Real Analytic Version of Lévy’s Theorem

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Abstract: We obtain real analytic version of the classical theorem of Lévy on absolutely convergent power series. Whence, as a consequence, its harmonic version.

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1. Introduction

Let \( A \) be a complex Banach algebra with the involution \( x \mapsto x^* \) and unit \( e \). The spectrum of an element \( x \) of \( A \) will be denoted by \( Spx \). An element \( h \) of \( A \) is called hermitian if \( h^* = h \). The set of all Hermitian elements of \( A \) will be denoted by \( H(A) \). We say that the Banach algebra \( A \) is Hermitian if the spectrum of every element of \( H(A) \) is real ([9]). For scalars \( \lambda \), we often write simply \( \lambda e \) for the element \( \lambda e \) of \( A \). Let \( p \in ]1, +\infty[ \). We say that \( \omega \) is a weight on \( \mathbb{Z} \) if \( \omega : \mathbb{Z} \longrightarrow [1, +\infty[ \), is a map satisfying

\[
c(\omega) = \sum_{n \in \mathbb{Z}} \omega(n) \frac{1}{1-p} < +\infty. \tag{1}
\]

We consider the following weighted space:

\[
\mathcal{A}^p(\omega) = \{ f : \mathbb{R} \longrightarrow \mathbb{C} : f(t) = \sum_{n \in \mathbb{Z}} a_n e^{int}, \ a_n \in l^p(\mathbb{Z}, \omega) \}.
\]

Endowed with the norm \( \| . \|_{p,\omega} \) defined by:

\[
\| f \|_{p,\omega} = \left( \sum_{n \in \mathbb{Z}} |a_n|^p \omega(n) \right)^{\frac{1}{p}}, \text{ for every } f \in \mathcal{A}^p(\omega),
\]

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the space $A^p(\omega)$ becomes a Banach space. Moreover, if there exists a constant $\gamma = \gamma(\omega) > 0$ such that
\[
\omega^{\frac{1}{1-p}} \ast \omega^{\frac{1}{1-p}} \leq \gamma \omega^{\frac{1}{1-p}}
\] (2)
then $(A^p(\omega), ||.||_{p,\omega})$ is closed under pointwise multiplication and it is a commutative semi-simple Banach algebra with unity element $\hat{e}$ given by $\hat{e}(t) = 1$ ($t \in \mathbb{R}$) ([4]). For the weight function $\omega$ on $\mathbb{Z}$ satisfying (2) and $\omega(n+m) \leq \omega(n)\omega(m)$, for every $n, m \in \mathbb{Z}$, it is also shown in ([4]), that the character space of $(A^p(\omega), ||.||_{p,\omega})$ can be identified with the closed annulus:
\[
\Gamma_\omega(\rho_1, \rho_2) = \{ \xi \in \mathbb{C} : \rho_1(\omega) \leq |\xi| \leq \rho_2(\omega) \},
\]
in such a way that each character has the form $f \mapsto \sum_{n \in \mathbb{Z}} a_n \xi^n$ for some $\xi \in \Gamma_\omega(\rho_1, \rho_2)$, where $f = \sum_{n \in \mathbb{Z}} a_n u^n \in A^p(\omega)$ with $u(t) = e^{it}$, for every $t \in \mathbb{R}$. For $\rho_1$ and $\rho_2$, they are given by:
\[
\rho_1 = e^{-\sigma_2} \quad \text{and} \quad \rho_2 = e^{-\sigma_1}
\]
where
\[
\sigma_1 = \sup \left\{ \frac{-1}{np} \ln(\omega(n)), \quad n \geq 1 \right\} \quad \text{and} \quad \sigma_2 = \inf \left\{ \frac{1}{np} \ln(\omega(-n)), \quad n \geq 1 \right\}.
\]
The real analytic functional calculus is defined and studied in [1]. To make the paper self-contained, we recall the fundamental properties of this calculus. Let $U$ be an open subset of $\mathbb{R}^2$ and $F : U \rightarrow \mathbb{C}$ be real analytic function. Then there exists an open subset $V$, of $\mathbb{C}^2$, and an holomorphic function $\tilde{F} : V \rightarrow \mathbb{C}$ such that
\[
V \cap \mathbb{R}^2 = U \quad \text{and} \quad \tilde{F}|_U = F.
\]
For the construction of $V$, we have $V = \bigcup_{x \in U} \Omega_x$, where $\Omega_x$ is an open of $\mathbb{C}^2$ centered at $x$. We denote by $\Lambda_0(U)$ the set of all open subset $V$ described us above and we consider, in $\Lambda_0(U)$, the order given in the following way:
\[
V \preceq W \iff W \subset V.
\]
For $V \in \Lambda_0(U)$, we denote by $\mathcal{O}(V)$ the set of holomorphic functions on $V$. Now we consider the family $(\mathcal{O}(V))_{V \in \Lambda_0(U)}$ of algebras and for every $V, W \in \Lambda_0(U)$ with $V \subseteq W$, let

$$\pi_{W,V} : \mathcal{O}(V) \rightarrow \mathcal{O}(W) : F \mapsto F|_W$$

The family of algebras $(\mathcal{O}(V))_{V \in \Lambda_0(U)}$ with the maps $\pi_{W,V}$ is an inductive system of algebras and it is denoted by $(\mathcal{O}(V), \pi_{W,V})$. Let $\varprojlim (\mathcal{O}(V), \pi_{W,V})$ its inductive limit. We shall denote this simply by $\varprojlim \mathcal{O}(V)$ and we have:

$$\varprojlim \mathcal{O}(V) = \bigcup_{V \in \Lambda_0(U)} \mathcal{O}(V)$$

In the sequel, we denote by $A(U)$ the algebra of real analytic functions on $U$. By lemma 2.1.1 of [1], the map

$$\Psi : A(U) \rightarrow \varprojlim \mathcal{O}(V) : f \mapsto \Psi(f)$$

is an isomorphism algebra. Now let $A$ be a commutative and unital Hermitian Banach algebra (with continuous involution) and $a \in A$. Then $a = h + ik$ with $h, k \in H(A)$. Put $a' = (h, k)$ and $Sp_Aa'$ the joint spectrum of $(h, k)$. We denote by $\Theta_{a'}$ the map that defined the holomorphic functional calculus for $a'$. One has $Sp_A(h, k) \subset Sp_Ah \times Sp_Ak \subset \mathbb{R}^2$. By the identification $\mathbb{R}^2 \cong \mathbb{C}$, via the map $(x, y) \mapsto x + iy$, we can consider that

$$Sp_Aa \simeq Sp_A(h, k)$$

and this motivates the following definition:

**Definition 1.1. ([1], Définition 2.1.2)** Let $A$ be a commutative and unital Hermitian Banach with continuous involution, $a \in A, U$ an open subset, of $\mathbb{R}^2$, containing $Sp_Aa$ and $f \in A(U)$. We denote by $f(a)$ the element of $A$ defined by:

$$f(a) = \Theta_{a'}(\Psi(f)) = \Psi(f) (h, k),$$

where $a = h + ik$ and $a' = (h, k)$ with $h, k \in H(A)$.

The fundamental properties of this functional calculus are contained in the following result:
PROPOSITION 1.2. ([1]) 1. The mapping $f \mapsto f(a)$ is a homomorphism of $\mathcal{A}(U)$ into $\mathcal{A}$ that extends the involutive homomorphism from $\mathcal{h}(U)$ into $\mathcal{A}$, where $\mathcal{h}(U)$ is the set of all harmonic functions on $U$.

2. “Spectral mapping theorem”:

$$Sp_\mathcal{A}f(a) = f(Sp_\mathcal{A}a), \text{ for every } f \in \mathcal{A}(U).$$

Let $f(t) = \sum_{n \in \mathbb{Z}} a_ne^{int}$ be a periodic function such that $\sum_{n \in \mathbb{Z}} |a_n| < +\infty$. If $F$ is an holomorphic function defined on an open set containing the image of $f$, then $F(f)$ can be developed in trigonometric series $F(f)(t) = \sum_{n \in \mathbb{Z}} c_ne^{int}$ such that $\sum_{n \in \mathbb{Z}} |c_n| < +\infty$. This result due to P. Lévy ([7]) generalizes the famous theorem of N. Wiener ([10]) which states that the reciprocal of a nowhere vanishing absolutely convergent trigonometric series is also an absolutely convergent trigonometric series. In this paper, we consider the general case of a weight $\omega$ on $\mathbb{Z}$ which satisfies (2), (3) and

$$\lim_{|n| \to +\infty} (\omega(|n|))^{1/p} = 1. \quad (4)$$

We then consider $f \in \mathcal{A}^p(\omega)$ and $F$ an analytic function in two real variables on a neighborhood $U$ of $Spf$. In this case, we obtain a weighted analogues of Lévy’s theorem which states that $F(f)$ can be developed in trigonometric series $F(f)(t) = \sum_{n \in \mathbb{Z}} c_ne^{int}$ such that

$$\sum_{n \in \mathbb{Z}} |c_n|^p \omega(n) < +\infty.$$

To proceed, we consider the Banach algebra $(\mathcal{A}^p(\omega), \|\cdot\|_{p,\omega})$ endowed with the involution $f \mapsto f^*$ defined by:

$$f^*(t) = \sum_{n \in \mathbb{Z}} a_{-n}e^{int}, \text{ for every } f \in \mathcal{A}^p(\omega).$$

We prove that $(\mathcal{A}^p(\omega), \|\cdot\|_{p,\omega})$ is Hermitian. In the particular case where $F$ is a harmonic function in a neighborhood of $f(\mathbb{R})$, we prove that the expression of $F(f)$ is also given by the Poisson integral formula ([1]).

2. REAL ANALYTIC VERSION OF LEVY’S THEOREM

Now we are ready to generalize Levy’s theorem for real analytic functions.
Theorem 2.1. (Real analytic version of Lévy’s theorem) Let $p \in ]1, +\infty[$ and $\omega$ be a weight on $\mathbb{Z}$ satisfying (2), (3) and (4). Let $f(t) = \sum_{n \in \mathbb{Z}} a_n e^{int}$ be a periodic function such that

$$\sum_{n \in \mathbb{Z}} |a_n|^p \omega(n) < +\infty.$$ 

Let $F$ be an analytic function in two real variables on an open $U$ containing the image of $f$, then the function $F(f)$ also can be developed in a trigonometric series $F(f)(t) = \sum_{n \in \mathbb{Z}} c_n e^{int}$ such that

$$\sum_{n \in \mathbb{Z}} |c_n|^p \omega(n) < +\infty.$$ 

Proof. We consider the Banach algebra $(\mathcal{A}^p(\omega), \| \cdot \|_{p, \omega})$ endowed with the involution $f \mapsto f^*$ defined by:

$$f^*(t) = \sum_{n \in \mathbb{Z}} a_{-n} e^{int}, \text{ for every } f \in \mathcal{A}^p(\omega).$$

One can prove that the map $f \mapsto f^*$ is an algebra involution on $(\mathcal{A}^p(\omega), \| \cdot \|_{p, \omega})$. Moreover, it is continuous for the algebra is semi-simple. By the real analytic functional calculus given by Definition 1.1, the proof will be completed by proving that the last involution is hermitian in $(\mathcal{A}^p(\omega), \| \cdot \|_{p, \omega})$. By hypothesis, $\lim_{|n| \to +\infty} (\omega(|n|))^{\frac{1}{p}} = 1$. Then the character space $\mathcal{M}(\mathcal{A}^p(\omega))$ of $(\mathcal{A}^p(\omega), \| \cdot \|_{p, \omega})$ can be identified with $[0, 2\pi]$ in such a way that each character is an evaluation at some $t_0 \in [0, 2\pi]$. This implies that

$$Spf = \{f(t) : t \in [0, 2\pi]\}, \text{ for every } f \in \mathcal{A}^p(\omega).$$

Now, it is clear, that $f(t) = \sum_{n \in \mathbb{Z}} a_n e^{int}, t \in \mathbb{R}$, is a hermitian element of $\mathcal{A}^p(\omega)$ if and only if

$$a_{-n} = \overline{a_n}, \text{ for every } n \in \mathbb{Z}$$

and so $Sp(f) \subset \mathbb{R}$. Whence $(\mathcal{A}^p(\omega), \| \cdot \|_{p, \omega})$ is Hermitian with continuous involution. This completes the proof. 

Remark 2.2. Actually, the reader can prove that the algebra $(\mathcal{A}^p(\omega), \| \cdot \|_{p, \omega})$ is Hermitian if and only if

$$\lim_{|n| \to +\infty} (\omega(|n|))^{\frac{1}{p}} = 1.$$ 

Indeed, if the algebra $(\mathcal{A}^p(\omega), \| \cdot \|_{p, \omega})$ is Hermitian. Let $f : t \mapsto \sum_{n \in \mathbb{Z}} a_n e^{int}$ be a hermitian
element of \((A^p(\omega), \| \cdot \|_{p, \omega})\). Then \(Sp(f) \subset \mathbb{R}\). Hence
\[
\Phi_\zeta(f) = \overline{\Phi_\zeta(f)}, \text{ for every } \zeta \in \Gamma_\omega(\rho_1, \rho_2),
\]
where
\[
\Phi_\zeta(f) = \sum_{n \in \mathbb{Z}} a_n \zeta^n \quad \text{and} \quad \overline{\Phi_\zeta(f)} = \sum_{n \in \mathbb{Z}} a_n \overline{\zeta}^{-n}, \text{ for every } \zeta \in \Gamma_\omega(\rho_1, \rho_2).
\]
It follows that
\[
|\zeta| = 1, \text{ for every } \zeta \in \Gamma_\omega(\rho_1, \rho_2).
\]
This yields \(\rho_1 = \rho_2 = 1\), and one obtains that
\[
\lim_{|n| \to +\infty} (\omega(|n|))^{\frac{1}{p}} = 1.
\]
Harmonic functions are particular real analytic functions. In this case, we have the following:

\textbf{Corollary 2.3. (Harmonic version of Lévy’s theorem)} Let \( p \in ]1, +\infty[ \) and \( \omega \) be a weight on \( \mathbb{Z} \) satisfying (2), (3) and (4). Let \( f(t) = \sum_{n \in \mathbb{Z}} a_n e^{int} \) be a periodic function such
\[
\sum_{n \in \mathbb{Z}} |a_n|^p \omega(n) < +\infty.
\]
Let \( U \) be an open subset of \( \mathbb{C} \), \( z_0 \in U \) such that \( \overline{D(z_0, r)} \subset U \) \((r > 0)\) and \( f(\mathbb{R}) \subset D(z_0, r)\). If \( F \in h(U) \), then
\[
F(f) = \frac{1}{2\pi} \int_{|z - z_0| = r} F(z) \text{Re}[ (z + f - 2z_0)(z - f)^{-1} ] \frac{|dz|}{r}
\]
can be developed in a trigonometric series \( F(f)(t) = \sum_{n \in \mathbb{Z}} c_n e^{int} \) such that
\[
\sum_{n \in \mathbb{Z}} |c_n|^p \omega(n) < +\infty.
\]
References


