

## Several Open Problems on Banach Spaces (Castro Urdiales 2014)

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Received October 27, 2013

*Abstract:* We report on the meeting *Interpolation and Banach space constructions*, recently held in Castro Urdiales, and collect the questions proposed by the participants during the open problems session.

*Key words:* Banach space, open problems.

AMS *Subject Class.* (2010): 46B03, 46B04, 46B42, 47B60.

The conference “Interpolation and Banach space constructions” was held in Castro Urdiales (Spain), during the first week of June 2014. It was the eleventh edition of a series of meetings about Banach spaces and operator theory organised by the Universities of Extremadura and Cantabria.

The invited speakers of the conference were Pandelis Dodos (University of Athens), Valentin Ferenczi (Universidade de São Paulo), Jordi López Abad (Instituto de Ciencias Matemáticas) and Richard Rochberg (Washington University in St. Louis). There were also several sessions of short communications and plenty of discussion time. The venue of the conference was the Centro Cultural y de Congresos “La Residencia”, close to the fishing port of Castro Urdiales, which is the customary venue of the activities of the Centro Internacional de Encuentros Matemáticos (CIEM), a joint initiative of the University of Cantabria and the City Council of Castro Urdiales. The organisers of the conference were Jesús M. F. Castillo (Badajoz), Manuel González (Santander) and Javier Pello (Madrid). They acknowledge financial support from the University of Cantabria, from the town council of Castro Urdiales and from the Ministerio de Economía y Competitividad.

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Supported by MICINN (Spain), Grant MTM2010-20190.

For additional information we refer to the official web page of the conference: <http://www.ciem.unican.es/encuentros/banach/2014/>

Here we collect the questions that were posed and discussed during the Open Problems Session of the conference, and describe the solution obtained to one of the problems raised during the 2013 Castro Urdiales Meeting [7]. We thank the participants for their interest and enthusiasm, which provided a stimulating working environment.

## 1. HYPERPLANES AND COMPLEX STRUCTURE VALENTIN FERENCZI (UNIVERSIDADE DE SÃO PAULO)

In the sequel,  $X$  is always a real Banach space. We say that  $X$  has *complex structure* if there exists a bounded operator  $J$  acting on  $X$  such that  $J^2 = -I_X$ , where  $I_X$  is the identity on  $X$ . In this case we can see  $X$  as a complex space with the law  $(r + is) \cdot x = rx + sJx$ , where  $r, s \in \mathbb{R}$  [11, Section 1.3].

**DEFINITION 1.** ([12, DEFINITION 1]) A real Banach space  $X$  is said to be *even* if it has complex structure but its hyperplanes do not; it is said to be *odd* if it does not have complex structure but its hyperplanes do.

Even Banach spaces are a generalisation of finite-dimensional spaces of even dimension, and can be seen as a complex Banach spaces.

A long-standing open problem is whether the Kalton-Peck space  $Z_2$  from [22] is isomorphic to its hyperplanes. The following conjecture would give a negative answer to this problem.

**CONJECTURE 1.**  $Z_2$  is even.

Gowers and Maurey [16] constructed the first example of an infinite-dimensional Banach space not isomorphic to its hyperplanes, and now there are plenty of examples in the literature.

Suppose that  $X$  is a Banach space not isomorphic to its hyperplanes. Then there are four possibilities with respect to having complex structure:

- (1)  $X$  is even;
- (2)  $X$  is odd;
- (3) neither  $X$  nor its hyperplanes have complex structure;
- (4) both  $X$  and its hyperplanes have complex structure.

The complex Gowers-Maurey space  $GM(\mathbb{C})$  is an even space, while its hyperplanes are odd [12]. The Gowers-Maurey space  $GM$  is an example of a space without complex structure whose hyperplanes also lack complex structure, but there are no known examples of the fourth category.

QUESTION 1. Does there exist a space not isomorphic to its hyperplanes and having complex structure whose hyperplanes also have complex structure?

Constructing such a space is difficult, because there must be a complex structure  $J$  on  $X$  and a complex structure  $K$  on its hyperplanes  $H$ , but no isomorphism between  $X$  and  $H$ .

There are other questions regarding the direct sum of spaces having complex structure. If  $X$  and  $Y$  are even and totally incomparable, then it is known that  $X \oplus Y$  is also even.

QUESTION 2. Is this true in general?

QUESTION 3. Is being even a 3-space property?

QUESTION 4. Is having complex structure a 3-space property?

## 2. OPTIMAL RETRACTIONS

KAZIMIERZ GOEBEL (MARIA CURIE-SKŁODOWSKA UNIVERSITY)

Let  $X$  be a Banach space, let  $B$  be its unit ball and let  $S$  be its unit sphere. A *retraction* from  $B$  to  $S$  is a continuous mapping  $R: B \rightarrow S$  such that  $R$  is the identity on  $S$ . It is well known that there are no retractions from  $B$  to  $S$  in a finite-dimensional space. They do exist, however, in infinite-dimensional spaces [21], and such a retraction can always be found as a Lipschitzian mapping [6],

$$\|Rx - Ry\| \leq K\|x - y\|,$$

for some universal constant  $K$ . This leads to the question of finding the best Lipschitz constant  $k$  for a retraction on a given space,

$$k_0(X) = \inf \{k > 0 : \text{there exists } R: B \rightarrow S \text{ retraction, } R \in \text{Lip}(k)\}.$$

Every retraction  $R: B \rightarrow S$  satisfies  $R(0) \in S$  and  $R(-R(0)) = -R(0) \in S$ , so any such constant must be at least 2. In fact, it is known that  $k_0(X) \geq 3$

for every space  $X$  [15], and sharper bounds have been found for certain spaces [8, 1, 15, 4]:

$$4 \leq k_0(\ell_1) \leq 8, \quad 4.55 \leq k_0(\ell_2) < 29.$$

There is a very simple proof of such a bound, quoted by Benyamini and Lindenstrauss [5, p. 65], for  $C[0, 1]$ .

**THEOREM 1.** *There exists a Lipschitz retraction from  $B_{C[0,1]}$  to  $S_{C[0,1]}$  with Lipschitz constant at most 70.*

The proof is as follows: Define  $A: C[0, 1] \rightarrow C[0, 1]$  as

$$Af(t) = |f(t) + 1 - 2(1 - \|f\|)t| - 1 + 2(1 - \|f\|)t;$$

then  $A \in \text{Lip}(5)$ ,  $A$  is the identity on  $S_{C[0,1]}$  and  $\|Af\| \geq 1/7$  for every  $f \in B_{C[0,1]}$ . This means that  $R(f) = Af/\|Af\|$  is a retraction from  $B_{C[0,1]}$  to  $S_{C[0,1]}$  with Lipschitz constant at most 70.

### 3. ON BANACH LATTICES

PEDRO TRADACETE (UNIVERSIDAD CARLOS III DE MADRID)

Let  $X$  be a Banach lattice, and let  $T: X \rightarrow Y$  be an operator.  $T$  is said to be *lattice-strictly singular* (LSS) if it is not an isomorphism when restricted to any sublattice of  $X$ .  $T$  is said to be *disjointly strictly singular* if it is not an isomorphism when restricted to any subspace of  $X$  spanned by a disjoint sequence.

We are interested in the problem of whether  $LSS \neq DSS$ . For this, we need to find a pair of basic sequences  $(x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}} \subseteq X$  such that the difference sequence  $z_n := x_n - y_n$  satisfies:

- (1)  $(z_n)_{n \in \mathbb{N}}$  is unconditional;
- (2)  $\|\sum_{n \in \mathbb{N}} a_n z_n\| \simeq \max \left\{ \|\sum_{n \in \mathbb{N}} a_n x_n\|, \|\sum_{n \in \mathbb{N}} a_n y_n\| \right\}$ ;
- (3)  $(z_n)_{n \in \mathbb{N}}$  does not contain any unconditional positive blocks, i.e., no sequence of the form

$$w_n = \sum_{j \in A_n} a_j x_j + \sum_{j \in B_n} b_j y_j, \quad a_j, b_j \geq 0,$$

can be unconditional.

A somewhat related question is the following one. Let us say that a disjoint sequence is *complemented* if it spans a complemented subspace.

QUESTION 5. ([13]) Let  $E$  be a (reflexive) separable Banach lattice. Does there exist a disjoint sequence in  $E$  which is complemented?

The answer is positive in many cases: every disjoint sequence is complemented when  $E = L_p$ , and has a complemented subsequence when  $E = L_{p,q}$  or some other Lorentz spaces. For rearrangement-invariant spaces, the expectation operator defines a continuous projection onto the subspace spanned by a disjoint sequence of characteristic functions, and a similar result is valid for Banach spaces with an unconditional basis.

#### 4. ON KOTTMAN'S CONSTANT

JESÚS M.F. CASTILLO (UNIVERSIDAD DE EXTREMADURA)

The *Kottman's constant* of a Banach space  $X$ , with unit ball  $B_X$  and unit sphere  $S_X$ , is defined as follows:

$$K(X) = \sup \{ \sigma > 0 : \exists (x_n)_{n \in \mathbb{N}} \subseteq B_X, \forall n \neq m, \|x_n - x_m\| \geq \sigma \}.$$

It was introduced and studied by Kottman [23, 24]. It is clear that  $K(X) = 0$  if and only if  $X$  is finite-dimensional. A well-known, although highly non-trivial, result of Elton and Odell [10] (see also [9, p. 241]) establishes that  $K(X) > 1$  for every infinite-dimensional Banach space. Kottman's constant has been considered in several papers and its exact value has been found for a number of classical Banach spaces.

A variation of the Kottman constant is the *symmetric separation constant*:

$$K^s(X) = \sup \{ \sigma > 0 : \exists (x_n)_{n \in \mathbb{N}} \subseteq B_X, \forall n \neq m, \|x_n \pm x_m\| \geq \sigma \}.$$

Equality between  $K$  and  $K^s$  holds in several classical spaces such as  $\ell_p$  spaces, where  $K(\ell_p) = K^s(\ell_p) = 2^{1/p}$  for  $1 \leq p < \infty$ ; or  $c_0$ , where  $K^s(c_0) = 2$  as the sequence  $x_n = e_{n+1} - \sum_{j=1}^n e_j$  shows. A natural question is:

PROBLEM 1. Does the Elton-Odell theorem hold for  $K^s$ ? Precisely, is it always  $K^s(X) > 1$  for every infinite-dimensional Banach space?

The questions that we are mostly interested in are:

PROBLEM 2. Does  $K(X^{**}) = K(X)$  hold for every Banach space?

PROBLEM 3. Does  $K^s(X^{**}) = K^s(X)$  hold for every Banach space?

Since both  $K^s(c_0) = K(c_0) = 2 = K(\ell_1) = K^s(\ell_1)$ , and a Banach space containing an isomorphic copy of either of those spaces also contains an almost-isometric copy, it is clear that a Banach space such that  $K(X) < K(X^{**})$  or  $K^s(X) < K^s(X^{**})$ , if it exists, cannot contain either  $c_0$  or  $\ell_1$ , hence it cannot have an unconditional basis.

## 5. ULTRATYPES

JESÚS M.F. CASTILLO (UNIVERSIDAD DE EXTREMADURA)

The following notion was introduced by Henson and Moore [19, p. 106].

DEFINITION 2. We say that two Banach spaces  $X$  and  $Y$  are ultra-isomorphic, or have the same ultratype, if there is an ultrafilter  $\mathcal{U}$  such that  $X_{\mathcal{U}}$  and  $Y_{\mathcal{U}}$  are isomorphic.

It can be shown that “having the same ultratype” is an equivalence relation. Recall that a Banach space is an  $\mathcal{L}_{\infty}$ -space if and only if some (or every) ultrapower is. The question of the classification of  $\mathcal{L}_{\infty}$ -spaces was posed in [19, p. 106] and [17, p. 315] and was considered in [18].

QUESTION 6. How many ultratypes of  $\mathcal{L}_{\infty}$ -spaces are there?

Currently it is known that there are at least two different ultratypes of  $\mathcal{L}_{\infty}$ : that of  $C(K)$ -spaces and that of Gurariĭ space. More precisely [3]:

THEOREM 2. *All  $C(K)$ -spaces and all complemented subspaces of a  $C(K)$ -space which are isomorphic to their square are ultra-isomorphic.*

THEOREM 3. *No ultrapower of Gurariĭ space is isomorphic to a complemented subspace of a  $C(K)$ -space.*

## 6. FOUR SPECIAL BANACH SPACES

JESÚS M.F. CASTILLO (UNIVERSIDAD DE EXTREMADURA)

Consider this property for a Banach space  $X$ : If  $X$  is a subspace of  $Z$  such that  $Z/X$  is isomorphic to  $X$ , then  $X$  is complemented in  $Z$ . Currently, there are four known examples of spaces with this property:

- $c_0$ , by Sobczyk’s theorem [28];

- injective spaces, by their very definition;
- $L_1(\mu)$  spaces, by Lindenstrauss's lifting [25];
- $C(\mathbb{N}^*)$  [2].

PROBLEM 4. Find more examples of spaces with this property.

PROBLEM 5. Characterise those compact sets  $K$  such that  $C(K)$  has this property.

## 7. CLASSICAL SPACES

VALENTIN FERENCZI (UNIVERSIDADE DE SÃO PAULO)

While it is known that  $\ell_p$  contains a continuum of non-isomorphic subspaces for  $1 \leq p < 2$ , for  $p > 2$  it is only known that there are uncountably many.

QUESTION 7. Does  $\ell_p$  contain a continuum of non-isomorphic subspaces for  $p > 2$ ?

Define  $SB$  as the set of all separable Banach spaces, which amounts to the set of subspaces of  $C[0, 1]$ . This set is Borel and has topological structure. Define  $\simeq_{\ell_2}$  as the set of all spaces isomorphic to  $\ell_2$ , which is a subset of  $SB$  and is Borel.

QUESTION 8. Is there any other example of a Borel isomorphism class?

QUESTION 9. Is the set of all spaces isomorphic to  $\ell_p$  a Borel set for  $p \neq 2$ ?

QUESTION 10. Characterise  $\ell_p$  by local properties.

It is known that  $L_p$  contains uncountably many non-isomorphic subspaces.

QUESTION 11. Does  $L_p$  contain a continuum of non-isomorphic subspaces?

## 8. RUD/RUC BASES IN BANACH SPACES

PEDRO TRADACETE (UNIVERSIDAD CARLOS III DE MADRID)

A series  $\sum_n x_n$  in a Banach space is *randomly unconditionally convergent* when  $\sum_n \varepsilon_n x_n$  converges almost surely on signs  $(\varepsilon_n)_{n \in \mathbb{N}}$  with respect to the Haar probability measure on  $\{-1, 1\}^{\mathbb{N}}$ .

A Schauder basis  $(e_n)_{n \in \mathbb{N}}$  in a Banach space  $X$  is called (RUC) if the expansion  $\sum_{i=1}^{\infty} e_i^*(x)e_i$  of every  $x \in X$  is randomly unconditionally convergent. This is equivalent to the existence of a constant  $K$  such that, for every  $x \in X$ ,

$$\sup_n \int_0^1 \left\| \sum_{i=1}^n r_i(t)e_i^*(x)e_i \right\| dt \leq K\|x\|.$$

It is therefore natural to say that the basis  $(e_n)_{n \in \mathbb{N}}$  is (RUD) (for random unconditionally divergent) if it satisfies the converse inequality: There exists a constant  $K$  such that, for every  $x \in X$ ,

$$\|x\| \leq K \int_0^1 \left\| \sum_{i=1}^n r_i(t)e_i^*(x)e_i \right\| dt.$$

RUC and RUD bases in Banach spaces were studied in [26].

Any RUC basis in  $\ell_1$  is equivalent to its unit vector basis.

QUESTION 12. Is every basis of  $\ell_1$  RUD?

Any sequence which is both RUC and RUD is unconditional.

## 9. CONVERSE AHARONI PROBLEM PANDELIS DODOS (UNIVERSITY OF ATHENS)

It is known that any Banach space that embeds into  $c_0$  must itself contain a copy of  $c_0$ . This means that  $c_0$  is a small space in a linear sense, but what happens from a non-linear point of view?

Aharoni proved that every separable Banach space is Lipschitz-isomorphic to a subset of  $c_0$ , and Kalton, Godefroy and Lancien [14] proved that any space  $X$  that is Lipschitz-isomorphic to a subspace of  $c_0$  must be isomorphic to a subspace of  $c_0$ .

We will say that a Banach space  $Y$  is an Aharoni space if every separable Banach space is Lipschitz-isomorphic to a subspace of  $Y$ .

CONJECTURE 2. Does every separable Aharoni space contain a *linear* copy of  $c_0$ ?

CONJECTURE 3. Let  $f: S_{c_0} \rightarrow \mathbb{R}$  be a uniformly continuous map, where  $S_{c_0} = A \cup B$ , and let  $\varepsilon > 0$ . Then there exist infinitely many block subspaces  $Z$  of  $c_0$  such that  $S_Z \subseteq A_\varepsilon$ .



Let  $X \neq \emptyset$  a finite set. A collection  $\mathcal{A}$  of subsets of  $X$  is said to be an antichain if for every  $A, B \in \mathcal{A}$  with  $A \neq B$ , we have  $A \not\subseteq B$  and  $B \not\subseteq A$ . The simplest example of an antichain is the set  $\binom{X}{k} = \{Y \subseteq X : |Y| = k\}$  of all subsets of  $X$  of cardinality  $k$ .

**THEOREM 4. (SPENNER'S LEMMA)** *Let  $\mathcal{A}$  be an antichain in  $[n] := \{1, 2, \dots, n\}$ . Then  $|\mathcal{A}| \leq \binom{n}{n/2}$ .*

Since  $\binom{n}{n/2} \sim 2^n / \sqrt{n}$ , this means that any set  $\mathcal{F}$  of cardinality greater than  $2^n/2$  cannot be an antichain.

**CONJECTURE 4.** Given  $0 < \delta \leq 1$ , there exists  $n_0(\delta) \in \mathbb{N}$  such that for every  $n \geq n_0$  and every collection  $\mathcal{A}$  of subsets of  $[n]^2$  with  $|\mathcal{A}| \geq \delta 2^{n^2}$ , there are  $A, B \in \mathcal{A}$  such that (i)  $A \subseteq B$ , (ii)  $B \setminus A = X^2$  for some  $X \subseteq [n]$ .

#### 10. ANSWER TO A PROBLEM IN [7] MANUEL GONZÁLEZ AND JAVIER PELLO

Amir-Bahman Nasserri formulated the following question during the open problems session of the 2013 Meeting in Castro Urdiales [7]:

- (1) Let  $T: L^1(0, 1) \rightarrow L^1(0, 1)$  be a bounded operator. Suppose that for each subspace  $M$  of  $L^1(0, 1)$  isometric to  $\ell^1$ , the restriction  $T|_M$  is an isomorphism. Is  $T$  itself an isomorphism?
- (2) Let  $T: \ell^\infty \rightarrow \ell^\infty$  be a bounded operator with dense range. Is  $T$  surjective?

It is possible to show that both problems are equivalent, and recently they received a negative answer [20].

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