thus continuous from the hypothesis. Then from Proposition 5.4, δ_ϕ is $Z(A[B_0])-$linear. So from $Z(A[B_0]) \cong Z(M)$, we have that δ is $Z(M)-$linear and thus from Theorem 5.5, δ is implemented by an element of $LS(M)$. ∎

The next result and Corollary 5.9 that follows inform us that the spatiality of a derivation in the previous corollary can in fact be improved to innerness.

Theorem 5.7. ([9, Proposition 5.17]) *Let B be a $*-$subalgebra of $LS(M)$ with $M \subset B$, such that if $x \in LS(M)$, $y \in B$ and $|x| \leq |y|$, then $x \in B$. If $w \in LS(M)$ is such that $wx - xw \in B$ for all $x \in B$, then there exists $v \in B$ such that $vx - xv = wx - xw$ for all $x \in B$.*

In proving Corollary 5.9, we need the following simple fact. Lemma 5.8 below is known and exists as Proposition 2.3.3 in the monograph [28], written in Russian. We include a proof for convenience of the reader.

Lemma 5.8. *If $x \in LS(M)$, then $|x| \in LS(M)$.*

Proof. Let $x = u|x|$ be the polar decomposition of x. Since x is affiliated with M, it follows from [25, Theorem 6.1.11] that $u \in M$ and that $|x|$ is affiliated with M. We note that since $|x|$ is closed, $|x| = u^*x = \overline{u^*x} = u^* \cdot x$ (see Definition 2.5). Now, since $M \subset LS(M)$, and given the fact that $LS(M)$ is a $*-$algebra [36, p. 260], we get that $|x| \in LS(M)$. ∎

Corollary 5.9. *Let $A[\tau]$ be a GB$^*-$algebra with $A[B_0]$ a type I W$^*-$algebra. If all derivations of A are continuous, then all derivations of A are inner.*

Proof. From Corollary 5.6, A is identifiable with an EW$^*-$algebra B over the von Neumann algebra $M \cong A[B_0]$, such that all derivations on B are spatial. Let $x \in LS(M)$, $y \in B$ and $|x| \leq |y|$. From Lemma 5.8, $|x| \in LS(M)$. Also from [21, Proposition 2.12], we get that $|y| \in B$. Recall that B is a full $*-$subalgebra of $LS(M)$. Therefore, we have that $|x| \in B$. By the polar decomposition of x, we then get that $x = u|x| \in MB \subset B$. It follows from Theorem 5.7 that every derivation of B is inner and thus every derivation of A is inner. ∎

Every commutative W*-algebra is of type I, and so the following result follows immediately from Corollary 5.9.

Corollary 5.10. *Let $A[\tau]$ be a commutative GB$^*-$algebra with $A[B_0]$ a W$^*-$algebra. Then the zero derivation is the only continuous derivation of A.*

If M is a type I von Neumann algebra, then, for any $x \in LS(M)$, there exists a sequence (z_n) of mutually orthogonal central projections in M such that $\bigvee_{n\in\mathbb{N}} z_n = 1$ and $z_n x \in M$ for all $n \in \mathbb{N}$. Let B be a $*$-subalgebra of $LS(M)$ such that $M \subset B$. If $D : B \to B$ is a derivation, then D can be extended to a derivation of $LS(M)$ by the formula $\tilde{D}(x) = \sum_{n=1}^{\infty} z_n D(z_n x)$, where $x \in LS(M)$ [3]. We summarize this in the following result, which we require in order to prove Proposition 5.12 and Proposition 5.13 below.

Proposition 5.11. ([3]) *Let M be a type I von Neumann algebra, and B a $*$-subalgebra of $LS(M)$ such that $M \subset B$. Then every derivation of B can be extended to a derivation of $LS(M)$.*

The following proposition shows that, under extra conditions, the continuity assumption for the derivation in the previous corollary can be dropped. We say that a von Neumann algebra M has an *atomic projection lattice* if for every nonzero projection $p \in M$, there exists a minimal projection $q \in M$ such that $q \leq p$.

Proposition 5.12. *Let $A[\tau]$ be a commutative GB$^*-$algebra such that $A[B_0]$ is a W$^*-$algebra having an atomic projection lattice. Then the zero derivation is the only derivation of A.*

Proof. By [13, Corollary 2 and Theorem 1], A is algebraically $*-$isomorphic to an EW$^*-$algebra B over a von Neumann algebra, say M, which is a full

$*-$subalgebra of $LS(M)$. By Proposition 5.11, every derivation of B can be extended to $LS(M)$. Since $LS(M)$ is commutative and M, being isomorphic with $A[B_0]$, has an atomic projection lattice, the zero derivation is the only derivation of $LS(M)$ ([8, Theorem 3.4] and [27, Theorem 2]). Therefore, B, and consequently A, has no nonzero derivations. ∎

An example of a GB*-algebra, with the hypothesis of the previous proposition, is Example 2.4, with the additional assumptions that M has an atomic projection lattice and is commutative.

Also, if (X, Σ, μ) is an atomic measure space satisfying the conditions of [14, Corollary 1.5.7(ii)], then, for $M = L_\infty(X, \Sigma, \mu)$, we have that $LS(M) = \{M_f : f \text{ finite almost everywhere}\}$ is also a GB*-algebra of the kind in Proposition 5.12.

If M is a von Neumann algebra of type I_∞, then every derivation $\delta : LS(M) \to LS(M)$ is inner [4]. This is needed in the proof of our next proposition.

PROPOSITION 5.13. *Let $A[\tau]$ be a GB*$-$algebra with $A[B_0]$ a type I_∞ W*$-$algebra. Then all derivations of A are inner and thus continuous.*

Proof. By [13, Corollary 2 and Theorem 1], A is algebraically $*-$isomorphic to an EW*$-$algebra B whose underlying von Neumann algebra is a type I_∞ von Neumann algebra $M \cong A[B_0]$, and B is a $*-$subalgebra of $LS(M)$. By Proposition 5.11, every derivation can be extended to a derivation of $LS(M)$, which is inner. Therefore every derivation of B is spatial in $LS(M)$. Thus from Theorem 5.7, every derivation of B is inner. ∎

If $M = L_\infty(X, \Sigma, \mu)\overline{\otimes}B(l^2)$, where (X, Σ, μ) is a localizable measure space, then M is a type I_∞ von Neumann algebra, and $LS(M) = L_0(X, \Sigma, \mu) \otimes B(l^2)$ is, under certain conditions (see [14, Section 1.5]), a GB*-algebra of the kind in Proposition 5.13.

Remark. An open problem is whether or not every derivation of a GB*-algebra is continuous. The authors are currently working on this problem for Fréchet GB*-algebras (see also [20, discussion after Theorem 5.2]).

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