EXTRACTA MATHEMATICAE Vol. **28**, Núm. 1, 13 – 31 (2013)

Spectrum and Numerical Range of a Compact Set

A. Bouchen, M.K. Chraïbi

Cadi Ayyad University, Faculty of Science Semlalia, Department of Mathematics, Marrakech, bouchen@uca.ma, chraibik@uca.ma

Presented by David Yost Received December 9, 2011

Abstract: In this paper, we define the multivalued entire series in a Banach algebra $\mathcal{A}$ as well as the exponential, the spectrum and the numerical range of a compact set of $\mathcal{A}$. We provide properties for these two sets which are also verified in the univalued case.

Key words: Banach algebra, Hausdorff distance, spectrum and numerical range.

AMS *Subject Class.* (2010): 30B10.

1. Introduction

The concept of the exponential of a set has been useful in the study of differential inclusions and Lipschitz selections. Firstly, it was considered (independently) by A. L. Dontchev and E. M. Farkhi [9] in 1989 and P. R. Wolenski [19] in 1990. In 2003, E. O. Ayoola has developed this concept for the study of quantum stochastic differential inclusions [3]. In 2006 and in various ways the extension of multivalued case exponential function was developed in [1], [5] and [6].

At the begining of this paper, we study the multivalued entire series $S(K) = \sum a_n K^n$ (where K is in $\mathbb{K}(\mathcal{A})$, the set of all compact sets of a Banach algebra $\mathcal{A}$) which is used to define e^K.

Then, for $K \in \mathbb{K}(\mathcal{A})$, we define $\sigma(K)$, the spectrum of K, as the union of all spectrum $\sigma(a)$ when a runs K. If $\mathcal{A} = \mathcal{B}(H)$, i.e., the set of all bounded linear operators on a complex Hilbert space H, and K is in $\mathbb{K}(\mathcal{B}(H))$, we define $W(K)$, the numerical range of K, as the convex hull of the union of $W(A)$ when A varies over K and

$$W(A) = \big\{ \langle Ax, x\rangle : \|x\| = 1 \big\}.$$

The last set is called the numerical range of A which is always a convex set of $\mathbb{C}$ whose closure contains the convex hull of $\sigma(A)$ or $co\sigma(A)$ [14]. In general, in the noncommutative case, the spectrum is not continuous with respect to the

Hausdorff metric [2]. (For more recent work on this topic, see, for example, [18]). We show a range of properties for $\sigma(K)$ and $W(K)$ which are verified in the single valued case, such as continuity of the numerical range in the sense of Hausdorff [8] and the continuity of the spectrum in the case where $\mathcal{A}$ is commutative. We also show for $K \in \mathbb{K}(\mathcal{B}(H))$ that:

$$|K| \leq 2\omega(K) - \frac{\omega'^2(K)}{|K|}, \tag{1}$$

where

$$\omega'(K) = \inf\{\|z\| : z \in W(A),\ A \in K\},$$

and

$$\omega(K) = \sup\{\|z\| : z \in W(A),\ A \in K\},$$

is the K numerical radius. The last inequality is optimal and generalized in the single valued case the following classical inequality [13]:

$$\|A\| \leq 2\omega(A), \quad A \in \mathcal{B}(H).$$

As an application of (1) we show that for K and K' in $\mathbb{K}(\mathcal{B}(H))$

$$|KK'| \leq \left(w(K) - \frac{w'^2(K)}{2|K|}\right)|K'| + \left(w(K') - \frac{w'^2(K')}{2|K'|}\right)|K|. \tag{2}$$

The previous inequality is an improvement in the single valued case of the following theorem from Dragomir [10]:

Theorem 1. ([10]) *Let $A, B \in \mathcal{B}(H)$ and $\alpha, \beta, \gamma, \lambda \in \mathbb{C}$ be such that for every $x \in H$,*

$$\langle (A^* - \overline{\alpha}I)(\beta I - A)x, x\rangle \geq 0 \text{ and } \langle (B^* - \overline{\gamma}I)(\lambda I - B)x, x\rangle \geq 0.$$

Then,

$$\|AB\| \leq w(A)\|B\| + w(B)\|A\| + w(A)\,w(B) + \frac{1}{4}|\beta - \alpha|\,|\lambda - \gamma|. \tag{3}$$

In [11], and [12], Dragomir said that's an open problem whether or not the constant $\frac{1}{4}$ is best possible in the inequality (3). The inequality (2) is the solution of this problem.

Dragomir in 2008 [11] showed that

$$\|A\|^2 \leq \omega^2(A) + d^2(A), \quad A \in \mathcal{B}(H),$$

with

$$d(A) = \sup\big\{\|\langle Ax, y\rangle\| : \|x\| = \|y\| = 1,\ \langle x, y\rangle = 0\big\}.$$

We also generalize this result in the set valued case by showing that for $K, K' \in \mathcal{B}(H)$

$$\omega(KK') \le \omega(K)\omega(K') + d(K)d(K'),$$

where

$$d(K) = \sup\big\{d(A) :\ A \in K\big\}.$$

Finally, when

$$\mathbb{K}_1(\mathcal{A}) = \big\{K \in \mathbb{K}(\mathcal{A}) : \forall a, b \in K,\ ab = ba\big\},$$

we show the following spectral theorem:

THEOREM 2. *For each $K \in \mathbb{K}_1(\mathcal{A})$, we have*

$$\sigma\big(S(K)\big) \subset S\big(\sigma(K)\big).$$

2. Definitions and preliminaries

In this paper $\mathcal{A}$ is a Banach algebra over $\mathbb{C}$, with unit element I. The following definitions are useful in the sequel.

DEFINITION 3. Let K and K' be two elements of $\mathbb{K}(\mathcal{A})$ and α a complex number. We denote

$$\begin{aligned}
K \cdot K' &= \{x \cdot y : x \in K,\ y \in K'\},\\
K + K' &= \{x + y : x \in K,\ y \in K'\},\\
\alpha K &= \{\alpha I\} \cdot K = \{\alpha \cdot x : x \in K\},\\
\alpha + K &= \{\alpha I\} + K = \{\alpha I + x : x \in K\},\\
|K| &= \sup_{X \in K} \|X\|,\\
K^0 &= \{I\}, \quad K^n = K \cdot K^{n-1},\ \forall n \in \mathbb{N}^*.
\end{aligned}$$

We note that in general $K \cdot K'$ is not equal to $K' \cdot K$ and $K^n = K^p K^q$, with $p + q = n$ and $p, q, n \in \mathbb{N}$.

DEFINITION 4. Let $K, K' \in \mathbb{K}(\mathcal{A})$. The Hausdorff distance between K and K' denoted by $h(K, K')$ is the maximum of the excess $e(K, K')$ and $e(K', K)$ where

$$e(K, K') = \sup_{X \in K} \inf_{Y \in K'} \|X - Y\|.$$

Definition 5. Let F be a multifunction from $\mathcal{A}$ into $\mathbb{K}(\mathcal{A})$ and let $X_0 \in \mathcal{A}$. F is called Hausdorff upper semicontinuous at X_0 ("F is Hscs" at X_0) if for any sequence $(X_n)_{n\in\mathbb{N}}$ of elements of $\mathcal{A}$, which converges to X_0, we have

$$\forall\epsilon > 0,\ \exists N \in \mathbb{N} \text{ such that } \forall n \geq N,\ F(X_n) \subset F(X_0) + B(0, \epsilon), \qquad (4)$$

where $B(0, \epsilon)$ is the open ball in $\mathcal{A}$ with center 0 and radius ϵ.

It follows immediately from (4) that

$$\forall\epsilon > 0,\ \exists\eta > 0 \text{ such that } \forall X \in B(X_0, \eta),\ e\big(F(X), F(X_0)\big) \leq \epsilon. \qquad (5)$$

3. Multivalued power series in $\mathcal{A}$

Definition 6. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers, and let $K \in \mathbb{K}(\mathcal{A})$. We set

$$S_n(K) = \sum_{i=0}^{n} a_i K^i = a_0 + a_1 K + a_2 K^2 + \cdots + a_n K^n = \left\{ \sum_{i=0}^{n} a_i x_i : x_i \in K^i \right\}.$$

Definition 7. Let $K \in \mathbb{K}(\mathcal{A})$ be such that the sequences $\sum_{i=0}^{n} a_i x_i$ converges for all $x_i \in K^i$. We set

$$S(K) = \left\{ \sum_{n=0}^{+\infty} a_n x_n : x_n \in K^n \right\} = \sum_{i=0}^{\infty} a_n K^n.$$

In the remainder of this section, K denotes an element of $\mathbb{K}(\mathcal{A})$ and $(a_n)_{n\in\mathbb{N}}$ a sequence of complex numbers such that

$$\sum_{n=0}^{+\infty} a_n x_n \text{ converges and } \forall n \in \mathbb{N},\ x_n \in K^n.$$

Theorem 8. *Let r be the radius of convergence of the complex power series $\sum a_n z^n$. If $K \in \mathbb{K}(\mathcal{A})$, with $K \subset B(0, \delta)$ and $0 < \delta < r$, then $S(K)$ is a compact set of $\mathcal{A}$.*

Proof. Let $(Y_p)_{p\in\mathbb{N}}$ be a sequence of elements of $S(K)$. We show that $(Y_p)_{p\in\mathbb{N}}$ admits a subsequence $(Y_{\varphi(p)})_{p\in\mathbb{N}}$ which converges in $S(K)$. For all $p \in \mathbb{N}$, we have

$$Y_p = \sum_{i=0}^{+\infty} a_i X_{i,p},$$

with $X_{i,p} \in K^i$ and $X_{0,p} = I$. We set

$$Z_p = \big(a_0 X_{0,p},\, a_1 X_{1,p},\, \ldots, a_i X_{i,p},\, \ldots\big) \in \prod_{i=0}^{\infty} a_i K^i.$$

This set is a compact set product. By Tychonov theorem [17], this is a compact set for the norm $\|\cdot\|_\pi$, where for all p in $\mathbb{N}$,

$$\|Z_p\|_\pi = \sum_{i=0}^{\infty} \frac{1}{2^{i+1}} \min\big\{1, \|a_i X_{i,p}\|\big\}.$$

We extract a subsequence $(Z_{\varphi(p)})_{p\in\mathbb{N}}$ which converges to

$$Z = (a_0 X_0, a_1 X_1, \ldots, a_i X_i, \ldots) \in \prod_{i=0}^{\infty} a_i K^i.$$

Let us show that $(Y_{\varphi(p)})_{p\in\mathbb{N}}$ converges to $Y = \sum_{i=0}^{\infty} a_i X_i$. Let $\varepsilon \in\,]0,1[$. The sequence $(Z_{\varphi(p)})_{p\in\mathbb{N}}$ converges to Z, and then, for all $\varepsilon_1 > 0$, there exists $p_1 > 0$ such that for all $p > p_1$,

$$\sum_{n=0}^{+\infty} \frac{1}{2^{n+1}} \min\big\{1, \big\|a_n X_{n,\varphi(p)} - a_n X_n\big\|\big\} \le \varepsilon_1,$$

and then,

$$\frac{1}{2^{n+1}} \min\big\{1, \big\|a_n X_{n,\varphi(p)} - a_n X_n\big\|\big\} \le \varepsilon_1$$

for any $n \ge 0$. Since $\delta < r$, $\sum_{n=0}^{+\infty} |a_n \delta^n|$ is convergent. Thus, there exists $n_2 > 0$ such that for all $n \ge n_2$,

$$\sum_{i=n+1}^{+\infty} |a_i \delta^i| \le \frac{\varepsilon}{3}.$$

Let $\varepsilon_1 = \frac{1}{3} \frac{1}{2^{n_2+1}} \frac{\varepsilon}{n_2+1}$. Then, there exists p_{n_2} such that $\frac{1}{2^{n+1}} > \varepsilon_1$ and

$$\frac{1}{2^{n+1}} \min\big\{1, \big\|a_n X_{n,\varphi(p)} - a_n X_n\big\|\big\} = \frac{\big\|a_n X_{n,\varphi(p)} - a_n X_n\big\|}{2^{n+1}} \le \varepsilon_1,$$

for all $p > p_{n_2}$ and $n \le n_2$. Then, for all $n \le n_2$,

$$\big\|a_n X_{n,\varphi(p)} - a_n X_n\big\| \le \frac{\varepsilon}{3(n_2+1)},$$

and thus, for all $p > p_{n_2}$,

$$\begin{aligned}\left\|Y_{\varphi(p)} - Y\right\| &\leq \sum_{n=0}^{n_2} \left\|a_n X_{n,\varphi(p)} - a_n X_n\right\| + \sum_{n=n_2+1}^{+\infty} \left\|a_n X_{n,\varphi(p)}\right\| + \sum_{n=n_2+1}^{+\infty} \|a_n X_n\| \\ &\leq \sum_{n=0}^{n_2} \left\|a_n X_{n,\varphi(p)} - a_n X_n\right\| + \frac{2}{3}\varepsilon \\ &\leq \sum_{n=0}^{n_2} (n_2+1)\frac{\varepsilon}{3(n_2+1)} + \frac{2}{3}\varepsilon = \varepsilon.\end{aligned}$$

∎

DEFINITION 9. Let $K \in \mathbb{K}(\mathcal{A})$. We define the set valued exponential of K, denoted e^K, by

$$e^K = \sum_{n=0}^{+\infty} \frac{1}{n!} K^n = \left\{ \sum_{n=0}^{+\infty} \frac{1}{n!} x_n : \forall n \in \mathbb{N}, \quad x_n \in K^n \right\}.$$

Remark 10. Since the radius of convergence of complex series $\sum \frac{z^n}{n!}$ is infinite, then for every $K \in \mathbb{K}(\mathcal{A})$, e^K is well defined. Using Theorem 8, e^K is in $\mathbb{K}(\mathcal{A})$.

THEOREM 11. *Let $K \in \mathbb{K}(\mathcal{A})$, with $K \subset B(0,\delta)$, r the radius of convergence of the complex power series $\sum a_n z^n$ and $0 < \delta < r$. Then, the sequence $S_n(K)$ converges in the sense of Hausdorff to $S(K)$.*

Proof. Let $Y_n \in S_n(K)$ and $Y \in S(K)$, with $Y_n = \sum_{i=0}^{n} a_i x_i$, $Y = Y_n + \sum_{i=n+1}^{\infty} a_i x_i$, and $x_i \in K^i$ for all $i \in \mathbb{N}$. We have

$$\|Y - Y_n\| \leq \sum_{i=n+1}^{\infty} |a_i|\, \delta^i,$$

and then

$$h\big(S(K), S_n(K)\big) \leq \sum_{i=n+1}^{\infty} |a_i|\, \delta^i.$$

Hence the result. ∎

The following lemma is useful in the proof of Theorem 13.

Lemma 12. *Let* $\sum a_n z^n$ *be a complex entire series. Then for any* $n \in \mathbb{N}$, *the mapping* S_n *from* $\mathbb{K}(\mathcal{A})$ *to* $\mathbb{K}(\mathcal{A})$, *which associates to each* K *the set* $S_n(K)$, *is continuous in the sense of Hausdorff.*

Proof. It is easy to see that the product and sum of two compact sets of $\mathcal{A}$ are compact sets. For the continuity of S_n, it suffices to show that if $(K_p)_{p\in\mathbb{N}}$ and $(K'_p)_{p\in\mathbb{N}}$ are two sequences of compact set of $\mathcal{A}$ which converge in the sense of Hausdorff respectively to two compact set K and K' then the sequences $(K_pK'_p)_{p\in\mathbb{N}}$ and $(K_p + K'_p)_{p\in\mathbb{N}}$ converge in the sense of Hausdorff respectively to KK' et $K + K'$.

By the triangle inequality, we have

$$h(K_pK'_p, KK') \leq |K_p|\, h(K'_p, K') + \left|K'\right| h(K_p, K).$$

The sequence $(K_p)_{p\in\mathbb{N}}$ is convergent, and therefore $(|K_p|)_{p\in\mathbb{N}}$ is bounded from above. As a result, $(K_pK'_p)_{p\in\mathbb{N}}$ converges to KK'.

For the other convergence, by triangle inequality, we have

$$h(K_p + K'_p, K + K') \leq h(K'_p, K') + h(K_p, K).$$

∎

Theorem 13. *Let* r *be the radius of convergence of the complex entire series* $\sum a_n z^n$ *and* $\delta < r$. *Then the mapping* $S : \mathbb{K}(\mathcal{A}) \to \mathbb{K}(\mathcal{A})$, *which to* $K \subset B(0, \delta)$ *associates* $S(K)$, *is continuous in the sense of Hausdorff.*

Proof. Let us consider a sequence $(K_p)_{p\in\mathbb{N}}$ of compact sets of $\mathcal{A}$ included in $B(0, \delta)$, which converges in the sense of Hausdorff to a compact set K. Let us show that $h(S(K_p), S(K))$ tends to 0.

The series $\sum |a_p|\, \delta^p$ is convergent, and so the sequence $R_n = \sum_{p=n}^{\infty} |a_p|\, \delta^p$ tends to 0. Thus, for all $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$\sum_{p=n}^{\infty} |a_p|\, \delta^p \leq \frac{\varepsilon}{3}.$$

Hence

$$\begin{aligned} h\big(S(K_p), S(K)\big) \leq{}& h\big(S(K_p), S_{n_0}(K_p)\big) + h\big(S_{n_0}(K_p), S_{n_0}(K)\big) \\ &+ h\big(S_{n_0}(K), S(K)\big). \end{aligned}$$

By Lemma 12, the mapping S_{n_0} is continuous, and so for all $\varepsilon > 0$, there exists $p_0 \in \mathbb{N}$ such that for all $p \geq p_0$,

$$h\big(S_{n_0}(K_p), S_{n_0}(K)\big) \leq \frac{\varepsilon}{3}.$$

We have

$$h\big(S(K_p), S_{n_0}(K_p)\big) \leq \sum_{p=n}^{\infty} |a_p|\, \delta^p \leq \frac{\varepsilon}{3}.$$

And, similarly, for $h\big(S_{n_0}(K), S(K)\big)$. Thus, for every $p \geq p_0$, $h(S(K_p), S(K)) \leq \varepsilon$. ∎

4. Spectrum and numerical range of a compact set

Definition 14. Let K be an element of $\mathbb{K}(\mathcal{A})$. We define the spectrum of K, denoted $\sigma(K)$, and the algebraic numerical range of K, denoted $V(K)$, by:

$$\sigma(K) = \big\{\lambda \in \mathbb{C} : \exists X \in K,\ \lambda \in \sigma(X)\big\} = \bigcup_{X \in K} \sigma(X)$$

and

$$V(K) = co\big\{\emptyset(t) : \emptyset \in S(\mathcal{A}),\ t \in K\big\},$$

respectively, with

$$S(\mathcal{A}) = \big\{\emptyset \in \mathcal{A}^* : \emptyset(I) = \|\emptyset\| = 1\big\},$$

and $\sigma(X)$ the spectrum of X. Therefore, we have

$$V(K) = co \bigcup_{t \in K} V(t),$$

where

$$V(t) = \big\{\emptyset(t) : \emptyset \in S(\mathcal{A})\big\}.$$

The last set is called the algebraic numerical range of t in the single-valued case, which is always a closed and convex set in $\mathbb{C}$ [16]. It is also located in the disk with center 0 and radius $\|t\|$, and satisfies $V(A) = \overline{W(A)}$ for all $A \in \mathcal{B}(H)$ [4].

Definition 15. If $\mathcal{A} = \mathcal{B}(H)$, we define the numerical domain of K by:

$$W(K) = co\big\{\, \langle Ax, x\rangle : \|x\| = 1,\ A \in K\big\} = co \bigcup_{A \in K} W(A).$$

For $K \in \mathbb{K}(\mathcal{A})$, we define the numerical radius of K, denoted $\omega(K)$, and the spectral radius of K, denoted $\rho(K)$, by:

$$\omega(K) = |V(K)| \text{ and } \rho(K) = |\sigma(K)|.$$

Similarly, if $\mathcal{A} = \mathcal{B}(H)$, the numerical radius of K is

$$\omega(K) = |W(K)|.$$

THEOREM 16. *If $K \in \mathbb{K}(\mathcal{A})$, then $\sigma(K)$ is a compact set in $\mathbb{C}$.*

The proof of this theorem is a consequence of Lemma 17 since in the single valued case, the spectrum mapping from $\mathcal{A}$ to $\mathbb{K}(\mathbb{C})$ is Husc [2].

LEMMA 17. *Let $(E, \| \|)$ be a normed space, F a Husc multifunction from $\mathcal{A}$ into $\mathbb{K}(E)$ and K a compact set of $\mathcal{A}$. Assume that there exists $\alpha > 0$ such that for all $x \in K$, $|F(x)| \leq \alpha \|x\|$. Then, $D = \cup F(x)$ is a closed bounded subset of E.*

Proof. D is bounded since for all $\lambda \in D$ there exists $x \in K$ such that $\lambda \in F(x)$. Thus $\|\lambda\| \leq |F(x)| \leq \alpha\,|K|$. D is closed since if $(\lambda_n)_{n\in\mathbb{N}}$ is a sequence of elements of D which converges to $\lambda \in E$, then for all $n \in \mathbb{N}$, there exists $x_n \in K$ such that $\lambda_n \in F(x_n)$. Let (x_{n_k}) be a subsequence of (x_n) which converges to $\overline{x}$ in K. Let us show that $\lambda \in F(\overline{x})$. For this, it suffices to prove that $e(\{\lambda\}, F(\overline{x})) = 0$ since $F(\overline{x})$ is a compact set. Fix $\varepsilon > 0$.

1) Since $\lambda_{n_k} \longrightarrow \lambda$, then there exists $N_0 \in \mathbb{N}$ such that for all $k \geq N_0$, $\|\lambda - \lambda_{n_k}\| \leq \frac{\varepsilon}{2}$.
2) By the inequality (5) and since F is Hscs at $\overline{x}$, then there exists $\eta > 0$ such that for all $x \in B(\overline{x}, \eta)$, $e(F(x), F(\overline{x})) \leq \frac{\varepsilon}{2}$.
3) Also $x_{n_k} \longrightarrow \overline{x}$ ensures that there exists $N_1 \in \mathbb{N}$ such that for all $k \geq N_1$, $x_{n_k} \in B(\overline{x}, \eta)$.

Take $k \geq \max(N_0, N_1) = N_2$, and use 1) and 2). We deduce that for all $k \geq N_2$, $e(F(x_{n_k}), F(\overline{x})) \leq \frac{\varepsilon}{2}$, and, consequently, for all $\epsilon > 0$ and all $k \geq N_2$,

$$\begin{aligned} e\big(\{\lambda\}, F(\overline{x})\big) &\leq \|\lambda - \lambda_{n_k}\| + e\big(\{\lambda_{n_k}\}, F(\overline{x})\big) \\ &\leq \|\lambda - \lambda_{n_k}\| + e\big(F(x_{n_k}), F(\overline{x})\big) \leq \varepsilon. \end{aligned}$$

Thus, $\lambda \in F(\overline{x})$. ∎

Definition 18. Let $K \in \mathbb{K}(\mathcal{B}(H))$. We say that K is positive (resp. self adjoint, normal) if each element of K is positive (resp. self adjoint, normal).

In the following Propositions 19 and 20 we show some properties for the spectral mapping and the numerical range of a compact set in $\mathcal{A}$ which are also verified in the case of single valued mappings.

Proposition 19. *Consider $K, K' \in \mathbb{K}(\mathcal{A})$ and $\alpha, \beta \in \mathbb{C}$. Then*

1) $\sigma(\alpha K + \beta K') \subset \alpha\sigma(K) + \beta\sigma(K')$, *if $ab = ba$ for all $(a, b) \in K \times K'$.*
2) $V(\alpha K + \beta K') \subset \alpha V(K) + \beta V(K')$.

If $\mathcal{A} = \mathcal{B}(H)$, we further have

3) $W(\alpha K + \beta K') \subset \alpha W(K) + \beta W(K')$.
4) $w(K) = 0 \Leftrightarrow K = \{0\}$.
5) $co\sigma(K) \subset \overline{W(K).}$
6) *If K is positive (resp. self adjoint), then $W(K) \subset \mathbb{R}^+$ (resp. $W(K) \subset \mathbb{R}$).*

Proof. Since $\sigma(\alpha a + \beta b) \subset \alpha\sigma(a) + \beta\sigma(b)$, $V(\alpha a + \beta b) \subset \alpha V(a) + \beta V(b)$ for $a, b \in \mathcal{A}$, and $W(\alpha A + \beta B) \subset \alpha W(A) + \beta W(B)$ for $A, B \in \mathcal{B}(H)$, then 1), 2) and 3) are fulfilled. Property 4) can be obtained from the fact that if $A \in \mathcal{B}(H)$, then $w(A) \leq \|A\| \leq 2w(A)$ [13]. Thus

$$w(A) = 0 \Leftrightarrow A = 0.$$

Property 5) is deduced from $co\sigma(A) \subset \overline{W(A)}$ if $A \in \mathcal{B}(H)$ [14]. Finally, the last property is trivial. ∎

Proposition 20. *Let $K, K' \in \mathbb{K}(\mathcal{A})$ be such that $ab = ba$ for all $(a, b) \in K \times K'$. Then*

1) $\sigma(KK') \subset \sigma(K)\sigma(K')$.
2) *If further $\mathcal{A} = \mathcal{B}(H)$ and K or K' is normal, then we have $\overline{W(KK')} \subset \overline{coW(K)W(K')}$.*

Proof. 1) is deduced from $\sigma(ab) \subset \sigma(a)\sigma(b)$ if $(a, b) \in K \times K'$ and $ab = ba$. If $A, B \in \mathcal{B}(H)$, $AB = BA$ and A or B is normal, then $\overline{W(AB)} \subset \overline{coW(A)W(B)}$ [7]. Thus, 2). ∎

EXAMPLE 21. In this example, we have $K = K'$, $KK' = K'K$, but the elements of K do not commute with each other. As a consequence, Proposition 20 is not verified. Indeed, if $K = \{A, B\}$, with

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

we have $\sigma(KK') = \{0, 1, 3, 4\}$, $\sigma(K)\sigma(K') = \{0, 1, 2, 4\}$. If $x = \frac{1}{\sqrt{2}}$ and $y = \frac{i}{\sqrt{2}}$, then $\langle AB\binom{x}{y}, \binom{x}{y}\rangle = \frac{3-i}{2} \in W(AB) \subset W(KK')$ and $coW(K)W(K') = [0, 4]$.

DEFINITION 22. An operator A in $B(H)$ is said to be convexoid (resp. normaloid, spectraloid) if $\overline{W(A)} = co\sigma(A)$ (resp. $w(A) = \|A\|$, $|\sigma(A)| = w(A)$).

DEFINITION 23. Let $K \in \mathbb{K}(\mathcal{B}(H))$, we say that K is a convexoid (resp. normaloid, spectraloid) if each element of K is a convexoid (resp. normaloid, spectraloid).

The following lemma, whose proof is obvious, is useful to demonstrate Proposition 25.

LEMMA 24. *Let $(\Gamma_i)_{i\in J}$ be a family of subsets of $\mathbb{C}$ which indexed by a set J. We have:*

$$co\overline{\Gamma_i} = \overline{co\Gamma_i}, \quad \overline{\bigcup_{i\in J}\overline{\Gamma_i}} = \overline{\bigcup_{i\in J}\Gamma_i}, \quad \textit{and} \quad co\bigcup_{i\in J} co\Gamma_i = co\bigcup_{i\in J}\Gamma_i.$$

PROPOSITION 25. *Let $K \in \mathbb{K}(\mathcal{B}(H))$ be a convexoid (resp. normaloid, spectraloid), then $\overline{W(K)} = co\sigma(K)$ (resp. $w(K) = |K|$, $|\sigma(K)| = w(K)$).*

Proof. In this proof we use the three equalities in the previous lemma. We consider only the case where K is a convexoid. The other two cases are obvious. For every $A \in K$, we have $\overline{W(A)} = co\sigma(A)$. So

$$\bigcup_{A\in K} co\sigma(A) = \bigcup_{A\in K}\overline{W(A)},$$

and

$$\overline{\bigcup_{A\in K} co\sigma(A)} = \overline{\bigcup_{A\in K}\overline{W(A)}}.$$

As a result, we have

$$\overline{co\bigcup_{A\in K} co\sigma(A)} = \overline{co\bigcup_{A\in K} \overline{W(A)}}.$$

This means

$$\overline{co\bigcup_{A\in K} co\sigma(A)} = \overline{co\bigcup_{A\in K} W(A)},$$

and thus

$$\overline{co\bigcup_{A\in K} \sigma(A)} = \overline{co\bigcup_{A\in K} W(A)}.$$

This implies that

$$\overline{co\sigma(K)} = \overline{W(K)}.$$

By Theorem 16, $\sigma(K)$ is closed, so it is the same for $co\sigma(K)$, and hence the desired equality. ∎

The following theorem shows the continuity of the multifunction $\overline{W(K)}$ and generalizes the univocal case [8].

THEOREM 26. *Let K_n be a sequence in $\mathbb{K}(\mathcal{B}(H))$ which converges in the Hausdorff sense to an element K of $\mathbb{K}(\mathcal{B}(H))$, then $\overline{W(K_n)}$ converges to $\overline{W(K)}$ in the sense of Hausdorff.*

Proof. We have

$$e(K_n, K) = \sup_{x\in K_n} d\,(x, K) \longrightarrow 0, \quad \text{with} \quad d(x,K) = e(\{x\}, K).$$

The continuity of the mapping $x \mapsto d(x,K)$ and the fact that K_n and K are compact set imply the existence of $x_n \in K_n$ and $z_n \in K$ such that:

$$e(K_n, K) = \|x_n - z_n\| \to 0.$$

We also have

$$\begin{aligned} e\big(\overline{W(K_n)}, \overline{W(K)}\big) &\le e\big(\overline{W(K_n)}, \overline{W\{z_n\}}\big) \\ &= \sup\big\{d\big(\alpha_n, \overline{W\{z_n\}}\big), \alpha_n \in \overline{W(K_n)}\big\} \\ &= d\big(t_n, \overline{W\{z_n\}}\big), \end{aligned}$$

with

$$t_n \in \overline{W(K_n)} = \bigcup_{A \in K_n} \overline{W\{A\}}.$$

Then

$$e\big(\overline{W(K_n)}, \overline{W(K)}\big) \le e\big(\overline{W(A)}, W\{z_n\}\big),$$

where

$$A \in K_n \quad \text{and} \quad t_n \in \overline{W(A)}.$$

And thus

$$e\big(\overline{W(K_n)}, \overline{W(K)}\big) \le \|A - z_n\| \le \|y_n - z_n\| \to 0.$$

∎

Proposition 27. *Let $K, K' \in \mathbb{K}(\mathcal{A})$. Suppose that for all $A \in K$ and $B \in K'$, $AB = BA$. Then*

$$h\big(\sigma(K), \sigma(K')\big) \le h(K, K').$$

Proof. The continuity of the norm in $\mathcal{A}$ and the compactness of K and K' provide

$$e(K, K') = \|y - z\|, \quad y \in K \text{ and } z \in K'.$$

We have

$$e\big(\sigma(K), \sigma(K')\big) \le e\big(\sigma(K), \sigma(z)\big) = e\big(\sigma(\{t_n\}), \sigma(z)\big),$$

where $t_n \in \sigma(K)$. Then, there exists $A \in K$ such that $t_n \in \sigma(A)$, and

$$\begin{aligned} e\big(\sigma(K), \sigma(K')\big) &\le e\big(\sigma(A), \sigma(z)\big), \\ &\le \|A - z\| \qquad ([2]) \\ &\le \|y - z\| = e\big(K, K'\big) \\ &\le h(K, K'). \end{aligned}$$

∎

The following corollary is satisfied in the univocal case [2, page 49].

Corollary 28. *Let $K_n, K \in \mathbb{K}(\mathcal{A})$ be such that for all $a_n \in K_n$ and all $b \in K$, $a_n b = b a_n$. If the sequence (K_n) converges in the sense of Hausdorff to K, then $\sigma((K_n))$ converges in the sense of Hausdorff to $\sigma(K)$.*

Definition 29. For $K \in \mathbb{K}(\mathcal{B}(H))$ we set

$$O(K) = \{\langle Ax, y\rangle : A \in K, \ \|x\| = \|y\| = 1, \ \langle x, y\rangle = 0\}$$

and

$$d(K) = \sup_{z \in O(K)} |z| = |O(K)|.$$

Proposition 30. *$O(K)$ is a disk centered at the origin and with radius $d(K)$.*

Proof. For all $A \in K$, $O(\{A\})$ is a disk centered at the origin and with radius $d(\{A\}) = \sup_{z \in O(\{A\})} |z|$, [8]. We have

$$O(K) = \bigcup_{A \in K} O(\{A\}) \text{ and } d(K) \leq |K|.$$

Then $O(K)$ is a disk centered at the origin and with radius $d(K)$. ∎

Proposition 31. *For $K \in \mathbb{K}(\mathcal{B}(H))$, we have*

$$d(K) = \inf_{\lambda \in \mathbb{C}} |K - \lambda\{I\}|.$$

Proof. Since

$$d(\{A\}) = \inf_{\lambda \in \mathbb{C}} \|A - \lambda\{I\}\| \leq \inf_{\lambda \in \mathbb{C}} |K - \lambda\{I\}|,$$

then

$$d(K) = \sup_{A \in K} d(\{A\}) \leq \inf_{\lambda \in \mathbb{C}} |K - \lambda\{I\}|.$$

For the reverse, we have that for all $\lambda \in \mathbb{C}$ and all $A \in K$,

$$|K - \lambda\{I\}| \geq \|A - \lambda\{I\}\|,$$

and then, for all $A \in K$,

$$\inf_{\lambda \in \mathbb{C}} |K - \lambda\{I\}| \geq d(A).$$

Thus

$$d(K) \leq \inf_{\lambda \in \mathbb{C}} |K - \lambda\{I\}|.$$

∎

PROPOSITION 32. *For $K \in \mathbb{K}(\mathcal{B}(H))$ we have*

$$|K| \leq 2w(K) - \frac{w't(K)}{|K|},$$

where

$$w'(K) = \inf\big\{|z| \in W(A) : \ A \in K\big\}.$$

Proof. Remark that

$$Ax = \langle Ax, x\rangle x + \langle Ax, y\rangle y, \quad \text{with } \langle x, y\rangle = 0,$$

then

$$\begin{aligned}\langle Ax, Ax\rangle &= \langle Ax, x\rangle\langle x, Ax\rangle + \langle Ax, y\rangle\langle y, Ax\rangle \\ &= |\langle Ax, x\rangle|^2 + |\langle Ax, y\rangle|^2.\end{aligned}$$

The product operator $M_{2,A,B}$ defined on the Hilbert-Schmidt space $C_2\,(H)$, fitted with the scalar product

$$\langle X, Y\rangle = trXY,$$

is given by

$$M_{2,A,B}\,(X) = AXB, \quad A, B \in \mathcal{B}(H),$$

and satisfies [15]

$$w\,(M_{2,A,B}) \leq w\,(A)\,\|B\|\,.$$

Set

$$X = \frac{\sqrt{2}}{2}x \otimes x + \frac{\sqrt{2}}{2}y \otimes y.$$

Then the norm of X in $C_2(H)$ is equal to 1. Then we have

$$\begin{aligned}\big\langle M_{2,A^*,A}(X), X\big\rangle &= \frac{1}{2}|\langle Ax, x\rangle|^2 + \frac{1}{2}|\langle Ax, y\rangle|^2 + \frac{1}{2}|\langle Ay, x\rangle|^2 + \frac{1}{2}|\langle Ay, y\rangle|^2 \\ &= \frac{1}{2}\|Ax\|^2 + \frac{1}{2}|\langle Ay, y\rangle|^2 + \frac{1}{2}|\langle Ax, x\rangle|^2 \\ &\leq w(A)\|A\|.\end{aligned}$$

Thus

$$\|Ax\|^2 \leq 2w(A)\|A\| - |\langle Ay, y\rangle|^2,$$

and

$$\|A\|^2 \leq 2w(A)\|A\| - w'^2(A).$$

We conclude

$$\|A\| \leq 2w(A) - \frac{w'^2(A)}{\|A\|},$$

and

$$\sup_{A\in K} \|A\| \leq 2 \sup_{A\in K} w(A) - \frac{\inf_{A\in K} w'^2(A)}{\sup_{A\in K} \|A\|},$$

that is to say

$$|K| \leq 2w(K) - \frac{w'^2(K)}{|K|}. \tag{6}$$

∎

In the single valued case the inequality (6) generalizes the following inequality [13]:

$$\|A\| \leq 2w(A). \tag{7}$$

COROLLARY 33. *If $w'(K) \neq 0$, then*

$$|K| < 2w\,(K)\,.$$

In the following example we have equality in (6) but not in (7): let $r > 0$, then for $K = \left\{re^{i\theta}I :\, \theta \in [0, 2\pi[\right\}$ we have $|K| = r = w(A) = w'(A)$.

PROPOSITION 34. *For $K, K' \in \mathbb{K}(\mathcal{B}(H))$ we have*

$$|KK'| \leq \left(w(K) - \frac{w'^2(K)}{2|K|}\right)|K'| + \left(w(K') - \frac{w'^2(K')}{2|K'|}\right)|K|.$$

Proof. By (6) we have $\frac{1}{2}|K| \leq w(K) - \frac{w'^2(K)}{2|K|}$ and $\frac{1}{2}|K'| \leq w(K') - \frac{w'^2(K')}{2|K'|}$. On the other hand, we have $|KK'| \leq |K||K'|$, hence the desired inequality. ∎

PROPOSITION 35. *Let $K, K' \in \mathbb{K}(\mathcal{B}(H))$. Then*

$$W(KK') \subset I_{K,K'} + O(K)O(K'),$$

and

$$w(KK') \leq w(K)w(K') + d(K)d(K'), \tag{8}$$

where

$$I_{K,K'} = \left\{\langle Ax, x\rangle\langle Bx, x\rangle :\, \|x\| = 1,\ A \in K,\ B \in K'\right\}.$$

Proof. Let $x \in H$ be such that $\|x\| = 1$. Then, $Bx = \langle Bx, x\rangle x + \langle Bx, y\rangle y$, with $\|y\| = 1$ and $\langle x, y\rangle = 0$, and thus,

$$\langle ABx, x\rangle = \langle Bx, x\rangle\langle Ax, x\rangle + \langle Bx, y\rangle\langle Ay, x\rangle,$$

and the result follows. ∎

Remark 36. If in the inequality (8) K and K' are, respectively, replaced byA^* and A we obtain the following inequality due to Dragomir [11]:

$$\|A\|^2 \leq w^2(A) + d^2(A).$$

PROPOSITION 37. *Let K be an element of $\mathbb{K}_1(\mathcal{A})$, and let P be the polynomial with complex coefficients defined by $P(X) = \sum_{i=0}^{n} a_i X^i = a_0 + a_1 X + a_2 X^2 + \cdots + a_n X^n$. Then*

$$\sigma\big(P(K)\big) \subset P\big(\sigma(K)\big).$$

If further $\mathcal{A} = \mathcal{B}(H)$ and K is normal, then

$$\overline{W\big(P(K)\big)} \subset co\overline{P\big(W(K)\big)}.$$

Proof. It suffices to use (1) and (3) of Propositions 19 and 20, respectively. ∎

Finally we end with the following spectral theorem:

THEOREM 38. *Let K be an element of $\mathbb{K}_1(\mathcal{A})$, then*

$$\sigma\big(S(K)\big) \subset S\big(\sigma(K)\big). \tag{9}$$

If further, $\mathcal{A} = \mathcal{B}(H)$ and K is normal, then

$$\overline{W\big(S(K)\big)} \subset co\overline{S\big(W(K)\big)} \tag{10}$$

Proof. Firstly, we prove (9). For this, let $\lambda \in \sigma(S(K))$ and verify $\lambda \in S(\sigma(K))$. There exists $A \in S(K)$ such that $A - \lambda I$ is not invertible. That is to say, $A = \sum_{i=0}^{\infty} a_i x_i$, $x_i \in K$ and $\lambda \in \sigma(A)$. However, $A = \lim A_n$ with $A_n = \sum_{i=0}^{n} a_i x_i$, $x_i \in K$ and $A_n \in S_n(K)$. Then $A_n A_p = A_p A_n$, for all $n, p \in \mathbb{N}$, $h(\sigma(A), \sigma(A_n)) \longrightarrow 0$ [2]. We have

$$e\big(\{\lambda\}, \sigma(A_n)\big) \leq h\big(\sigma(A), \sigma(A_n)\big) \longrightarrow 0.$$

Therefore $e(\{\lambda\}, \sigma(A_n)) = \|\lambda - \lambda_n\|$, where $\lambda_n \in \sigma(A_n)$ and $\lambda = \lim \lambda_n$. Thus,

$$\lambda_n \in \sigma(A_n) \subset \sigma\big(S_n(K)\big) \subset S_n\big(\sigma(K)\big).$$

The last inclusion is due to Proposition 37. Therefore,

$$e\big(\{\lambda\}, S(\sigma(K))\big) \leq e\big(\{\lambda\}, \{\lambda_n\}\big) + e\big(\{\lambda_n\}, S_n(\sigma(K))\big) + e\big(S_n(\sigma(K)), S(\sigma(K))\big).$$

By Theorem 11, we have

$$e\big(S_n(\sigma(K)), S(\sigma(K))\big) \longrightarrow 0.$$

In addition,

$$e\big(\{\lambda\}, \{\lambda_n\}\big) = \|\lambda - \lambda_n\| \longrightarrow 0,$$

and

$$e\big(\{\lambda_n\}, S_n(\sigma(K))\big) = 0, \text{ since } \lambda_n \in S_n\big(\sigma(K)\big).$$

So $\lambda \in \overline{S(\sigma(K)} = S(\sigma(K))$. The last equality follows from Theorem 8. Inclusion (10) is the same as (9) by replacing the multifunction $\sigma(K)$ by the multifunction $\overline{W(K)}$, with values in $\mathbb{K}(\mathbb{C})$. ∎

References

[1] A. Amri, A. Seeger, Exponentiating a bundle of linear operators, *Set-Valued Anal.* **14** (2) (2006), 159 – 185.

[2] B. Aupetit, "A Primer on Spectral Theory", Springer-Verlag, New York, 1991.

[3] E.O. Ayoola, Exponential formula for the reachable sets of quantum stochastic differential inclusions, *Stochastic Anal. Appl.* **21** (3) (2003), 515 – 543.

[4] F.F. Bonsall, J. Duncan, "Numerical Ranges of Operators on Normed Spaces and Elements of Normed Algebras", London Math. Soc. Lecture Note Series 2, Cambridge University Press, London-New-York, 1971.

[5] A. Cabot, A. Seeger, Multivalued exponentiation analysis. Part I: Maclaurin exponentials, *Set-Valued Anal.* **14** (4) (2006), 347 – 379.

[6] A. Cabot, A. Seeger, Multivalued exponentiation analysis. Part II: Recursive exponentials, *Set-Valued Analysis* **14** (4) (2006), 381 – 411.

[7] M.K. Chraïbi, Domaine numérique du produit AB avec A normal, *Serdica Math. J.* **32** (1) (2006), 1 – 6.

[8] M.K. Chraïbi, Domaine numérique de l'opérateur produit $M_{2,A,B}$ et de la dérivation généralisée $\delta_{2,A,B}$, *Extracta Math.* **17** (1) (2002), 59 – 68.

[9] A.L. Dontchev, E.M. Farkhi, Error estimates for discretized differential inclusions, *Computing* **41** (4) (1989), 349 – 358.

[10] S.S. Dragomir, Some inequalities for norm the and the numerical radius of linear operators in Hilbert spaces, *Tamkamg J. Math.* **39** (1) (2008), 1 – 7.

[11] S.S. Dragomir, Norm and numerical radius inequalities for a product of two linear operators in Hilbert spaces, *J. Math. Inequal.* **2** (4) (2008), 499 – 510.

[12] S.S. Dragomir, Norm and numerical radius inequalities for two linear operators in Hilbert spaces: A survey of recent results, in “Functional Equations in Mathematical Analysis”, (T. M. Rassias, J. Brzdęk, eds.), Springer Optimization and Its Applications 52, Springer, New York, 2012, 427 – 490.

[13] K.E. Gustafson, D.K.M. Rao, “Numerical Range. The Field of Values of Linear Operators and Matrices”, Universitext, Springer-Verlag, New York, 1996.

[14] P.R. Halmos, “A Hilbert Space Problem”, Van Nostrand, Princeton, 1967.

[15] J.A.R. Holbrook, On the power bounded operators of Sz.-Nagy and Foias, *Acta Sci. Math. (Szeged)* **29** (1968), 299 – 310.

[16] J. Kyle, Numerical ranges of derivations, *Proc. Edinburgh. Math. Soc. (2)* **21** (1) (1978/79), 33 – 39.

[17] W. Rudin, “Functional Analysis”, T M H Edition, New Delhi: TATA McGraw-Hill, 1978.

[18] A. Seeger, On stabilized point spectra of multivalued systems, *Integral Equations Operator Theory* **54** (2) (2006), 279 – 300.

[19] P.R. Wolenski, The exponential formula for the reachable set of a Lipschitz differential inclusion, *SIAM J. Control Optim.* **28** (5) (1990), 1148 – 1161.