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Remarks on Isometries of Products of Linear Spaces

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Abstract: Given two normed spaces X, Y, the aim of this paper is establish that the existence of an isomorphism isometric between $X \times \mathbb{R}$ and $Y \times \mathbb{R}$ is equivalent to the existence of an isometric isomorphism between X and Y, provided the norms satisfy an appropriate condition. By means of a counterexample, it is shown that this result fails for arbitrary norms even if $X = Y = \mathbb{R}^2$.

Key words: Normed space, isometry.

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1. Introduction

The motivation behind this problem traces its origin back to [4], in connection with the Banach-Stone theorem. There are natural situations under which, if two metric functions spaces $(\mathcal{A}(X), d)$ and $(\mathcal{A}(Y), d')$ –defined on the Banach spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ respectively– are isometrically isomorphic, then $(X \times \mathbb{R}, \|\cdot\|_{X\times\mathbb{R}})$ and $(Y \times \mathbb{R}, \|\cdot\|_{Y\times\mathbb{R}})$ are isometrically isomorphic as Banach spaces under the norms $\|(x, t)\|_{X\times\mathbb{R}} := \|x\|_X + |t|$, for $(x, t) \in X \times \mathbb{R}$ and $\|(y, t)\|_{Y\times\mathbb{R}} := \|y\|_Y + |t|$, for $(y, t) \in Y \times \mathbb{R}$. So, with the aim of obtaining a version of the Banach-Stone theorem, it is natural to investigate whether this situation implies that $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are isometrically isomorphic. Notice that if $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are isometrically isomorphic, then we generally have that $(\mathcal{A}(X), d)$ and $(\mathcal{A}(Y), d)$ are also isometrically isomorphic. This problem was solved positively in [4] for norms of the form $\|(x, t)\|_{X\times\mathbb{R}} := (\|x\|_X^p + |t|^p)^{\frac{1}{p}}$ for all $(x, t) \in X \times \mathbb{R}$ when $p \in [1, \infty[\setminus \{2\}$. The property proposed in this paper is more general.

We provide applications of our main theorem to the space of continuously differentiable functions and the spaces of affine functions. We also provide counterexamples for norms which do not satisfy our property.

2. The main theorem.

We use the following property.

Definition 1. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed vector spaces. Let $\|\cdot\|_{X\times\mathbb{R}}$ and $\|\cdot\|_{Y\times\mathbb{R}}$ be two norms defined on $X\times\mathbb{R}$ and $Y\times\mathbb{R}$ respectively. We say that the pair of norms $(\|\cdot\|_{X\times\mathbb{R}}, \|\cdot\|_{Y\times\mathbb{R}})$ satisfies the property (P) if

(i) $\|(x,t)\|_{X\times\mathbb{R}} \geq \|(x,0)\|_{X\times\mathbb{R}} = \|x\|_X$ for all $(x,t) \in X \times \mathbb{R}$ and $\|(y,t)\|_{Y\times\mathbb{R}} \geq \|(y,0)\|_{Y\times\mathbb{R}} = \|y\|_Y$ for all $(y,t) \in Y \times \mathbb{R}$.

(ii) for all $x \in X$ and $y \in Y$:

$$\|x\|_X = \|y\|_Y \Rightarrow \|(x,\lambda)\|_{X\times\mathbb{R}} = \|(y,\lambda)\|_{Y\times\mathbb{R}}, \ \ \forall\lambda \in \mathbb{R}.$$

(iii) Let $(a,u) \in X \times \mathbb{R}$ and $(b,v) \in Y \times \mathbb{R}$ such that $a \neq 0$ and $b \neq 0$. If for all $(\alpha,\beta) \in \mathbb{R}^2$, $\|(\alpha(0,1) + \beta(a,u)\|_{X\times\mathbb{R}} = \|(\beta(0,1) + \alpha(b,v)\|_{Y\times\mathbb{R}}$ then $u = v = 0$.

Remark 1. Let us note that property (P) in this paper does not appear to be related to the notion of orthogonality. However there exists some similarity between these two notions. For interesting results on the various notions of orthogonality in normed vector spaces, we refer to [1], [2] and [3].

Theorem 1. *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed vector spaces. Suppose that $(\|\cdot\|_{X\times\mathbb{R}}, \|\cdot\|_{Y\times\mathbb{R}})$ satisfies the property (P). Then $(X \times \mathbb{R}, \|\cdot\|_{X\times\mathbb{R}})$ and $(Y \times \mathbb{R}, \|\cdot\|_{Y\times\mathbb{R}})$ are isometrically isomorphic if and only if $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are isometrically isomorphic.*

Our theorem applies to many norms, see for instance Example 1 and more generally Proposition 1. We give now a generic counterexample showing that Theorem 1 fails for arbitrary norms.

Counterexample 1. Let $X = Y = \mathbb{R}^2$. For each norm $\|\cdot\|_X$ on X there exist a norm $\|\cdot\|_Y$ on Y, a norm $\|\cdot\|_{X\times\mathbb{R}}$ on $X \times \mathbb{R}$ and a norm $\|\cdot\|_{Y\times\mathbb{R}}$ on $Y \times \mathbb{R}$ such that:

(1) $(X, \|\cdot\|_X)$ is not isometrically isomorphic to $(Y, \|\cdot\|_Y)$.
(2) $(X \times \mathbb{R}, \|\cdot\|_{X\times\mathbb{R}})$ is isometrically isomorphic to $(Y \times \mathbb{R}, \|\cdot\|_{Y\times\mathbb{R}})$.
(3) $\|\cdot\|_{X\times\mathbb{R}}$ coincide with $\|\cdot\|_X$ on $X \times \{0\}$ and $\|\cdot\|_{Y\times\mathbb{R}}$ coincide with $\|\cdot\|_Y$ on $Y \times \{0\}$.

Proof. Let $p \in [1, +\infty[$. Let us define $\|\cdot\|_{X\times\mathbb{R}}$ and $\|\cdot\|_{Y\times\mathbb{R}}$ as follows:

$$\|(x_1, x_2, t)\|_{X\times\mathbb{R}} := \left(\|(x_1, x_2)\|_X^p + |t|^p\right)^{\frac{1}{p}}, \quad \forall (x_1, x_2, t) \in X \times \mathbb{R}$$

and

$$\|(y_1, y_2, s)\|_{Y\times\mathbb{R}} := \left(|y_2|^p + \frac{\|(y_1, s)\|_X^p}{c^p}\right)^{\frac{1}{p}}, \quad \forall (y_1, y_2, s) \in Y \times \mathbb{R},$$

where $c = \|(1, 0)\|_X$. Let us define the norm $\|\cdot\|_{Y,p}$ on Y as follows:

$$\|(y_1, y_2)\|_{Y,p} := \left(|y_1|^p + |y_2|^p\right)^{\frac{1}{p}},$$

for all $(y_1, y_2) \in Y$. Clearly,

$$\|(x_1, x_2, 0)\|_{Y\times\mathbb{R}} = \|(x_1, x_2)\|_X, \quad \forall (x_1, x_2) \in X$$

and

$$\|(y_1, y_2, 0)\|_{Y\times\mathbb{R}} = \left(|y_1|^p + |y_2|^p\right)^{\frac{1}{p}} := \|(y_1, y_2)\|_{Y,p}, \quad \forall (y_1, y_2) \in Y.$$

(Since $\frac{\|(y_1,0)\|_X^p}{c^p} = |y_1|^p \frac{\|(1,0)\|_X^p}{c^p} = |y_1|^p$). On the other hand, the following map is an isometric isomorphism:

$$\begin{array}{rcl} \Theta : (X \times \mathbb{R}, \|\cdot\|_{X\times\mathbb{R}}) & \to & (Y \times \mathbb{R}, \|\cdot\|_{Y\times\mathbb{R}}) \\ (x_1, x_2, t) & \mapsto & (cx_1, t, cx_2). \end{array}$$

There exist two cases:

Case 1: If every point of the sphere S_X of X is an extreme point, we choose $p = 1$ and so the sphere S_Y of Y has no extreme point: indeed, in this case $\|(y_1, y_2)\|_Y := \|(y_1, y_2)\|_{Y,1} = |y_1| + |y_2|$. (For example $(\frac{1}{2}, \frac{1}{2})$ is not extreme for $\|\cdot\|_{Y,1}$). Consequently $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ cannot be isometrically isomorphic.

Case 2: If there exists some point of the sphere S_X which is not extreme, then choosing $p = 2$ we get that every point of the sphere S_Y is an extreme point: indeed in this case $\|(y_1, y_2)\|_Y := \|(y_1, y_2)\|_{Y,2} = (|y_1|^2 + |y_2|^2)^{\frac{1}{2}}$ is the Euclidean norm. Again $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ cannot be isometrically isomorphic. ∎

The idea of the above counterexample can be extended to infinite dimensions.

COUNTEREXAMPLE 2. Let $(X, \|\cdot\|_X)$ be a smooth normed vector space (The norm $\|\cdot\|_X$ is Gâteaux differentiable outside 0). Then there exist another norm $\|\cdot\|$ on X and two norms N_1 and N_2 on $X \times \mathbb{R}$ such that:

(1) $(X, \|\cdot\|_X)$ is not isometrically isomorphic to $(X, \|\cdot\|)$.
(2) $(X \times \mathbb{R}, N_1)$ is isometrically isomorphic to $(X \times \mathbb{R}, N_2)$.
(3) N_1 coincide on $X \times \{0\}$ with $\|\cdot\|_X$ and N_2 coincide on $X \times \{0\}$ with $\|\cdot\|$.

Proof. Let H be an hyperplane of $(X, \|\cdot\|_X)$. So, there exists $e \in X$ such that for each $x \in X$, there exists a unique $(x_H, t_H) \in H \times \mathbb{R}$ such that $x = x_H + t_H e$. We define the norm $\|\cdot\|$ on X as follow $\|x\| := \|x_H\|_X + |t_H|$ for all $x \in X$. We define N_1 and N_2 as follows.

$$N_1(x,t) := \|x\|_X + |t|; \quad \forall (x,t) \in X \times \mathbb{R}$$

$$N_2(x,t) := \|x_H + te\|_X + |t_H|; \quad \forall (x,t) \in X \times \mathbb{R}.$$

Clearly,

$$N_1(x,0) = \|x\|_X; \quad \forall x \in X$$

and

$$N_2(x,0) = \|x_H\|_X + |t_H| = \|x\|; \quad \forall x \in X.$$

On the other hand, the following map is an isometric isomorphism

$$\begin{array}{rcl} \Theta : (X \times \mathbb{R}, N_1) & \to & (X \times \mathbb{R}, N_2) \\ (x,t) & \mapsto & (x_H + te, t_H) \end{array}$$

But $(X, \|\cdot\|_X)$ and $(X, \|\cdot\|)$ cannot be isometrically isomorphic since the norm $\|\cdot\|_X$ is smooth and the norm $\|\cdot\|$ is not smooth. ∎

We give now examples of norms satisfying our theorem.

EXAMPLE 1. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed vector spaces.
(1) Let $p \in [1, +\infty[\setminus \{2\}$; we define the l_p^2-norms as follows:

$$\|(x,t)\|_{X \times \mathbb{R}} := (\|x\|_X^p + |t|^p)^{\frac{1}{p}},$$

or

$$\|(x,t)\|_{X \times \mathbb{R}} := \max(\|x\|_X, |t|),$$

for all $(x,t) \in X \times \mathbb{R}$. We define $\|\cdot\|_{Y \times \mathbb{R}}$ in an analogous manner as we did for the norm $\|\cdot\|_{X \times \mathbb{R}}$. Then the pair $(\|\cdot\|_{X \times \mathbb{R}}, \|\cdot\|_{Y \times \mathbb{R}})$ satisfies the property (P).

Note that the part (iii) of the property (P) fails for the l_p^2-norms with $p=2$. However, we obtain from the Proposition 1 below a criterion which provides the property (P) through a general class of norms N on $\mathbb{R}^2$. Recall that a norm N on a vector lattice space E is absolute if $N(|x|) = N(x)$ for all $x \in E$; and monotone if $N(x) \leq N(y)$ whenever $0 \leq x \leq y$. If the norm is both absolute and monotone, it is called a Riesz norm. It is easy to show that a norm is Riesz if and only if $N(x) \leq N(y)$ whenever $|x| \leq |y|$. Any absolute norm on $\mathbb{R}^n$ (equipped with the usual order) is already monotone, hence a Riesz norm (see [6, Theorem 2], and [7]). We set now $E = \mathbb{R}^2$ and let N be a Riesz norm on $\mathbb{R}^2$. We denote by $\mathrm{Isom}(\mathbb{R}^2, N)$ the group of all automorphism isometric of $(\mathbb{R}^2, N)$. Let S_2 be the group of permutations of $\{1,2\}$. For $\sigma \in S_2$ and $\lambda := (\lambda_1, \lambda_2) \in (\mathbb{R}^*)^2$, we denote by $u_{\sigma,\lambda}$ the automorphism of $\mathbb{R}^2$ defined by $u_{\sigma,\lambda} : (t_1,t_2) \mapsto (\lambda_1 t_{\sigma(1)}, \lambda_2 t_{\sigma(2)})$. By I_2 we denote the following group of automorphisms

$$I_2 := \left\{ u_{\sigma,\lambda} / \sigma \in S_2; \lambda \in (\mathbb{R}^*)^2 \right\}.$$

Recall (see [5]) that the group of isometries of the l_p^2-norms on $\mathbb{R}^2$ for $p \in [1,+\infty] \setminus \{2\}$ is exactly the set $\left\{ u_{\sigma,\lambda} / \sigma \in S_2; \lambda \in \{-1,1\}^2 \right\} \subset I_2$.

Proposition 1. *For each Riesz norm N on $\mathbb{R}^2$, if $\mathrm{Isom}(\mathbb{R}^2, N) \subset I_2$ then the pair $(\|\cdot\|_{X\times\mathbb{R}}, \|\cdot\|_{X\times\mathbb{R}})$ satisfies the property (P) where $\|(x,t)\|_{X\times\mathbb{R}} := cN(\|x\|_X, |t|)$ for all $(x,t) \in X \times \mathbb{R}$ and $\|(y,t)\|_{Y\times\mathbb{R}} := cN(\|y\|_Y, |t|)$ for all $(y,t) \in Y \times \mathbb{R}$ with $c := \frac{1}{N(1,0)}$.*

Proof. The part (i) and (ii) in Definition 1 are easy to verify. Let us prove that if $\mathrm{Isom}(\mathbb{R}^2, N) \subset I_2$ then the part (iii) of Definition 1 is verified. Indeed, let $(a,u) \in X \times \mathbb{R}$ and $(b,v) \in Y \times \mathbb{R}$ be such that

$$N\big(|\alpha|\|a\|_X, |\alpha u + \beta|\big) = N\big(|\beta|\|b\|_Y, |\beta v + \alpha|\big); \quad \forall (\alpha,\beta) \in \mathbb{R}^2. \tag{1}$$

Suppose that $a \neq 0$ and $b \neq 0$, then the map $\varphi : (\alpha\|a\|_X, \alpha u + \beta) \mapsto (\beta\|b\|_Y, \beta v + \alpha)$ is an automorphism of $\mathbb{R}^2$. Since the norm N is absolute, from the formula (1) we get

$$\begin{aligned} N\big(\beta\|b\|_Y, \beta v + \alpha\big) &= N\big(|\beta|\|b\|_Y, |\beta v + \alpha|\big) \\ &= N\big(|\alpha|\|a\|_X, |\alpha u + \beta|\big) \\ &= N\big(\alpha\|a\|_X, \alpha u + \beta\big). \end{aligned} \tag{2}$$

Using a change of variables by putting $t_1 = \alpha\|a\|_X$ and $t_2 = \alpha u + \beta$ we have

$$\varphi : (t_1, t_2) \mapsto \left(\frac{-u\|b\|_Y}{\|a\|_X} t_1 + \|b\|_Y t_2, \frac{1 - uv}{\|a\|_X} t_1 + vt_2\right)$$

for all $(t_1, t_2) \in \mathbb{R}^2$ and from (2) we obtain:

$$N\big(\varphi(t_1, t_2)\big) = N(t_1, t_2), \quad \forall (t_1, t_2) \in \mathbb{R}^2. \tag{3}$$

The formula (3) means that φ is isometric for the norm N. So, by our hypothesis φ must be an element of I_2, which implies that $\varphi = u_{\sigma,\lambda}$ for some $\sigma \in S_2$ and $\lambda \in (\mathbb{R}^*)^2$. Since $b \neq 0$, the unique possibility is that $\frac{-u\|b\|_Y}{\|a\|_X} = 0$ and $v = 0$. Thus we have $u = v = 0$ which implies the property (P). ∎

Remark 2. Using Minkowski functional of appropriate convex and symmetric sets of $\mathbb{R}^2$, we can easily construct Riesz norms satisfying the hypothesis of Proposition 1. For example the set

$$C_\epsilon := \big([-1, 1] \times [-1 + \epsilon, 1 - \epsilon]\big) \bigcap \big\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\big\}$$

(for $0 < \epsilon < 1$) is the closed unit ball of some Riesz norm N_{C_ϵ} (the Minkowski functional of C_ϵ) satisfying $\mathrm{Isom}(N_{C_\epsilon}, \mathbb{R}^2) = \{\pm i, \pm u\} \subset I_2$ where i denotes the identity map of $\mathbb{R}^2$ and $u : (t_1, t_2) \to (-t_1, t_2)$ for all $(t_1, t_2) \in \mathbb{R}^2$. Note that $\|\cdot\|_2 \leq N_{C_\epsilon} \leq \frac{1}{1-\epsilon}\|\cdot\|_2$ where $\|\cdot\|_2$ denotes the euclidean norm on $\mathbb{R}^2$.

3. Notation

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed vector spaces and let $\|\cdot\|_{X\times\mathbb{R}}$ be a norm on $X \times \mathbb{R}$ and $\|\cdot\|_{Y\times\mathbb{R}}$ be a norm on $Y \times \mathbb{R}$. Let

$$\Theta : (X \times \mathbb{R}, \|\cdot\|_{X\times\mathbb{R}}) \to (Y \times \mathbb{R}, \|\cdot\|_{Y\times\mathbb{R}})$$

be an isomorphism. There exists two linear operators $T : X \to Y$ and $S : Y \to X$, two real valued linear maps $p : X \to \mathbb{R}$ and $q : Y \to \mathbb{R}$ and two vectors $(a, u) := \Theta^{-1}(0, 1) \in X \times \mathbb{R}$ and $(b, v) := \Theta(0, 1) \in Y \times \mathbb{R}$, such that:

$$\Theta(x, t) = \big(Tx + tb, p(x) + tv\big); \quad \forall (x, t) \in X \times \mathbb{R} \tag{4}$$

and

$$\Theta^{-1}(y, t) = \big(Sy + ta, q(y) + tu\big); \quad \forall (y, t) \in Y \times \mathbb{R}. \tag{5}$$

Using the fact that $\Theta^{-1}(\Theta(x,t)) = (x,t)$ for all $(x,t) \in X \times \mathbb{R}$, we obtain the following formulas

$$S \circ Tx + p(x)a = x; \quad \forall x \in X, \tag{6}$$
$$Sb = -va, \tag{7}$$
$$q(Tx) + p(x)u = 0; \quad \forall x \in X, \tag{8}$$
$$q(b) = 1 - uv. \tag{9}$$

By inverting, the roles of Θ and Θ^{-1} we obtain:

$$T \circ Sy + q(y)b = y; \quad \forall y \in Y, \tag{10}$$
$$Ta = -ub, \tag{11}$$
$$p(S(y)) + q(y)v = 0; \quad \forall y \in Y, \tag{12}$$
$$p(a) = 1 - uv. \tag{13}$$

Finally, we denote $Ker(p) := \big\{x \in X : p(x) = 0\big\}$ and $Ker(q) := \big\{y \in Y : q(y) = 0\big\}$.

4. Proof of the main theorem

Before proving our main theorem, we need the following lemmas.

Lemma 1. *Suppose that $\Theta : (X \times \mathbb{R}, \|\cdot\|_{X\times\mathbb{R}}) \to (Y \times \mathbb{R}, \|\cdot\|_{X\times R})$ is an isometric isomorphism. Then, the spaces $Ker(p) \times \{0\}$ and $Ker(q) \times \{0\}$ are isometrically isomorphic. More precisely, the map*

$$\begin{aligned} \Theta : \big(Ker(p) \times \{0\}, \|\cdot\|_{X\times\mathbb{R}}\big) &\to \big(Ker(q) \times \{0\}, \|\cdot\|_{Y\times\mathbb{R}}\big) \\ (x,0) &\mapsto \Theta(x,0) \end{aligned} \tag{14}$$

is an isometric isomorphism.

Proof. Since Θ is an isomorphism isometric, it suffices to show that Θ sends $Ker(p) \times \{0\}$ into $Ker(q) \times \{0\}$ and Θ^{-1} sends $Ker(q) \times \{0\}$ into $Ker(p) \times \{0\}$. Indeed, suppose that $x \in Ker(p)$. Using the formula (4) we have $\Theta(x,0) = (Tx,0)$. Now, by using the formula (8) we have $q(Tx) = 0$ since $p(x) = 0$. It follows that $Tx \in Ker(q)$ and so Θ sends $Ker(p) \times \{0\}$ into $Ker(q) \times \{0\}$. In a similar way we prove that Θ^{-1} sends $Ker(q) \times \{0\}$ into $Ker(p) \times \{0\}$. ∎

LEMMA 2. *We have that $a \neq 0$ if and only if $b \neq 0$.*

Proof. Let $a \neq 0$ and suppose that $b = 0$. By the formula (7) we have that $va = 0$, which implies that $v = 0$ since $a \neq 0$. Using the formula (9) we obtain that $q(b) = 1$ which is a contradiction since $b = 0$. So we have that $a \neq 0 \Rightarrow b \neq 0$. In a similar way we prove the converse. ∎

We give now the proof of the main result.

Proof of Theorem 1. For the "if" part, suppose that $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are isometrically isomorphic and let $T : (X, \|\cdot\|_X) \to (Y, \|\cdot\|_Y)$ be an isomorphism isometric. Let us define $\Theta : (X \times \mathbb{R}, \|\cdot\|_{X\times\mathbb{R}}) \to (Y \times \mathbb{R}, \|\cdot\|_{Y\times\mathbb{R}})$ by $\Theta(x, \lambda) = (T(x), \lambda)$ for all $(x, \lambda) \in X \times \mathbb{R}$. Then, clearly Θ is an isomorphism and by part (ii) of the property (P), it is also isometric.

We prove now the "only if part". Indeed, suppose that there exists an isomorphism isometric $\Theta : (X \times \mathbb{R}, \|\cdot\|_{X\times\mathbb{R}}) \to (Y \times \mathbb{R}, \|\cdot\|_{Y\times\mathbb{R}})$. We need to consider two cases: $a = 0$ or $a \neq 0$.

Case 1. If $a = 0$ then $b = 0$ by Lemma 2. Using the formulas (6) and (10) we obtain that T is an isomorphism from X onto Y with the inverse S. To see that T is an isometry, we use the formula (4) and (5) to obtain that $\Theta(x, 0) = (Tx, p(x))$ and $\Theta^{-1}(y, 0) = (T^{-1}(y), q(y))$ for all $x \in X$ and $y \in Y$. Replacing y by Tx we obtain $\Theta^{-1}(T(x), 0) = (x, q(Tx))$. Now, using the part (i) of the property (P) and the fact that Θ is isometric, we get

$$\|x\|_X = \|\Theta(x,0)\|_{Y\times\mathbb{R}} = \|(Tx, p(x))\|_{Y\times\mathbb{R}} \geq \|Tx\|_Y$$

and

$$\|Tx\|_Y = \|\Theta^{-1}(Tx,0)\|_{X\times\mathbb{R}} = \|(x, q(Tx))\|_{X\times\mathbb{R}} \geq \|x\|_X.$$

By combining the two inequalities we have that T is isometric. Thus $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are isometrically isomorphic.

Case 2. If $a \neq 0$ then $b \neq 0$ by Lemma 2. Since Θ is isometric then for all $(\alpha, \beta) \in \mathbb{R}^2$ we have

$$\begin{aligned}\big\|\alpha(a,u) + \beta(0,1)\big\|_{X\times\mathbb{R}} &= \big\|\Theta\big(\alpha(a,u) + \beta(0,1)\big)\big\|_{Y\times\mathbb{R}}\\ &= \big\|\alpha(0,1) + \beta(b,v)\big\|_{Y\times\mathbb{R}}.\end{aligned}$$

This implies that $u = v = 0$ by the part (iii) of the property (P). Thus we have $\Theta^{-1}(0,1) = (a,0)$ and $\Theta(0,1) = (b,0)$. We first show that $X \times \{0\} = Ker(p) \times \{0\} \oplus \mathbb{R}(a,0)$ and $Y \times \{0\} = Ker(q) \times \{0\} \oplus \mathbb{R}(b,0)$. Indeed, For each $x \in X$

we can write $x = (x - p(x)a) + p(x)a$. Since $u = 0$ then $p(a) = 1$ from formula (13). Thus $x - p(x)a \in Ker(p)$ and so $X \times \{0\} = Ker(p) \times \{0\} \oplus \mathbb{R}(a, 0)$. In a similar way we prove the second part. Now we prove that the map

$$\begin{aligned}\Delta : X \times \{0\} = Ker(p) \times \{0\} \oplus \mathbb{R}(a, 0) \; &\to \; Y \times \{0\} = Ker(q) \times \{0\} \oplus \mathbb{R}(b, 0)\\ (x, 0) + \lambda(a, 0) \; &\mapsto \; \Theta(x, 0) + \lambda(b, 0)\end{aligned}$$

is an isomorphism isometric. Indeed, the fact that Δ is linear and onto map is clear by using Lemma 1. Let us prove that Δ is isometric. For all $(x, 0) \in Ker(p) \times \{0\}$, by Lemma 1 there exists $(y, 0) \in Ker(q) \times \{0\}$ such that $\Theta(x, 0) = (y, 0)$ and we have $\|x\|_X = \|(x, 0)\|_{X\times\mathbb{R}} = \|(y, 0)\|_{Y\times\mathbb{R}} = \|y\|_Y$, since Θ is isometric. On the other hand, since

$$\begin{aligned}(x, 0) + \lambda(a, 0) &= \Theta^{-1}(\Theta(x, 0)) + \lambda\Theta^{-1}(0, 1)\\ &= \Theta^{-1}\left(\Theta(x, 0) + (0, \lambda)\right)\\ &= \Theta^{-1}(y, \lambda)\end{aligned}$$

then, using the fact that Θ^{-1} is isometric we have

$$\begin{aligned}\|(x, 0) + \lambda(a, 0)\|_{X\times\mathbb{R}} &= \|\Theta^{-1}(y, \lambda)\|_{X\times\mathbb{R}}\\ &= \|(y, \lambda)\|_{Y\times\mathbb{R}}\end{aligned} \tag{15}$$

On the other hand we know that $(b, 0) = \Theta(0, 1)$, so $\Theta(x, 0) + \lambda(b, 0) = \Theta(x, 0) + \lambda\Theta(0, 1) = \Theta(x, \lambda)$. Thus, using the fact that Θ is isometric we have,

$$\begin{aligned}\|\Delta\left((x, 0) + \lambda(a, 0)\right)\|_{Y\times\mathbb{R}} &= \|\Theta(x, 0) + \lambda(b, 0)\|_{Y\times\mathbb{R}}\\ &= \|\Theta(x, \lambda)\|_{Y\times\mathbb{R}}\\ &= \|(x, \lambda)\|_{X\times\mathbb{R}}.\end{aligned} \tag{16}$$

Now, since $\|x\|_X = \|y\|_Y$ and since $(\|\cdot\|_{X\times\mathbb{R}}, \|\cdot\|_{Y\times\mathbb{R}})$ satifies the property (P), then $\|(x, \lambda)\|_{X\times\mathbb{R}} = \|(y, \lambda)\|_{Y\times\mathbb{R}}$. Thus, using the formulas (15) and (16) we obtain that Δ is isometric. Finally, since $\|\cdot\|_{X\times\mathbb{R}}$, and $\|\cdot\|_{Y\times\mathbb{R}}$ coincide with $\|\cdot\|_X$ and $\|\cdot\|_Y$ on $X \times \{0\}$ and $Y \times \{0\}$ respectively, we obtain that $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are isometrically isomorphic. ∎

By induction, we can easily extend our main theorem (see the corollary below) to $X \times \mathbb{R}^n$ and $Y \times \mathbb{R}^n$ $(n \in N^*)$ with the norms $\|\cdot\|_{X\times\mathbb{R}^n}$ and $\|\cdot\|_{Y\times\mathbb{R}^n}$ defined inductively as follows: Let N be a Riesz norm on $\mathbb{R}^2$.

For all $(a, t_1, \dots, t_n) \in \mathbb{R}^+ \times \mathbb{R}^n$, we set $N_1(a, |t_1|) := N(a, |t_1|)$ and for all $k \in \{1, \dots, n-1\}$, $N_{k+1}(a, |t_1|, \dots, |t_{k+1}|) := N(N_k(a, |t_1|, \dots, |t_k|), |t_{k+1}|)$. We then define

$$\|(x, t_1, t_2, \dots, t_n)\|_{X\times\mathbb{R}^n} := N_n\big(\|x\|_X, |t_1|, \dots, |t_n|\big); \quad \forall (x, t_1, \dots, t_n) \in X\times\mathbb{R}^n$$

and

$$\|(y, t_1, t_2, \dots, t_n)\|_{Y\times\mathbb{R}^n} := N_n\big(\|y\|_Y, |t_1|, \dots, |t_n|\big); \quad \forall (y, t_1, \dots, t_n) \in Y\times\mathbb{R}^n.$$

Example 2. Let $p \in [1, +\infty[\setminus \{2\}$, and let N be the norm on R^2 defined by $N(s_1, s_2) := (|s_1|^p + |s_2|^p)^{\frac{1}{p}}$ $\big($respectively, $N(s_1, s_2) := \max(|s_1|, |s_2|)\big)$ for all $(s_1, s_2) \in \mathbb{R}^2$, then we have

$$\|(x, t_1, \dots, t_n)\|_{X\times\mathbb{R}^n} := N_n\big(\|x\|_X, |t_1|, \dots, |t_n|\big) = \left(\|x\|_X^p + \sum_{k=1}^{n} |t_k|^p\right)^{\frac{1}{p}},$$

$\big($respectively, $\|(x, t_1, \dots, t_n)\|_{X\times\mathbb{R}^n} := N_n(\|x\|_X, |t_1|, \dots, |t_n|) = \max(\|x\|_X, |t_1|, \dots, |t_n|)\big)$ for all $(x, t_1, \dots, t_n) \in X \times \mathbb{R}^n$.

Corollary 1. *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed vector spaces. Let N be a Riesz norm on $\mathbb{R}^2$ such that $\mathrm{Isom}(\mathbb{R}^2, N) \subset I_2$ and let $n \in \mathbb{N}^*$. Then $(X \times \mathbb{R}^n, \|\cdot\|_{X\times\mathbb{R}^n})$ and $(Y \times \mathbb{R}^n, \|\cdot\|_{Y\times\mathbb{R}^n})$ are isometrically isomorphic if and only if $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are isometrically isomorphic.*

Remark 3. Note that the spaces l_p^n and l_q^n are not isometric if $p \neq q$ except in the case where $p, q \in \{1, \infty\}$ and $n = 2$. Here, l_p^n denotes the space $\mathbb{R}^n$ endowed with the norm $\|t\|_p := (\sum_{k=1}^n |t_k|^p)^{\frac{1}{p}}$ $\big($respectively $\|t\|_\infty := \max(|t_1|, \dots, |t_n|)\big)$ for all $t = (t_1, \dots, t_n) \in \mathbb{R}^n$. For more information see [10]. So from Corollary 1 we deduce that for all $p, q \in [1, +\infty]$, $r \in [1, +\infty] \setminus \{2\}$ and $n, m \in \mathbb{N}^*$ the spaces $l_p^n \underbrace{\oplus}_{l_r^2} l_r^m$ and $l_q^n \underbrace{\oplus}_{l_r^2} l_r^m$ are not isometric except in the case where $p, q \in \{1, \infty\}$ and $n = 2$.

5. Applications

We give in this section some applications of Theorem 1. In what follows, N denotes a Riesz norm on $\mathbb{R}^2$, such that $\mathrm{Isom}(\mathbb{R}^2, N) \subset I_2$. We denote by $C^1[0, 1]$ the space of continuously differentiable functions on $[0, 1]$, by

$(C[0,1], \|\cdot\|_\infty)$ the space of continuous functions on $[0,1]$ endowed with the supremum norm. Let $(X, \|\cdot\|_X)$ be a Banach space. We consider the following norms on $C^1[0,1]$, $X \times \mathbb{R}$ and $C[0,1] \times \mathbb{R}$ respectively (see Proposition 1).

$$\begin{aligned} \|f\|_{C^1[0,1]} &:= N(\|f'\|_\infty, |f(0)|) \\ \|(x,t)\|_{X\times\mathbb{R}} &:= N(\|x\|_X, |t|) \\ \|(g,t)\|_{C[0,1]\times\mathbb{R}} &:= N(\|g\|_\infty, |t|). \end{aligned}$$

Proposition 2. *We have* $(X \times \mathbb{R}, \|\cdot\|_{X\times\mathbb{R}}) \cong (C^1[0,1], \|\cdot\|_{C^1[0,1]})$ *if and only if* $(X, \|\cdot\|_X) \cong (C[0,1], \|\cdot\|_\infty)$.

Proof. Let us consider the map

$$\begin{array}{rcl} \chi : (C^1[0,1], \|\cdot\|_{C^1[0,1]}) & \to & (C[0,1] \times \mathbb{R}, \|\cdot\|_{C[0,1]\times\mathbb{R}}) \\ f & \mapsto & (f', f(0)) \end{array}$$

Clearly, χ is an isomorphism isometric. So we have $(X \times \mathbb{R}, \|\cdot\|_{X\times\mathbb{R}}) \cong (C^1[0,1], \|\cdot\|_{C^1[0,1]})$ if and only if $(X \times \mathbb{R}, \|\cdot\|_{X\times\mathbb{R}}) \cong (C[0,1] \times \mathbb{R}, \|\cdot\|_{C[0,1]\times\mathbb{R}})$. Since the pair $(\|\cdot\|_{X\times\mathbb{R}}, \|\cdot\|_{C[0,1]\times\mathbb{R}})$ satisfies the property (P) by Proposition 1 then using Theorem 1 we obtain that $(X \times \mathbb{R}, \|\cdot\|_{X\times\mathbb{R}}) \cong (C[0,1] \times \mathbb{R}, \|\cdot\|_{C[0,1]\times\mathbb{R}})$ if and only if $(X, \|\cdot\|_X) \cong (C[0,1], \|\cdot\|_\infty)$. ∎

Finally we give an application to the spaces of affine functions. Let K and C be convex subsets of vector spaces. A function $T : K \to C$ is said to be affine if for all $x, y \in K$ and $0 \leq \lambda \leq 1$, $T(\lambda x + (1-\lambda)y) = \lambda T(x) + (1-\lambda)T(y)$. The set of all continuous real-valued affine functions on a convex subset K of a topological vector space will be denoted by $\mathit{Aff}(K)$. Clearly, all translates of continuous linear functionals are elements of $\mathit{Aff}(K)$, but the converse in not true in general (see [9, page 22]). However, we do have the following relationship.

Proposition 3. ([9, Proposition 4.5]) *Assume that K is a compact convex subset of a separated locally convex space X then*

$$\left\{ a \in \mathit{Aff}(K) : a = r + x^*_{|K} \text{ for some } x^* \in X^* \text{ and some } r \in R \right\}$$

is dense in $(\mathit{Aff}(K), \|\cdot\|_\infty)$*, where* $\|\cdot\|_\infty$ *denotes the norm of uniform convergence.*

But in the particular case when $(X, \|\cdot\|_X)$ is a Banach space and $K = (B_{X^*}, w^*)$ is the unit ball of the dual space X^* endowed with the weak star topology, the well known result due to Banach and Dieudonné states that:

THEOREM 2. (Banach-Dieudonné). *The space $(\mathit{Aff}_0(B_{X^*}), \|\cdot\|_\infty)$ of all affine weak star continuous functions that vanish at 0, is isometrically identified to $(X, \|\cdot\|_X)$. In other words, $\mathit{Aff}_0(B_{X^*}) = \left\{\hat{z}_{|B_{X^*}} : z \in X\right\}$. Where $\hat{z} : p \mapsto p(z)$ for all $p \in X^*$ and $\hat{z}_{|B_{X^*}}$ denotes the restriction of $\hat{z}$ to B_{X^*}.*

Now, let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two Banach spaces. We consider the following norms on $\mathit{Aff}(B_{X^*})$ and $\mathit{Aff}_0(B_{X^*}) \times \mathbb{R}$ respectively (and in a similar way, we define norms on $\mathit{Aff}(B_{Y^*})$ and $\mathit{Aff}_0(B_{Y^*}) \times \mathbb{R}$ by replacing X by Y). (See Proposition 1).

$$\|f\|_{\mathit{Aff}(B_{X^*})} := N(\|f - f(0)\|_\infty, |f(0)|); \quad \forall f \in \mathit{Aff}(B_{X^*})$$

$$\|(f_0, t)\|_{\mathit{Aff}_0(B_{X^*}) \times \mathbb{R}} := N(\|f_0\|_\infty, |t|); \quad \forall (f_0, t) \in \mathit{Aff}_0(B_{X^*}) \times \mathbb{R}.$$

We obtain the following version of the Banach-Stone theorem for affine functions. For more information about the Banach-Stone theorem see [8].

PROPOSITION 4. *The following assertions are equivalent.*

(1) *$(\mathit{Aff}(B_{X^*}), \|\cdot\|_{\mathit{Aff}(B_{X^*})})$ and $(\mathit{Aff}(B_{Y^*}), \|\cdot\|_{\mathit{Aff}(B_{Y^*})})$ are isometrically isomorphic.*

(2) *$(\mathit{Aff}_0(B_{X^*}), \|\cdot\|_\infty)$ and $(\mathit{Aff}_0(B_{Y^*}), \|\cdot\|_\infty)$ are isometrically isomorphic.*

(3) *$(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are isometrically isomorphic.*

Proof. Let us consider the map,

$$\begin{array}{rcl} \chi : \left(\mathit{Aff}(B_{X^*}), \|\cdot\|_{\mathit{Aff}(B_{X^*})}\right) & \to & \left(\mathit{Aff}_0(B_{X^*}) \times \mathbb{R}, \|\cdot\|_{\mathit{Aff}_0(B_{X^*}) \times \mathbb{R}}\right) \\ f & \mapsto & (f - f(0), f(0)). \end{array}$$

Clearly, χ is an isometric isomorphism. Thus, $(\mathit{Aff}(B_{X^*}), \|\cdot\|_{\mathit{Aff}(B_{X^*})})$ and $(\mathit{Aff}(B_{Y^*}), \|\cdot\|_{\mathit{Aff}(B_{Y^*})})$ are isometrically isomorphic if and only if $\left(\mathit{Aff}_0(B_{X^*}) \times \mathbb{R}, \|\cdot\|_{\mathit{Aff}_0(B_{X^*}) \times \mathbb{R}}\right)$ and $\left(\mathit{Aff}_0(B_{Y^*}) \times \mathbb{R}, \|\cdot\|_{\mathit{Aff}_0(B_{X^*}) \times \mathbb{R}}\right)$ are isometrically isomorphic. Using Proposition 1 and Theorem 1, this is equivalent to the fact that $(\mathit{Aff}_0(B_{X^*}), \|\cdot\|_\infty)$ and $(\mathit{Aff}_0(B_{Y^*}), \|\cdot\|_\infty)$ are isometrically isomorphic, which is equivalent by Theorem 2 to the fact that $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are isometrically isomorphic. ∎

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References

[1] J. Alonso, C. Benitez, The Joly's construction: A common property of some generalized orthogonalities, *Extracta Math.* **2** (3) (1987), 117-119.

[2] J. Alonso, C. Benitez, Orthogonality in normed linear spaces: A survey. Part I: Main properties, *Extracta Math.* **3** (1) (1988), 1-15.

[3] J. Alonso, C. Benitez, Orthogonality in normed linear spaces: A survey. Part II: Relations between main orthogonalities, *Extracta Math.* **4** (3) (1989), 121-131.

[4] M. Bachir, The inf-convolution as a law of monoid. An analogue to the Banach-Stone theorem, *J. Math. Anal. Appl.* **420** (1) (2014), 145-166.

[5] S. Banach, "Théorie des Opérations Lineaires", Monografie Matematyczne 1, Warszowa 1932. Reprinted, Clelsea Publishing Company, New York 1963.

[6] F.L. Bauer, J. Stoer, C. Witzgall, Absolute and monotonic norms, *Numer. Math.* **3** (1961), 257-264.

[7] J.M. Borwein, D.T. Yost, Absolute norms on vector lattices", *Proc. Edinburgh Math. Soc.* **27** (2) (1984), 215-222.

[8] M.I. Garrido, J.A. Jaramillo, Variations on the Banach-Stone Theorem, *Extracta Math.* **17** (3) (2002), 351-383.

[9] R. Phelps, "Lectures on Choquet's Theorem", Second edition, Lecture Notes in Mathematics 1757, Berlin, 2001.

[10] F.C. Sánchez, J.M.F. Castillo, Isometries of finite-dimensional normed spaces, *Extracta Math.* **10** (2) (1995), 146-151.