Moore-Penrose Inverse and Operator Inequalities

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Abstract: In this note, we shall give complete characterizations of the class of all normal operators with closed range, and the class of all selfadjoint operators with closed range multiplied by scalars in terms of some operator inequalities.

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1. Introduction and preliminaries results

Let $\mathfrak{B}(H)$ be the C*-algebra of all bounded linear operators acting on a complex Hilbert space $H$, and let $\mathcal{N}(H)$, and $\mathcal{S}(H)$ denote the class of all normal operators, and the class of all selfadjoint operators in $\mathfrak{B}(H)$, respectively.

We denote by

- $\mathcal{I}(H)$, the group of all invertible elements in $\mathfrak{B}(H)$,
- $\mathcal{S}_0(H) = \mathcal{S}(H) \cap \mathcal{I}(H)$, the set of all invertible selfadjoint operators in $\mathfrak{B}(H)$,
- $\mathcal{N}_0(H) = \mathcal{N}(H) \cap \mathcal{I}(H)$, the set of all invertible normal operators in $\mathfrak{B}(H)$,
- $\mathcal{R}(H)$, the set of all operators with closed ranges in $\mathfrak{B}(H)$,
- $x \otimes y$ (where $x, y \in H$), the one rank operator on $H$ defined by $(x \otimes y)z = \langle z, y \rangle x$, for every $z \in H$,
- $|S|$ the positive square root of the positive operator $S^*S$ (where $S \in \mathfrak{B}(H)$),
- $\{S\}' = \{X \in \mathfrak{B}(H) : SX = XS\}$ the commutant of $S$ (where $S \in \mathfrak{B}(H)$).
For \( S \in \mathfrak{B}(H) \), let \( R(S) \) and \( \ker S \) denote the range and the kernel of \( S \), respectively. It is known that \( S \in R(H) \) if and only if there exits a unique operator \( S^+ \in R(H) \) satisfying the following four equations

\[
SS^+S = S, \quad S^+SS^+ = S^+, \quad (SS^+)^* = SS^+, \quad (S^+S)^* = S^+S.
\]

Then, the operator \( S^+ \) is called the Moore-Penrose inverse of \( S \), and it satisfies that \( SS^+ \) and \( S^+S \) are orthogonal projections onto \( R(S) \) and \( R(S^*) \), respectively. It is clear that if \( S \in \mathfrak{I}(H) \), then \( S^+ = S^{-1} \), and if \( S \in \mathfrak{B}(H) \) is a surjective operator (resp. injective with closed range), then \( SS^+ = I \) (resp. \( S^+S = I \)).

For every \( S \) in \( R(H) \), we associate the \( 2 \times 2 \) matrix representation \( S = \begin{bmatrix} S_1 & S_2 \\ 0 & 0 \end{bmatrix} \) on \( R(S) \oplus \ker S^* \). The operator \( S \) is called an EP operator if \( R(S^*) = R(S) \), or equivalently \( S_2 = 0 \) and \( S_1 \) is invertible; in this case \( S^+ = \begin{bmatrix} S_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} \) (see [2]). Any normal operator with a closed range in \( \mathfrak{B}(H) \) is an EP operator (see [4]).

One of the most essential inequalities in operator theory is the arithmetic-geometric mean inequality given by (see [1, 12])

\[ \forall A, B, X \in \mathfrak{B}(H), \quad \| A^*AX + XBB^* \| \geq 2 \| AXB \|. \quad (1) \]

From this inequality, we deduce immediately that for every \( S \in S_0(H) \) the following inequality holds

\[ \forall X \in \mathfrak{B}(H), \quad \| SXS^{-1} + S^{-1}XS \| \geq 2 \| X \|. \quad (2) \]

The inequality (2) was proved by Corach-Porta-Recht [6] with another motivation and independently of inequality (1). In [14], we give some characterization of some distinguished classes of operators in terms of operator inequalities. We proved that the class of all operators \( S \in \mathfrak{I}(H) \) satisfying (2) is exactly the class \( \mathbb{C}^*S_0(H) \) (that is the class of all rotations of invertible selfadjoint operators). So, the class \( \mathbb{C}^*S_0(H) \) is characterized by the following property

\[ \forall X \in \mathfrak{B}(H), \quad \| SXS^{-1} + S^{-1}XS \| \geq 2 \| X \|, \quad (S \in \mathfrak{I}(H)). \quad (3) \]

In [15], we have found two other characterizations of this last class given by
∀\(X \in \mathfrak{B}(H), \quad \|SXS^{-1} + S^{-1}XS\| = \|S^*XS^{-1} + S^{-1}XS^*\|, \quad (S \in \mathfrak{J}(H)). \quad (4)\)

∀\(X \in \mathfrak{B}(H), \quad \|SXS^{-1} + S^{-1}XS\| \geq \|S^*XS^{-1} + S^{-1}XS^*\|, \quad (S \in \mathfrak{J}(H)). \quad (5)\)

For the class of all invertible normal operators \(N_0(H)\), we have showed in [15] that this class is characterized by each of the following three properties

∀\(X \in \mathfrak{B}(H), \quad \|SXS^{-1}\| + \|S^{-1}XS\| \geq 2\|X\|, \quad (S \in \mathfrak{J}(H)). \quad (6)\)

∀\(X \in \mathfrak{B}(H), \quad \|SXS^{-1}\| + \|S^{-1}XS\| = \|S^*XS^{-1}\| + \|S^{-1}XS^*\|, \quad (S \in \mathfrak{J}(H)). \quad (7)\)

∀\(X \in \mathfrak{B}(H), \quad \|SXS^{-1}\| + \|S^{-1}XS\| \geq \|S^*XS^{-1}\| + \|S^{-1}XS^*\|, \quad (S \in \mathfrak{J}(H)). \quad (8)\)

In this note, we consider the following extensions of the above six properties from the domain \(\mathfrak{J}(H)\) to the domain \(\mathfrak{R}(H)\)

∀\(X \in \mathfrak{B}(H), \quad \|SXS^+ + S^+XS\| \geq 2\|SS^+XS^+\|, \quad (S \in \mathfrak{R}(H)). \quad (9)\)

∀\(X \in \mathfrak{B}(H), \quad \|SXS^+ + S^+XS\| = \|S^*XS^+ + S^+XS^*\|, \quad (S \in \mathfrak{R}(H)). \quad (10)\)

∀\(X \in \mathfrak{B}(H), \quad \|SXS^+ + S^+XS\| \geq \|S^*XS^+ + S^+XS^*\|, \quad (S \in \mathfrak{R}(H)). \quad (11)\)

∀\(X \in \mathfrak{B}(H), \quad \|SXS^+ + S^+XS\| \geq 2\|SS^+XS^+S\|, \quad (S \in \mathfrak{R}(H)). \quad (12)\)

∀\(X \in \mathfrak{B}(H), \quad \|SXS^+\| + \|S^+XS\| \geq 2\|SS^+XS^+S\|, \quad (S \in \mathfrak{R}(H)). \quad (13)\)

∀\(X \in \mathfrak{B}(H), \quad \|SXS^+\| + \|S^+XS\| \geq \|S^*XS^+\| + \|S^+XS^*\|, \quad (S \in \mathfrak{R}(H)). \quad (14)\)

We consider also in this note the following two properties

∀\(X \in \mathfrak{B}(H), \quad \|S^2X + XS^2\| \geq 2\|SXS\|, \quad (S \in \mathfrak{R}(H)). \quad (15)\)

∀\(X \in \mathfrak{B}(H), \quad \|S^2X\| + \|XS^2\| \geq 2\|SXS\|, \quad (S \in \mathfrak{R}(H)). \quad (16)\)
In the finite dimensional case, we showed in [13] that each of properties (9) and (15) characterizes the class $CS(H)$ and each of properties (12) and (16) characterizes the class $N(H)$ (here $R(H) = B(H)$). In this note and in general situation, we shall show that any one of properties (9), (10), (11) and (15) characterizes the class $C(S(H) \cap R(H))$ and any one of properties (12), (13), (14) and (16) characterizes the class $N(H) \cap R(H)$.

2. Characterizations and Moore-Penrose inverse

To achieve our new characterizations, we need the following lemma.

**Lemma 1.** Let $S \in B(H)$. If $S$ is injective with a closed range (or is surjective) and satisfies property (16), then it is normal.

**Proof.** Assume that $S$ is injective. It is clear that $S^2$ is also injective with a closed range. Hence, $S^+ S = I = (S^2)^+ S^2$.

Put $P = |S|$, $Q = |S^*|$ and $R = |S^2|$. Since $S$ is injective with a closed range, then $\ker P = \ker S = \{0\}$, and $R(P) = R(S^* S)$ is closed (since $R(S^*)$ is also closed). Thus $\ker P = \{0\}$ and $R(P) = (\ker P)^\perp = H$. So, $P$ is invertible.

On the other hand, since $S^2$ is also injective with a closed range, using the same argument as used before $R$ is invertible.

The proof is given in six steps.

**Step 1.** $\{R\}' = \{P\}'$.

From (16), it follows that

$$\forall X \in B(H), \quad \|X\| + \|S^2 X (S^2)^+\| \geq 2 \|SX (S^2)^+ S\|.$$  

Since $\|SS^+\| = 1$, the last inequality shows that

$$\forall X \in B(H), \quad \|X\| + \|S^2 X (S^2)^+\| \geq 2 \|SX (S^2)^+ S^2 S^+\|.$$  

Thus the following inequality holds

$$\forall X \in B(H), \quad \|X\| + \|S^2 X (S^2)^+\| \geq 2 \|SX S^+\|.$$  

By taking the polar decomposition of each of the two operators $S$ and $S^2$ in this last inequality, we obtain

$$\forall X \in B(H), \quad \|X\| + \|RX R^{-1}\| \geq 2 \|PX P^{-1}\|. $$
Hence from [14, Lemma 3.2], \( \{ R \}^{' \prime} \subset \{ P \}^{' \prime} \). So, from this last inequality, we obtain also the following inequality

\[
\forall X \in \mathcal{B}(H), \quad \| P^{-1} XP \| + \left\| (RP^{-1}) X (RP^{-1})^{-1} \right\| \geq 2 \| X \| .
\]

where \( RP^{-1} \) is a positive invertible operator. So, from [14, Theorem 3.3], \( \{ P \}^{' \prime} = \{ RP^{-1} \}^{' \prime} \). Hence \( \{ R \}^{' \prime} = \{ P \}^{' \prime} \).

**Step 2.** \( (S^2)^{+} S = S^{+} \).

From the property (16) the following inequality holds

\[
\forall X \in \mathcal{B}(H), \quad \| XS \| + \| S^2 XS^+ \| \geq 2 \| SX \|. \quad (\ast)
\]

It is known that \( S^{+} \) is the unique solution of the following four equations: \( SXS = S, \ XSX = X, \ (XS)^{*} = XS, \ (SX)^{*} = SX \). It is easy to see that \( (S^2)^{+} S \) satisfies the first three equations.

Now we prove that \( (S^2)^{+} S \) also satisfies the last equation. Since the operator \( S (S^2)^{+} S \) is a projection, it suffices to prove that its norm is less than or equal to one. By taking \( X = (S^2)^{+} S \) in (\ast), we obtain

\[
2 \geq \left\| (S^2)^{+} S \right\| + \left\| S^2 (S^2)^{+} SS^+ \right\| \geq 2 \left\| S (S^2)^{+} S \right\| .
\]

Hence \( \left\| S (S^2)^{+} S \right\| \leq 1 \). Therefore \( (S^2)^{+} S = S^{+} \).

**Step 3.** \( (S^2)^{+} = (S^{+})^2 \).

Since \( S^2 (S^2)^{+} = SS^+ S \), then \( S^2 (S^2)^{+} = S^2 (S^2)^{+} SS^+ \). So from Step 2, we obtain \( S^2 (S^2)^{+} = S^2 (S^{+})^2 \). Since \( S^2 \) is injective, we have \( (S^2)^{+} = (S^{+})^2 \).

**Step 4.** \( P \) and \( R \) are \( 2 \times 2 \) diagonal matrices with respect to the orthogonal direct sum \( H = R(S) \oplus \ker S^{*} \).

All matrices given here are given with respect to the orthogonal direct sum \( H = R(S) \oplus \ker S^{*} \). Put \( S = \begin{bmatrix} S_1 & S_2 \\ 0 & 0 \end{bmatrix} \). Then \( S^2 = \begin{bmatrix} S_1^2 & S_1 S_2 \\ 0 & 0 \end{bmatrix} \). Since \( (S^2)^{+} = (S^{+})^2 \), then the operators \( S^{*} S \) and \( SS^{+} \) commute (see [3, 10]). Hence \( P^{2} = \begin{bmatrix} S_1 S_1 & 0 \\ 0 & S_2 S_2 \end{bmatrix} \). So that \( P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \), where \( P_1 = |S_1| \) and \( P_2 = |S_2| \). From Step 1 and \( SS^{+} \in \{ S^{*} S \}^{\prime} \), we deduce that \( (S^2)^{*} S^2 \) and \( SS^{+} \)
commute. Therefore $R^2 = \begin{bmatrix} (S_2^2)^* S_1^2 & 0 \\ 0 & (S_2^2)^* S_2^2 \end{bmatrix}$. Hence $R = \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix}$.

where $R_1 = |S_1^2|$ and $R_2 = |S_2^2|.$

**Step 5.** $\ker S^* = \{0\}$.

Since $S$ is injective, $\ker S^* = \{0\}$ if and only if $S_1 = 0$.

Assume that $\ker S^* \neq \{0\}$. It is easy to see that $SS^* = \begin{bmatrix} S_1 S_1^* + S_2^2 S_2^2 & 0 \\ 0 & 0 \end{bmatrix}$.

Then $Q = \begin{bmatrix} Q_1 & 0 \\ 0 & 0 \end{bmatrix}$, where $Q_1 = (S_1 S_1^* + S_2^2 S_2^2)$.

Using the polar decomposition of the operators $S$, $S^*$ and $S^2$ in $(\ast)$ we obtain the following inequality

$$\forall X \in \mathfrak{B}(H), \quad \|XQ\| + \|RXP^{-1}\| \geq 2\|PX\|.$$  \hfill (\ast\ast)

By putting $X = 0 \oplus Y P_2$ (where $Y \in \mathfrak{B}(\ker S^*)$) in inequality (\ast\ast), we obtain

$$\forall Y \in \mathfrak{B}(\ker S^*), \quad \|R_2 Y\| \geq 2\|P_2 Y P_2\|.$$  

Hence $\|S_2^2\| = \|R_2\| \geq 2\|P_2\|^2 = 2\|S_2\|^2$. So that $\|S_2\|^2 \geq 2\|S_2\|^2$. Thus $S_2 = 0$, which is a contradiction with $\ker S^* \neq \{0\}$.

Therefore $\ker S^* = \{0\}$.

**Step 6.** $S$ is normal.

From Step 5, $S$ is surjective. So that $S$ is invertible and satisfies property (16). Thus $S$ satisfies property (6). Hence $S$ is normal.

With the second assumption "$S$ surjective", $S^*$ is injective with a closed range satisfying also property (16), so that $S^*$ is normal. Hence $S$ is normal.

**Theorem 1.** Let $S \in \mathcal{N}(H)$. Then the following properties are equivalent:

(i) $S \in \mathcal{N}(H)$,

(ii) $\forall X \in \mathfrak{B}(H), \quad \|SXS^+\| + \|S^+XS\| = \|S^*XS^+\| + \|S^+XS^*\|,$

(iii) $\forall X \in \mathfrak{B}(H), \quad \|SXS^+\| + \|S^+XS\| \geq \|S^*XS^+\| + \|S^+XS^*\|,$

(iv) $\forall X \in \mathfrak{B}(H), \quad \|SXS^+\| + \|S^+XS\| \geq 2\|SS^+XS^+S\|,$

(v) $\forall X \in \mathfrak{B}(H), \quad \|S^2X\| + \|XS^2\| \geq 2\|SXS\|.$

**Proof.** The proof is trivial if $S = 0$. Assume now that $S \neq 0$. 


(ii) implies (vi). This implication follows immediately using [11, Theorem 2.4]. (iv) implies (v). Assume (iv) holds. Then the following inequality holds

\[ \forall X \in \mathcal{B}(H), \quad \|S^2XS^+\| + \|S^+XS^2\| \geq 2 \|SS^+XS^2S^+\| . \]

From this inequality and since \(\|SS^+\| = \|S^+S\| = 1\), property (v) follows immediately.

(v) implies (i). Assume (v) holds. Let \( S = \begin{bmatrix} S_1 & S_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R(S) \\ \ker S^* \end{bmatrix} \) and let \( S^* = \begin{bmatrix} T_1 & T_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R(S^*) \\ \ker S \end{bmatrix} \). Put \( X = \begin{bmatrix} X_1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} R(S) \\ \ker S^* \end{bmatrix} \). By a simple computation, we obtain

\[ S^2X = \begin{bmatrix} S_1^2X_1 \\ 0 \\ 0 \end{bmatrix}, \quad XS^2 = \begin{bmatrix} X_1S_1^2 \\ X_1S_1S_2 \\ 0 \end{bmatrix}, \quad SXS = \begin{bmatrix} S_1X_1S_1 \\ S_1X_1S_2 \\ 0 \end{bmatrix} . \]

Put \( Y = \begin{bmatrix} Y_1 & Y_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R(S) \\ \ker S^* \end{bmatrix} \), where \( Y \) denotes one of the above three operators. Then \( \|Y\|^2 = \|YY^*\| = \begin{bmatrix} Y_1Y_1^* + Y_2Y_2^* \\ 0 \end{bmatrix} / 0 \| = \|Y_1Y_1^* + Y_2Y_2^*\| . \) Hence we have \( \|S^2X\| = \|S_1^2X_1\| , \)

\[ \|XS^2\|^2 = \| (X_1S_1^2)(X_1S_1^2)^* + (X_1S_1S_2)(X_1S_1S_2)^* \| = \| X_1S_1K^2(X_1S_1)^* \| = \| X_1S_1K \|^2 , \]

and

\[ \|SX\| = \| (S_1X_1S_1)(S_1X_1S_1)^* + (S_1X_1S_2)(S_1X_1S_2)^* \| = \| (S_1X_1)^2K^2(S_1X_1)^* \| = \| S_1X_1K \|^2 , \]

(where \( K \) is the positive square root of the positive operator \( S_1S_1^* + S_2S_2^* \)). Then using (v), we obtain the following inequality

\[ \forall X_1 \in \mathcal{B}(R(S)), \quad \|S_1^2X_1\| + \|X_1S_1K\| \geq 2 \|S_1X_1K\| \quad (*) \]

On the other hand, if we put \( X = x \otimes y \) (for \( x, y \in H \)) in (v), we obtain

\[ \forall x, y \in H, \quad \|y\| \|S^2x\| + \|x\| \|S^*y\| \geq 2 \|Sx\| \|S^*y\| \quad (**) \]
We shall prove (i) in three steps.

**Step 1.** $S_1$ or $T_1$ is bounded below.

Assume that it is not the case. With the condition “$S_1$ is not bounded below”, we may choose a sequence $(u_n)$ in $H$ such that

$$S^2 u_n \to 0 \quad \text{and} \quad \|S u_n\| = 1, \quad \text{for} \quad n \geq 1.$$

For every $n \geq 1$, there exist $x_n \in R(S^*)$ and $z_n \in \ker S$ such that $u_n = x_n + z_n$. Thus, we obtain

$$S^2 x_n = S^2 u_n \to 0, \quad \|S x_n\| = \|S u_n\| = 1, \quad \|x_n\| = \|S^+ S u_n\| \leq \|S^+\|, \quad \text{for} \quad n \geq 1.$$

With the second condition “$T_1$ is not bounded below”, by the same argument, we may choose a bounded sequence $(y_n)$ in $H$ satisfying

$$(S^*)^2 y_n \to 0, \quad \|S^* y_n\| = 1, \quad \text{for} \quad n \geq 1.$$

Applying (**) for $x = x_n$ and $y = y_n$, we obtain

$$\forall n \geq 1, \quad \|y_n\| \|S^2 x_n\| + \|x_n\| \|(S^*)^2 y_n\| \geq 2.$$

Letting $n \to \infty$, we have $0 \geq 2$, which is impossible. Therefore $S_1$ or $T_1$ is bounded below.

**Step 2.** $S_1$ or $T_1$ is surjective.

Assume that $T_1$ is bounded below. Then there exists a constant $k > 0$ such that

$$\forall x \in H, \quad \|(S^*)^2 x\| \geq k \|S^* x\|.$$

So we have $S^2(S^*)^2 \geq k^2 SS^*$. From [7], we obtain $R(S^2) \supset R(S)$. Thus $R(S^2) = R(S)$. So $S_1$ is surjective.

Also, if $S_1$ is bounded below, then by the same argument, we deduce that $T_1$ is surjective.

**Step 3.** $S$ is normal.

Assume that $S_1$ is surjective (on $R(S)$). Then $S^2_1$ is also surjective on $R(S)$. Since $R(S) \neq \{0\}$, thus $S_1 S_1^+ = I_1 = S^2_1 (S^2_1)^+$ (where $I_1$ is the identity operator on $R(S)$), $S^+_1 S_1$ and $(S^2_1)^+ S^2_1$ are nonzero orthogonal projections. By putting $X_1 = (S^2_1)^+$ in (*), we obtain

$$\left\| S^2_1 (S^2_1)^+ \right\| + \left\| (S^2_1)^+ S_1 K \right\| \geq 2 \left\| S_1 (S^2_1)^+ K \right\|$$
Hence, \( \|S_1^2(S_2^2)^+\| = 1 \). \( \| (S_2^2)^+ S_1 K \| = \| (S_2^2)^+ S_1^2 S_1^+ K \| \leq \| S_1^+ K \| \), and \( \| S_1(S_2^2)^+ K \| \geq \| S_1\| \| (S_2^2)^+ K \| \). Thus 1 \( \geq \| S_1^+ K \| \). Hence

\[
1 \geq \| S_1^+ K \|^2 = \| S_1^+ K^2 (S_1^+)^* \| = \| S_1^+ S_1 + (S_1^+ S_2)(S_1^+ S_2)^* \| \geq \| S_1^+ S_1 \| = 1.
\]

Hence \( \|S_1^+ S_1 + (S_1^+ S_2)(S_1^+ S_2)^*\| = 1 \). Since \( S_1^+ S_1 \) is an orthogonal projection, by a simple computation, we obtain that \( S_1^+ S_1 S_1^+ S_1^2 = 0 \). Hence \( S_2 = S_1 S_1^+ S_1 S_1^+ S_2 = 0 \). So, we obtain that \( S_1 \) is a surjective operator (as element in \( \mathcal{B} (R(S)) \) and satisfies the following inequality

\[
\forall X \in \mathcal{B} (\mathcal{R}(S)), \quad \| S_1^2 X \| + \| XS_1^2 \| \geq 2 \| S_1 X S_1 \|
\]

Utilizing Lemma 1, we obtain that \( S_1 \) is normal. Hence \( S \) is normal.

With the second assumption “\( T_1 \) surjective”, and since \( S^* \) satisfies (v), by using the same argument as used with the first assumption, we obtain also that \( S^* \) is normal. Thus \( S \) is normal. ■

**Corollary 1.** Assume \( \dim H < \infty \). The class \( \mathcal{N}(H) \) is characterized by each of the following properties

\[
\forall X \in \mathcal{B} (H), \quad \| SXS^+ + S^+ XS \| \geq 2 \| SS^+ XS^+ S \|, \quad (S \in \mathcal{B} (H)),
\]

\[
\forall X \in \mathcal{B} (H), \quad \| SXS^+ + S^+ XS \| = \| SXS^+ \| + \| S^+ XS \|, \quad (S \in \mathcal{B} (H)),
\]

\[
\forall X \in \mathcal{B} (H), \quad \| SXS^+ + S^+ XS \| \geq \| SXS^+ \| + \| S^+ XS \|, \quad (S \in \mathcal{B} (H)),
\]

\[
\forall X \in \mathcal{B} (H), \quad \| S^2 X \| + \| XS^2 \| \geq 2 \| SSXS \|, \quad (S \in \mathcal{B} (H)).
\]

**Theorem 2.** Let \( S \in \mathcal{R} (H) \). Then the following properties are equivalent:

(i) \( S \in \mathcal{C} S (H) \),

(ii) \( \forall X \in \mathcal{B} (H), \quad \| SXS^+ + S^+ XS \| = \| SXS^+ + S^+ XS \|, \)

(iii) \( \forall X \in \mathcal{B} (H), \quad \| SXS^+ + S^+ XS \| \geq \| SXS^+ + S^+ XS \|, \)

(iv) \( \forall X \in \mathcal{B} (H), \quad \| SXS^+ + S^+ XS \| \geq 2 \| SSXS \|, \)

(v) \( \forall X \in \mathcal{B} (H), \quad \| S^2 X + XS^2 \| \geq 2 \| SSXS \|. \)

**Proof.** The implications (i)\( \Rightarrow \) (ii) and (ii)\( \Rightarrow \) (iii) are trivial. The implication (iii)\( \Rightarrow \) (iv) follows immediately from [11, Theorem 2.4]. The implication (i)\( \Rightarrow \) (v) follows immediately from (1).

Assume now that (iv) or (v) holds. Applying the triangular inequality in (iv) or (v), we obtain from Theorem 1, that \( S \) is normal (with a closed range).
So that $S$ is an EP operator satisfying (iv) or (v). So $S = \begin{bmatrix} S_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R(S) \\ \ker S^* \end{bmatrix}$ where $S_1$ is invertible on $R(S)$. Hence we obtain the following inequality

$$\forall X \in \mathfrak{B}(R(S)), \quad \|S_1XS_1^{-1} + S_1^{-1}XS_1\| \geq 2\|X\|.$$ 

Hence $S_1$ is a selfadjoint operator in $\mathfrak{B}(R(S))$ multiplied by a nonzero scalar. Thus $S \in \mathfrak{CS}(H)$.

**Corollary 2.** Assume that $\dim H < \infty$. The class $\mathfrak{CS}(H)$ is characterized by each of the four following properties

$\forall X \in \mathfrak{B}(H), \|SXS^* + S^*XS\| \geq 2\|SS^*XS\|, \ (S \in \mathfrak{B}(H)),$

$\forall X \in \mathfrak{B}(H), \|SXS^* + S^*XS\| = \|S^*XS^* + S^*XS\|, \ (S \in \mathfrak{B}(H)),$

$\forall X \in \mathfrak{B}(H), \|SXS^* + S^*XS\| \geq \|S^*XS^* + S^*XS\|, \ (S \in \mathfrak{B}(H)),$

$\forall X \in \mathfrak{B}(H), \|S^2X + XS^2\| \geq 2\|SSXS\|, \ (S \in \mathfrak{B}(H)).$

**Remarks.** 1. Let the following extension of the property (15) to the domain $\mathfrak{B}(H)$:

$$\forall X \in \mathfrak{B}(H), \quad \|S^2X + XS^2\| \geq 2\|SSXS\|, \ (S \in \mathfrak{B}(H)). \quad (17)$$

The restriction of this property to the domain $\mathcal{R}(H)$ characterizes the class $\mathfrak{CS}(H) \cap \mathcal{R}(H)$ (by using Theorem 2). On the other hand, by using the inequality (1), the property (17) is satisfied for every $S \in \mathfrak{CS}(H)$. So, does property (17) characterize the class $\mathfrak{CS}(H)$?

2. Let the following extension of the property (16) to the domain $\mathfrak{B}(H)$:

$$\forall X \in \mathfrak{B}(H), \quad \|S^2X\| + \|XS^2\| \geq 2\|SSXS\|, \ (S \in \mathfrak{B}(H)). \quad (18)$$

The restriction of this property to the domain $\mathcal{R}(H)$ characterizes the class $\mathcal{N}(H) \cap \mathcal{R}(H)$ (by using Theorem 1). The property (18) is satisfied for every $S \in \mathcal{N}(H)$. Indeed, if $S \in \mathcal{N}(H)$, then $\|S^2X\| + \|XS^2\| = \|S^*SX\| + \|XS^2\| \geq \|SS^*SX + XS^2\| \geq 2\|SSXS\|$, for every $X \in \mathfrak{B}(H)$. So, does property (18) characterize the class $\mathcal{N}(H)$?

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References