On the Approximate Solution of D’Alembert Type Equation Originating from Number Theory

B. Bouikhalene1, E. Elqorachi2, A. Charifi1

1Sultan Moulay Slimane University, Polydisciplinaire Faculty, Beni-Mellal, Morocco
2Ibn Zohr University, Faculty of Sciences, Agadir, Morocco
bbouikhalene@yahoo.fr, elqorachi@yahoo.fr, charifi2000@yahoo.fr

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Abstract: We solve the functional equation

\[ E(\alpha) : f(x_1x_2 + \alpha y_1y_2, x_1y_2 + x_2y_1) + f(x_1x_2 - \alpha y_1y_2, x_2y_1 - x_1y_2) = 2f(x_1, y_1)f(x_2, y_2), \]

where \((x_1, y_1), (x_2, y_2) \in \mathbb{R}^2, f : \mathbb{R}^2 \to \mathbb{C}\) and \(\alpha\) is a real parameter, on the monoid \(\mathbb{R}^2\). Also we investigate the stability of this equation in the following setting:

\[ |f(x_1x_2 + \alpha y_1y_2, x_1y_2 + x_2y_1) + f(x_1x_2 - \alpha y_1y_2, x_2y_1 - x_1y_2) - 2f(x_1, y_1)f(x_2, y_2)| \]

\[ \leq \min\{|\varphi(x_1), \psi(y_1), \phi(x_2), \zeta(y_2)|\}. \]

From this result, we obtain the superstability of this equation.

Key words: D’Alembert functional equation, monoid \(\mathbb{R}^2\), multiplicative function, stability, superstability.

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1. Introduction

For any \(\alpha \in \mathbb{R}\), Berrone and Dieulefait [5] equipped \(\mathbb{R}^2\) with the multiplication rule \(\cdot_\alpha\), defined by

\[(x_1, y_1) \cdot_\alpha (x_2, y_2) = (x_1x_2 + \alpha y_1y_2, x_1y_2 + x_2y_1), \quad (x_1, y_1), (x_2, y_2) \in \mathbb{R}^2.\]

For \(\alpha = -1\), the multiplication is the usual product of complex numbers in \(\mathbb{C} = \mathbb{R}^2\). The rule makes \(\mathbb{R}^2\) into a commutative monoid with neutral element \((1, 0)\) and \(\sigma(x, y) = (x, -y)\) (complex conjugation) as an involution.

Berrone and Dieulefait [5, Theorem 1] studied the homomorphisms \(m : (\mathbb{R}^2, \cdot_\alpha) \to (\mathbb{R},.)\), i.e., the multiplicative, real-valued functions on the monoid \((\mathbb{R}^2, \cdot_\alpha)\). We extend their investigations by finding the bigger set of all multiplicative, complex-valued functions \(M : (\mathbb{R}^2, \cdot_\alpha) \to (\mathbb{C},.)\). Combining
this information with Davison’s work [9] about D’Alembert’s functional equation on monoids, we obtain an explicit description of the solutions \( f : \mathbb{R}^2 \to \mathbb{C} \) of D’Alembert’s functional equation

\[
E(\alpha) : f(a \cdot \alpha b) + f(a \cdot \sigma(b)) = 2f(a)f(b), \quad a, b \in \mathbb{R}^2,
\]
on the monoid \((\mathbb{R}^2, \cdot, \sigma)\). The description falls into three different cases, according to whether \( \alpha > 0 \) or \( \alpha < 0 \). The equation \( E(\alpha) \) is a common generalization of many functional equations of type D’Alembert

\[
f(ab) + f(a\sigma(b)) = 2f(a)f(b), \quad a, b \in \mathbb{R}^2 \quad (1.1)
\]
on the monoid \( \mathbb{R}^2 \), like, e.g.,

1) If \( \alpha = 0 \),

\[
E(0) : f(x_1x_2, x_1y_2 + x_2y_1) + f(x_1x_2, x_2y_1 - x_1y_2) = 2f(x_1, y_1)f(x_2, y_2),
\]
for all \((x_1, y_1), (x_2, y_2) \in \mathbb{R}^2\). Setting \( x_1 = x_2 = 1 \) and \( F(y) = f(1, y) \) for any \( y \in \mathbb{R} \) respectively \( y_1 = y_2 = 0 \) and \( m(x) = f(x, 0) \) for any \( x \in \mathbb{R} \) in \( E(0) \), we get the classical D’Alembert functional equation

\[
F(y_1 + y_2) + F(y_1 - y_2) = 2F(y_1)F(y_2), \quad y_1, y_2 \in \mathbb{R} \quad (1.2)
\]
on \( \mathbb{R} \) (see [1], [4], [15] and [23]) respectively the classical Cauchy equation

\[
m(x_1x_2) = m(x_1)m(x_2), \quad x_1, x_2 \in \mathbb{R} \quad (1.3)
\]
on \( \mathbb{R} \). We call \( m \) a multiplicative function on \( \mathbb{R} \) (see[1]).

2) If \( \alpha = -1 \),

\[
E(-1) : f(x_1x_2 - y_1y_2, x_1y_2 + x_2y_1) + f(x_1x_2 + y_1y_2, x_2y_1 - x_1y_2)
= 2f(x_1, y_1)f(x_2, y_2),
\]
\((x_1, y_1), (x_2, y_2) \in \mathbb{R}^2\). The equation \( E(-1) \) is in connection with the identity

\[
(x_1x_2 - y_1y_2)^2 + (x_1y_2 + x_2y_1)^2 + (x_1x_2 + y_1y_2)^2 + (x_2y_1 - x_1y_2)^2
= 2(x_1^2 + y_1^2)(x_2^2 + y_2^2) \quad (1.4)
\]
for any \( x_1, x_2, y_1, y_2 \in \mathbb{R} \).
3) If $\alpha \neq 1$ is a square free integer and $Q(\sqrt{\alpha}) = \{x + y\sqrt{\alpha} : x, y \in \mathbb{Q}\}$ is the quadratic monoid equipped with the multiplicative rule

$$(x_1 + y_1\sqrt{\alpha})(x_2 + y_2\sqrt{\alpha}) = (x_1x_2 + \alpha y_1y_2) + (x_1y_2 + x_2y_1)\sqrt{\alpha}, \quad (1.5)$$

then $E(\alpha)$ reduces to D’Alembert functional equation (1.1) on the monoid $Q(\sqrt{\alpha})$. In [9] Davison solved the D’Alembert functional equation with involution on a monoid $A$: any solution $f : A \rightarrow \mathbb{C}$ has the general form $f = M + M \circ \nu$, where $M : A \rightarrow \mathbb{C}$ is a multiplicative function.

In 1940, Ulam [22] gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. Among these was the following question concerning the stability of homomorphisms.

**Question 1.1.** Let $(G_1, \ast)$ be a group and let $(G_1, \circ, d)$ be a metric group with the metric $d$. Given $\varepsilon > 0$, does there exist $\delta(\varepsilon) > 0$ such that if a mapping $h : G_1 \rightarrow G_2$ satisfies the inequality $d(h(x \ast y), h(x) \circ h(y)) < \delta$ for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \rightarrow G_2$ with $d(h(x), H(x)) < \delta(\varepsilon)$ for all $x \in G_1$?

In 1941, Hyers [12] answered this question for the case where $G_1$ and $G_2$ are Banach spaces. In 1978, Rassias [20] provided a generalization of Hyer’s theorem which allows the Cauchy difference to be unbounded. The interested reader may refer to the book by Hyers, Isac, Rassias [13] for an in depth account on the subject of stability of functional equations. In 1982, Rassias [19] solved the Ulam problem by involving a product of powers of norms. Since then, the stability problems of various functional equations have been investigated by many authors (see [10], [11] and [14]). In [3] and [7] Baker et al. and Bourgin respectively, introduced the notion that by now is frequently referred to as superstability or Baker’s stability: if a function $f$ satisfies the stability inequality $|E_1(f) - E_2(f)| \leq \varepsilon$, then either $f$ is bounded or $E_1(f) = E_2(f)$. The superstability of D’Alembert’s functional equation $f(x + y) + f(x - y) = 2f(x)f(y)$ was investigated by Baker [4] and Cholewa [8]. Badora and Ger [2], and Kim ([16], [17] and [18]) proved its superstability under the condition $|f(x + y) + f(x - y) - 2f(x)f(y)| \leq \varphi(x)$ or $\varphi(y)$. In a previous work, Bouikhalene et al. [6] investigated the superstability of the cosine functional equation on the Heisenberg group. Following this investigation we study the superstability of the functional equation $E(\alpha)$ on the monoid $(\mathbb{R}^2, \cdot_\alpha)$. Also we say that a function $f : \mathbb{R}^2 \rightarrow \mathbb{C}$ is of approximate a cosine type function,
if there is $\delta > 0$ such that
\[ |f(a \cdot_b b) + f(a \cdot_i b) - 2f(a)f(b)| < \delta, \quad a, b \in \mathbb{R}^2. \] (1.6)

In the case where $\delta = 0$, $f$ satisfies the functional equation $E(\alpha)$. We call $f$ a cosine type function on $\mathbb{R}^2$. The paper is organized as follows: In the first section after this introduction we solve the functional equation $E(\alpha)$. In the second section we study the superstability equation $E(\alpha)$.

2. Solution of equation $E(\alpha)$

According to [9] we drive the following lemma.

**Lemma 2.1.** The solution $f : \mathbb{R}^2 \rightarrow \mathbb{C}$ of $E(\alpha)$ is of the form
\[
 f = \frac{M + M \circ \sigma}{2},
\]
where $M : (\mathbb{R}^2, \cdot_\alpha) \rightarrow (\mathbb{C}, \cdot)$ is a multiplicative function.

By extending Berrone-Dieulefait’s result [5] to complex-valued multiplicative functions, we get the following lemmas.

**Lemma 2.2.** The multiplicative functions $M : (\mathbb{R}^2, \cdot_1) \rightarrow (\mathbb{C}, \cdot)$ are the functions
\[
 M(x, y) = m_1(x + y)m_2(x - y), \quad x, y \in \mathbb{R},
\]
where $m_1, m_2 : \mathbb{R} \rightarrow \mathbb{C}$ are multiplicative functions.

**Lemma 2.3.** The multiplicative functions $M : (\mathbb{R}^2, \cdot_0) \rightarrow (\mathbb{C}, \cdot)$ are the trivial function $M = 1$ and $M(0, y) = 0$ for any $y \in \mathbb{R}$ and $M(x, y) = m(x)\gamma(\frac{y}{x})$ for any $(x, y) \in \mathbb{R}^2$, with $x \neq 0$, where $m : \mathbb{R} \rightarrow \mathbb{C}$ is a multiplicative function and $\gamma : (\mathbb{R}, +) \rightarrow \mathbb{C}$ is an arbitrary character.

**Lemma 2.4.** The multiplicative functions $M : (\mathbb{C}, \cdot_{-1}) \rightarrow (\mathbb{C}, \cdot)$ are the trivial functions $M = 0$ and $M = 1$ and
\[
 M(z) = \begin{cases} 
 \tilde{m}(|z|)\Gamma(\exp(i\theta)), & \text{for } z = |z|\exp(i\theta) \neq 0 \\
 0, & \text{for } z = 0,
\end{cases}
\]
where $\tilde{m} : (\mathbb{R}^+, \cdot) \rightarrow \mathbb{C}^*$ and $\Gamma : \{\exp(i\theta), \theta \in \mathbb{R}\} \rightarrow \mathbb{C}^*$ are arbitrary characters.
Proof. When $\alpha = -1$, the multiplicative rule $\cdot_{-1}$ becomes the usual product numbers in $\mathbb{C}$. By using the polar decomposition $z = |z|\exp(i\theta)$ for any $z \in \mathbb{C}^*$ where $\theta = \arg(z)$, we get

$$M(|z_1||z_2|) = M(|z_1|)M(|z_2|), \quad z_1, z_2 \in \mathbb{C}^*$$

(2.1)

and

$$M(\exp(i(\theta_1 + \theta_2))) = M(\exp(i\theta_1))M(\exp(i\theta_2)), \quad \theta_1, \theta_2 \in \mathbb{R}.$$  

(2.2)

By letting $\hat{m}(|z|) = M(|z|)$, for any $z \in \mathbb{C}^*$, and $\Gamma(\exp(i\theta)) = M(\exp(i\theta))$ for any $\theta \in \mathbb{R}$ it follows that $\hat{m} : (\mathbb{R}^+ , \cdot) \rightarrow \mathbb{C}^*$ and $\Gamma : \{\exp(i\theta), \theta \in \mathbb{R}\} \rightarrow \mathbb{C}^*$ are characters. If $z = 0$, we set $M(z) = 0$.

In the next corollary we give the set of all multiplicative complex-valued functions $M : (\mathbb{R}^2 , \cdot_\alpha) \rightarrow \mathbb{C}$.

**Corollary 2.5.** The multiplicative functions $M : (\mathbb{R}^2 , \cdot_\alpha) \rightarrow (\mathbb{C}, \cdot)$ are given by the following list:

I) If $\alpha > 0$, then

$$M(x, y) = m_1(x + y\sqrt{\alpha})m_2(x - y\sqrt{\alpha}), \quad (x, y) \in \mathbb{R}^2.$$

II) If $\alpha = 0$, then

a) $M(x, y) = 1$, for any $(x, y) \in \mathbb{R}^2$.

b) $M(0, y) = 0$, for any $y \in \mathbb{R}$.

c) $M(x, y) = m(x)\gamma(\frac{y}{x})$, for any $(x, y) \in \mathbb{R}^2$ with $x \neq 0$.

III) If $\alpha < 0$, then

a) $M(x, y) = 0$, for any $(x, y) \in \mathbb{R}^2$.

b) $M(x, y) = 1$, for any $(x, y) \in \mathbb{R}^2$.

\[ c) \quad M(x, y) = \begin{cases} \hat{m}(\sqrt{x^2 - \alpha y^2})\Gamma(\arg(x + iy)), & \text{for } (x, y) \neq (0, 0) \\ 0, & \text{for } (x, y) = (0, 0). \end{cases} \]

where $m_1, m_2, m : \mathbb{R} \rightarrow \mathbb{C}$ are multiplicative functions, and $\hat{m} : (\mathbb{R}^+ , \cdot) \rightarrow \mathbb{C}^*$, $\Gamma : \{\exp(i\theta), \theta \in \mathbb{R}\} \rightarrow \mathbb{C}^*$ and $\gamma : (\mathbb{R}, +) \rightarrow \mathbb{C}$ are arbitrary characters.
The next theorem is the main result of this section.

**Theorem 2.6.** The set of solutions of the functional equation \( E(\alpha) \) consists of the following three cases:

A) If \( \alpha > 0 \), then

\[
f(x, y) = \frac{m_1(x)m_2(y)}{2} \left\{ m_1(y\sqrt{\alpha})m_2(-y\sqrt{\alpha}) + m_1(-y\sqrt{\alpha})m_2(y\sqrt{\alpha}) \right\},
\]

for any \((x, y) \in \mathbb{R}^2\).

B) If \( \alpha = 0 \), then

a) \( f(x, y) = 1 \), for any \((x, y) \in \mathbb{R}^2\).

b) \( f(0, y) = 0 \), for any \( y \in \mathbb{R} \).

c) \( f(x, y) = \frac{m(x)}{2} \left\{ \gamma\left(\frac{x}{2}\right) + \gamma\left(-\frac{y}{x}\right), \ (x, y) \in \mathbb{R}^2, \ x \neq 0 \right\} \)

C) If \( \alpha < 0 \), then \( f(0, 0) = 0 \) and

\[
f(x, y) = \frac{\tilde{m}\left(\sqrt{x^2 - \alpha y^2}\right)}{2} \left\{ \Gamma(\arg(x + iy)), \ (x, y) \in \mathbb{R}^2 \setminus (0, 0) \right\},
\]

where \( m_1, m_2, m : \mathbb{R} \rightarrow \mathbb{C} \) are multiplicative functions, and \( \tilde{m} : (\mathbb{R}^+, \cdot) \rightarrow \mathbb{C}^* \), \( \Gamma : \{\exp(i\theta), \ \theta \in \mathbb{R}\} \rightarrow \mathbb{C}^* \) and \( \gamma : \mathbb{R} \rightarrow \mathbb{C} \) are arbitrary characters.

**Proof.** According to Lemma 2.1 and Corollary 2.5 we get the proof of theorem. \( \square \)

3. **Superstability of equation \( E(\alpha) \)**

In the next theorem we establish the stability of \( E(\alpha) \).

**Theorem 3.1.** Let \( \varphi, \psi, \phi, \zeta : \mathbb{R} \rightarrow [0, +\infty[ \) be functions and let \( f : \mathbb{R}^2 \rightarrow \mathbb{C} \) be a function such that

\[
|f(x_1x_2 + \alpha y_1y_2, x_1y_2 + x_2y_1) + f(x_1x_2 - \alpha y_1y_2, x_2y_1 - x_1y_2) - 2f(x_1, y_1)f(x_2, y_2)| \leq \min \{\varphi(x_1), \psi(y_1), \phi(x_2), \zeta(y_2)\}
\]

(3.1)
for all \((x_1, y_1), (x_2, y_2) \in \mathbb{R}^2\) and \(\alpha\) is a real parameter. Then either \(f\) is bounded or \(f\) satisfies the functional equation

\[
E(\alpha) : f(x_1 x_2 + \alpha y_1 y_2, x_1 y_2 + x_2 y_1) + f(x_1 x_2 - \alpha y_1 y_2, x_2 y_1 - x_1 y_2) = 2f(x_1, y_1) f(x_2, y_2)
\]

for all \((x_1, y_1), (x_2, y_2) \in \mathbb{R}^2\).

**Proof.** For all \((x_1, y_1), (x_2, y_2) \in \mathbb{R}^2\) and \(\alpha\) a real parameter we get from the inequality (3.1) that

\[
\left| f(x_1 x_2 + \alpha y_1 y_2, x_1 y_2 + x_2 y_1) + f(x_1 x_2 - \alpha y_1 y_2, x_2 y_1 - x_1 y_2) - 2f(x_1, y_1) f(x_2, y_2) \right| \leq \varphi(x_1) \text{ or } \psi(y_1).
\]

(3.2)

Since \(f\) is unbounded then we can choose a sequence \((x_n, y_n)_{n \geq 1}\) in \(\mathbb{R}^2\) such that \(f(x_n, y_n) \neq 0\) and \(\lim_{n \to +\infty} |f(x_n, y_n)| = +\infty\). Taking \((x_2, y_2) = (x_n, y_n)\) in (3.2) we obtain

\[
\left| f(x_1 x_n + \alpha y_1 y_n, x_1 y_n + x_n y_1) + f(x_1 x_n - \alpha y_1 y_n, x_n y_1 - x_1 y_n) - 2f(x_1, y_1) f(x_n, y_n) \right| \leq \varphi(x_1) \text{ or } \psi(y_1)
\]

and

\[
\left| \frac{f(x_1 x_n + \alpha y_1 y_n, x_1 y_n + x_n y_1) + f(x_1 x_n - \alpha y_1 y_n, x_n y_1 - x_1 y_n)}{2f(x_n, y_n)} - f(x_1, y_1) \right| \leq \frac{\varphi(x_1)}{2|f(x_n, y_n)|} \text{ or } \frac{\psi(y_1)}{2|f(x_n, y_n)|}.
\]

That is we get

\[
f(x_1, y_1) = \lim_{n \to +\infty} \frac{f(x_1 x_n + \alpha y_1 y_n, x_1 y_n + x_n y_1) + f(x_1 x_n - \alpha y_1 y_n, x_n y_1 - x_1 y_n)}{2f(x_n, y_n)}.
\]

(3.3)

Setting \(X_n = x_2 x_n + \alpha y_2 y_n, Y_n = x_2 y_n + x_n y_2, \bar{X}_n = x_2 x_n - \alpha y_2 y_n, \bar{Y}_n = x_2 y_n - x_n y_2\). For any \((x_1, y_1), (x_2, y_2) \in \mathbb{R}^2\) it follows that
\[ \begin{align*}
&\left| f((x_1 x_2 + \alpha y_1 y_2)x_n + \alpha(x_1 y_2 + x_2 y_1) y_n, \\
&\quad (x_1 x_2 + \alpha y_1 y_2)y_n + x_n(x_1 y_2 + x_2 y_1)) \right| \\
+&\left| f((x_1 x_2 + \alpha y_1 y_2)x_n - \alpha(x_1 y_2 + x_2 y_1) y_n, \\
&\quad x_n(x_1 y_2 + x_2 y_1) - (x_1 x_2 + \alpha y_1 y_2)y_n) \right| \\
- &2f(x_1, y_1)f(x_2 x_n + \alpha y_2 y_n, x_2 y_n + x_n y_2) \\
+&\left| f((x_1 x_2 - \alpha y_1 y_2)x_n + \alpha(x_2 y_1 - x_1 y_2) y_n, \\
&\quad (x_1 x_2 - \alpha y_1 y_2)y_n + x_n(x_2 y_1 - x_1 y_2)) \right| \\
- &2f(x_1, y_1)f(x_2 x_n - \alpha y_2 y_n, x_2 y_n - x_n y_2) \\
\leq &\left| f((x_1 x_2 + \alpha y_1 y_2)x_n + \alpha(x_1 y_2 + x_2 y_1) y_n, \\
&\quad (x_1 x_2 + \alpha y_1 y_2)y_n + x_n(x_1 y_2 + x_2 y_1)) \right| \\
+&\left| f((x_1 x_2 - \alpha y_1 y_2)x_n - \alpha(x_2 y_1 - x_1 y_2) y_n, \\
&\quad x_n(x_2 y_1 - x_1 y_2) - (x_1 x_2 - \alpha y_1 y_2)y_n) \right| \\
- &2f(x_1, y_1)f(x_2 x_n + \alpha y_2 y_n, x_2 y_n + x_n y_2) \\
+&\left| f((x_1 x_2 - \alpha y_1 y_2)x_n + \alpha(x_2 y_1 - x_1 y_2) y_n, \\
&\quad (x_1 x_2 - \alpha y_1 y_2)y_n + x_n(x_2 y_1 - x_1 y_2)) \right| \\
+ &2f(x_1, y_1)f(x_2 x_n - \alpha y_2 y_n, x_2 y_n - x_n y_2) \\
= &\left| f(x_1 X_n + \alpha y_1 Y_n, x_1 Y_n + X_n y_1) + f(x_1 X_n - \alpha y_1 Y_n, X_n y_1 - x_1 Y_n) \\
&\quad - 2f(x_1, y_1)f(X_n, Y_n) \right| \\
+ &\left| f(x_1 \tilde{X}_n + \alpha y_1 \tilde{Y}_n, x_1 \tilde{Y}_n + \tilde{X}_n y_1) + f(x_1 \tilde{X}_n - \alpha y_1 \tilde{Y}_n, \tilde{X}_n y_1 - x_1 \tilde{Y}_n) \\
&\quad - 2f(x_1, y_1)f(\tilde{X}_n, \tilde{Y}_n) \right| \\
\leq & 2\varphi(x_1) \text{ or } 2\psi(y_1). 
\end{align*}\]
So that

\[
\begin{aligned}
&f((x_1x_2 + \alpha y_1y_2)x_n + \alpha(x_1y_2 + x_2y_1)y_n, \\
&\qquad\quad \frac{(x_1x_2 + \alpha y_1y_2)y_n + x_n(x_1y_2 + x_2y_1))}{f(x_n, y_n)} \\
&+ f((x_1x_2 + \alpha y_1y_2)x_n - \alpha(x_1y_2 + x_2y_1)y_n, \\
&\qquad\quad \frac{x_n(x_1y_2 + x_2y_1) - (x_1x_2 + \alpha y_1y_2)y_n)}{f(x_n, y_n)} \\
&+ f((x_1x_2 - \alpha y_1y_2)x_n + \alpha(x_2y_1 - x_1y_2)y_n, \\
&\qquad\quad \frac{x_n(x_2y_1 - x_1y_2) + (x_1x_2 - \alpha y_1y_2)y_n)}{f(x_n, y_n)} \\
&+ f((x_1x_2 - \alpha y_1y_2)x_n - \alpha(x_2y_1 - x_1y_2)y_n, \\
&\qquad\quad \frac{x_n(x_2y_1 - x_1y_2) - (x_1x_2 - \alpha y_1y_2)y_n)}{f(x_n, y_n)} \\
&- 2f(x_1, y_1) \left\{ \frac{f(x_2x_n + \alpha y_2y_n, x_2y_n + x_ny_2)}{f(x_n, y_n)} + \frac{f(x_2x_n - \alpha y_2y_n, x_2y_n - x_ny_2)}{f(x_n, y_n)} \right\} \leq 2 \frac{\varphi(x_1)}{|f(x_n, y_n)|} \text{ or } 2 \frac{\psi(y_1)}{|f(x_n, y_n)|}
\end{aligned}
\]

for any \((x_1, y_1), (x_2, y_2) \in \mathbb{R}^2\). Since \(|f(x_n, y_n)| \rightarrow +\infty\) as \(n \rightarrow +\infty\) we get that \(f\) satisfies \(E(\alpha)\).

By letting \(\min\{\varphi(x_1), \psi(y_1), \phi(x_2), \zeta(y_2)\} = \delta\) we get the Baker’s stability ([3], [4]) for the functional equation \(E(\alpha)\).

**Corollary 3.2.** Let \(\delta > 0\) and let \(f : \mathbb{R}^2 \rightarrow \mathbb{C}\) be a function such that

\[
|f(x_1x_2 + \alpha y_1y_2, x_1y_2 + x_2y_1) + f(x_1x_2 - \alpha y_1y_2, x_2y_1 - x_1y_2) \\
- 2f(x_1, y_1)f(x_2, y_2)| \leq \delta
\]

for all \((x_1, y_1), (x_2, y_2) \in \mathbb{R}^2\) and \(\alpha\) is a real parameter. Then either \(f\) is bounded and \(|f(x, y)| \leq \frac{1+\sqrt{1+4\delta^2}}{2}\) for all \((x, y) \in \mathbb{R}^2\) or \(f\) satisfies the functional equation \(E(\alpha)\).
References


