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On a ρ_n-Dilation of Operator in Hilbert Spaces †

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Abstract: In this paper we define the class of ρ_n-dilations for operators on Hilbert spaces. We give various properties of this new class extending several known results $\rho-$contractions. Some applications are also given.

Key words: ρ_n-dilation, ρ-dilation.

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1. Introduction

Sz-Nagy and Foias introduced in [8], the subclass $\mathcal{C}_\rho$ of the algebra $\mathcal{L}(\mathcal{H})$ of all bounded operators on a given complex Hilbert space $\mathcal{H}$. More precisely, for each fixed $\rho > 0$, an operator $\mathbf{T} \in \mathcal{C}_\rho$ if there exists a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ as a subspace and a unitary transformation $\mathbf{U}$ on $\mathcal{K}$ such that;

$$\mathbf{T}^n = \rho Pr \mathbf{U}^n_{|\mathcal{H}} \quad \text{for all } n \in \mathbb{N}^*. \tag{1}$$

Where $Pr : \mathcal{K} \to \mathcal{H}$ is the orthogonal projection on $\mathcal{H}$. The unitary operator $\mathbf{U}$ is then called a unitary ρ-dilation of $\mathbf{T}$, and the operator $\mathbf{T}$ is a ρ-contraction.

Recall that T is power bounded if $\|T^n\| \leq M$ for some fixed M and every nonegative integer n. From Equation (1), it follows that every ρ-contraction is power bounded since $\|\mathbf{T}^n\| \leq \rho$ for all $n \in \mathbb{N}^*$. Computing the spectral radius of $\mathbf{T}$, it comes that the spectrum of the operator $\mathbf{T}$ satisfies $\sigma(\mathbf{T}) \subset \overline{\mathrm{D}}$, where $\mathrm{D} = D(0,1)$ is the open unit disc of the set of complex numbers $\mathbb{C}$.

Operators in the class $\mathcal{C}_\rho$ enjoy several nice properties, we list below the most known, we refer to [7] for proofs and further information.

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(1) The function $\rho \mapsto \mathcal{C}_\rho$ is nondecreasing, that is $\mathcal{C}_\rho \subsetneq \mathcal{C}_{\rho'}$ if $\rho < \rho'$. We will denote by $\mathcal{C}_\infty = \bigcup_{\rho>0} \mathcal{C}_\rho$.

(2) $\mathcal{C}_1$ coincides whith the class of contractions (see [6]) and $\mathcal{C}_2$ is the class of operators $\mathbf{T}$ having a numerical radius less or equal to 1 (see [1]). The numerical radius is given by the expression, $\mathrm{w}(\mathbf{T}) = \sup\{|\langle \mathbf{T}h; h\rangle| : \|h\| = 1\}$.

(3) If $T \in \mathcal{C}_\rho$ so is T^n. It is however not true in general that the product of two operators in $\mathcal{C}_\rho$ is in $\mathcal{C}_\rho$. Also it is not always true that $\xi\mathbf{T}$ belongs to $\mathcal{C}_\rho$ when $\mathbf{T} \in \mathcal{C}_\rho$ for $|\xi| \neq 1$.

(4) For any M a $\mathbf{T}-$invariant subspace, the restriction of $\mathbf{T}$ to the subspace M is in the class $\mathcal{C}_\rho$ whenever $\mathbf{T}$ is.

(5) Any operator $\mathbf{T}$ such that $\sigma(\mathbf{T}) \subset \mathrm{D}$ belongs to $\mathcal{C}_\infty$.

Numerous papers where devoted the the study of differents aspects of $\mathcal{C}_\rho$; we refer to [2, 4, 5] for more information.

The next theorem provides a useful characterization of the class $\mathcal{C}_\rho$ in term of some positivity conditions,

THEOREM 1.1. *Let $\mathbf{T}$ be a bounded operator on the Hilbert space $\mathcal{H}$ and ρ be a nonnegative real. The following are equivalent*

(1) *The operator $\mathbf{T}$ belongs to the class $\mathcal{C}_\rho$;*

(2) *for all $h \in \mathcal{H}; z \in D(0;1)$*

$$(\frac{2}{\rho} - 1)\|z\mathbf{T}h\|^2 + (2 - \frac{2}{\rho})\mathrm{Re}\,(z\mathbf{T}h, h) \leq \|h\|^2; \tag{2}$$

(3) *for all $h \in \mathcal{H}; z \in D(0;1)$*

$$(\rho - 2)\|h\|^2 + 2\mathrm{Re}\,((I - z\mathbf{T})^{-1}h, h) \geq 0. \tag{3}$$

2. UNITARY ρ_n-DILATION

We extend the notion of ρ-contractions to a more general setting. More precisely, let $(\rho_n)_{n\in\mathbb{N}}$ be a sequence of nonnegative numbers. We will say that the operator $\mathbf{T}$ on a complex Hilbert space $\mathcal{H}$ belongs to the class $\mathcal{C}_{\rho_n}$ if, there exists a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ as a subspace and a unitary operator $\mathbf{U}$ such that

$$\mathbf{T}^n = \rho_n Pr \mathbf{U}^n_{|\mathcal{H}} \text{ for all } n \in \mathbb{N}^*. \tag{4}$$

We say in this case that the unitary operator $\mathbf{U}$ is a ρ_n-dilation for the operator $\mathbf{T}$ and the operator $\mathbf{T}$ will be called a ρ_n-contraction.

Remark 2.1.

(1) For any bounded operator $\mathbf{T}$, the operator $\frac{\mathbf{T}}{\|\mathbf{T}\|}$ is a contraction and hence admits a unitary dilation. We deduce that,

$$\mathbf{T} \in \mathcal{C}_{\rho_n} \text{ for } \rho_n = \|\mathbf{T}\|^n \text{ for all } n \in \mathbb{N}.$$

We notice at this level that, without additional restrictive assumptions on the sequence $(\rho_n)_{n\in\mathbb{N}}$, there is no hope to construct a reasonable ρ_n-dilation theory. Our goal will be to extend the most usefull properties of $\rho-$contraction to this more general setting.

(2) From Equation 4, for $\mathbf{T} \in \mathcal{C}_{\rho_n}$ with $\mathbf{U}$ a ρ_n-dilation, we obtain

$$\|\mathbf{T}^n\| \leq \|\rho_n Pr \mathbf{U}^n_{|\mathcal{H}}\| \leq \rho_n.$$

Therefore the condition $\lim_{n\to\infty} (\rho_n)^{\frac{1}{n}} \leq 1$ will ensure that $\sigma(\mathbf{T}) \in \overline{D(0;1)}$.

(3) In contrast with the class $\mathcal{C}_\rho$, the class $\mathcal{C}_{(\rho_n)}$ is not stable by powers. However, if $\mathbf{T} \in \mathcal{C}_{\rho_n}$ and $k \geq 1$ is a given integer, we obtain $\mathbf{T}^k \in \mathcal{C}_{\rho_{kn}}$. This latter fact can be seen as a trivial extension of the case $\rho_n = \rho_0$ for every n.

In the remaining part of this paper, we will assume that $(\rho_n)_{n\in\mathbb{N}}$ is a sequence of nonnegative numbers satisfying

$$\lim_{n\to\infty} (\rho_n)^{\frac{1}{n}} \leq 1. \tag{5}$$

We associate with the sequence $(\rho_n)_{n\in\mathbb{N}}$, the following function,

$$\rho(z) = \sum_{n\geq 0} \frac{z^n}{\rho_n}.$$

It is easy to see that condition $\lim_{n\to\infty} (\rho_n)^{\frac{1}{n}} \leq 1$ implies that $\rho \in \mathcal{H}(\mathrm{D})$. Here $\mathcal{H}(\mathrm{D})$ is the set of holomorphic functions on the open unit disc D. Also, the valued-operators function

$$\rho(z\mathbf{T}) = \sum_{n\geq 0} \frac{z^n \mathbf{T}^n}{\rho_n}$$

is well defined and converges in norm for every $|z| < 1$.

We give next a necessary and sufficient condition to the membership to the class $\mathcal{C}_{\rho_n}$;

THEOREM 2.2. *Let* $\mathbf{T}$ *be an operator on a Hilbert space* $\mathcal{H}$ *and* $(\rho_n)_{n\in\mathbb{N}}$ *is a sequence of nonnegative numbers. The operator* $\mathbf{T}$ *has a* ρ_n*-dilation if and only if*

$$(1 - \frac{2}{\rho_0})\|h\|^2 + 2\mathrm{Re}\,\langle \rho(z\mathbf{T})(h); h\rangle \geq 0 \text{ for all } h \in \mathcal{H}; z \in D(0;1). \tag{6}$$

We recall first the next well known lemma from [7, Theorem 7.1] that will be needed in the proof of the previous theorem.

LEMMA 2.3. *Let* $\mathcal{H}$ *be a Hilbert space,* G *be a multiplicative group and* Ψ *be an operator valued function* $s \in G \mapsto \Psi(s) \in \mathcal{L}(\mathcal{H})$ *such that*

$$\begin{cases} \Psi(e) = I, \ e \text{ is the identity element of } G \\ \Psi(s^{-1}) = \Psi(s)^* \\ \sum_{s\in G}\sum_{t\in G}(\Psi(t^{-1}s)h(s); h(t)) \geq 0 \end{cases}$$

for finitely non-zero function $h(s)$ *from* G.

Then, there exists a Hilbert space $\mathcal{K}$ *containing* $\mathcal{H}$ *as a subspace and a unitary representation* U *of* G*, such that*

$$\Psi(s) = Pr(U(s)) \qquad (s \in G)$$

and

$$\mathcal{K} = \bigvee_{s\in G} U(s)\mathcal{H}$$

Proof of Theorem 2.2. Let $\mathbf{T}$ be a bounded operator in the class $\mathcal{C}_{\rho_n}$ and $\mathbf{U}$ be the unitary ρ_n-dilation of $\mathbf{T}$, given by the expression 4. We have clearly,

$$I + 2\sum_{n\geq 1} z^n\mathbf{U}^n \text{ converges to } (I + z\mathbf{U})(I - z\mathbf{U})^{-1}$$

for all complex numbers z such that $|z| < 1$.
And

$$Pr(I + 2\sum_{n\geq 1} z^n\mathbf{U}^n) = I + 2\sum_{n\geq 1}\frac{z^n}{\rho_n}\mathbf{T}^n.$$

By writing,

$$\begin{aligned} I + 2\sum_{n\geq 1}\frac{z^n}{\rho_n}\mathbf{T}^n &= (1-\frac{2}{\rho_0})I + 2\sum_{n\geq 0}\frac{z^n}{\rho_n}\mathbf{T}^n \\ &= (1-\frac{2}{\rho_0})I + 2\rho(z\mathbf{T}), \end{aligned}$$

we get

$$Pr((I+z\mathbf{U})(I-z\mathbf{U})^{-1}) = (1-\frac{2}{\rho_0})I + 2\rho(z\mathbf{T}).$$

On the other hand,

$$\langle (I+z\mathbf{U})k; (I-z\mathbf{U})k\rangle = \|k\|^2 + \langle z\mathbf{U}k; k\rangle - \langle k; z\mathbf{U}k\rangle - \|z\mathbf{U}k\|^2$$

It follows that for every $k \in \mathcal{K}$, we have

$$\begin{aligned} \text{Re }\langle (I+z\mathbf{U})k; (I-z\mathbf{U})k\rangle &= \|k\|^2 - \|z\mathbf{U}k\|^2 \\ &= \|k\|^2 - |z|^2\|k\|^2 \\ &= \|k\|^2(1-|z|^2) \geq 0 \text{ since } |z|<1. \end{aligned}$$

Now if we take $h = (I - z\mathbf{U})k$ we will find,

$$\begin{aligned} \text{Re }\langle (I+z\mathbf{U})(I-z\mathbf{U})^{-1}h; h\rangle &= \text{Re }\langle Pr(I+z\mathbf{U})(I-z\mathbf{U})^{-1}h; h\rangle \\ &= \text{Re }\langle (1-\frac{2}{\rho_0})h + 2\rho(z\mathbf{T})(h); h\rangle, \end{aligned}$$

and hence for every $h \in \mathcal{H}$, we obtain

$$\text{Re }\langle (1-\frac{2}{\rho_0})h + 2\rho(z\mathbf{T})(h); h\rangle \geq 0$$

or equivalently,

$$(1-\frac{2}{\rho_0})\|h\|^2 + 2\text{Re }\langle \rho(z\mathbf{T})(h); h\rangle \geq 0$$

for every $h \in \mathcal{H}$ and all complex number z such that $|z| < 1$.

Conversely, let us show that condition (6) implies that the operator $\mathbf{T}$ belongs to the class $\mathcal{C}_{\rho_n}$. To this aim, assume that (6) is satisfied and take $0 \leq r < 1$ and $0 \leq \phi < 2\pi$. We introduce the next operator valued function

$$Q(r;\phi) = I + \sum_{n\geq 1}\frac{r^n}{\rho_n}(e^{in\phi}\mathbf{T}^n + e^{-in\phi}\mathbf{T}^{*n}).$$

Then $Q(r;\phi)$ converges in the norm operator for every r and ϕ. Moreover, from the inequality 6, we have

$$\langle Q(r;\phi)l;l\rangle \geq 0$$

for every $l \in \mathcal{H}$. Therefore

$$J = \frac{1}{2\pi}\int_0^{2\pi} \langle Q(r;\phi)h(\phi);h(\phi)\rangle d\phi \geq 0$$

for every $h(\phi) = \sum_{-\infty}^{+\infty} h_n e^{-in\phi}$ where $(h_n)_{n\in\mathbb{Z}}$ is a sequence with only finite number of nonzero elements in $\mathcal{H}$. We have

$$J =: \sum_{-\infty}^{+\infty} \|h_n\|^2 + \sum_m \sum_{n>m} \frac{r^{n-m}}{\rho_{n-m}} \langle \mathbf{T}^{n-m}h_n;h_m\rangle + \sum_m \sum_{n<m} \frac{r^{m-n}}{\rho_{m-n}} \langle \mathbf{T}^{*(m-n)}h_n;h_m\rangle$$

for every $0 \leq r < 1$. Now taking $r \to 1^-$ will imply

$$\sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} \langle \mathbf{\Psi}_{(\rho_n)}(n-m)h_n;h_m\rangle \geq 0,$$

where $\mathbf{\Psi}_{(\rho_n)} : \mathbb{Z} \to \mathcal{L}(\mathcal{H})$ is defined by $\mathbf{\Psi}_{(\rho_n)}(0) = I$, $\mathbf{\Psi}_{(\rho_n)}(n) = \frac{1}{\rho_n}\mathbf{T}^n$ and $\mathbf{\Psi}_{(\rho_n)}(-n) = \frac{1}{\rho_n}\mathbf{T}^{*n}$ for every $n > 0$.

It is immediate that $\mathbf{\Psi}_{(\rho_n)}(n)$ is nonnegative on the additive group $\mathbb{Z}$ of integers. Using Lemma 2.3, there exists a unitary operator $\mathbf{U}$ on a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ as a subspace and such that $\mathbf{\Psi}_{(\rho_n)}(n) = Pr\mathbf{U}(n)$ for all $n \in \mathbb{Z}$.

Therefore for all $n \in \mathbb{N}^*$

$$\mathbf{T}^n = \rho_n Pr \mathbf{U}^n_{|\mathcal{H}}.$$

The proof is completed. ∎

Remark 2.4. In the case where $(\rho_n)_{n\in\mathbb{N}}$ is a constant sequence, that is $\rho_n = \rho$ for all $n \in \mathbb{N}$ with $\rho > 0$, we obtain

$$\rho(z) = \frac{1}{1-z}$$

and hence, the inequality 6 becomes

$$(1 - \frac{2}{\rho})\|h\|^2 + 2\mathrm{Re}\,\langle (I - z\mathbf{T})^{-1}h;h\rangle \geq 0$$

for all $h \in \mathcal{H}$ and $z \in \mathbb{D}$. We substitute h by $l = (I - z\mathbf{T})^{-1}h$ to retrieve relation 3 and by Theorem (2.1) we obtain $\mathbf{T}$ is a ρ-contraction.

The next two corollaries are immediate consequences of Equation (6) and are related to analogous results of $\rho-$contraction.

COROLLARY 2.5. *Let* $\mathbf{T} \in \mathcal{C}_{\rho_n}$ *and* M *a* $\mathbf{T}-$*invariant subspace. Then* $\mathbf{T}_{|M} \in \mathcal{C}_{\rho_n}$.

Proof. It suffices to see that Equation 6 is close to restrictions. ∎

COROLLARY 2.6. *Let* $\mathbf{T}$ *be in the class* $\mathcal{C}_{\rho_n}$ *and* $r \geq 1$ *be a real number, then* $\mathbf{T}$ *is in the class* $\mathcal{C}_{r\rho_n}$.

Proof. The inequality 6 is equivalent to

$$(\rho_0 - 2)\|h\|^2 + 2\rho_0 \mathrm{Re}\, \langle \rho(z\mathbf{T})(h); h\rangle \geq 0.$$

Pluging $r\rho_n$ instead of ρ_n, we get

$$(r\rho_0 - 2)\|h\|^2 + 2r\rho_0 \mathrm{Re}\, \langle \frac{1}{r}\rho(z\mathbf{T})(h); h\rangle \geq 0,$$

and thus

$$(1 - \frac{2}{r\rho_0})\|h\|^2 + 2\mathrm{Re}\, \langle \frac{1}{r}\rho(z\mathbf{T})(h); h\rangle \geq 0.$$

Therefore $\mathbf{T} \in \mathcal{C}_{(r\rho_n)}$. ∎

We also have,

PROPOSITION 2.7. *Let* T *be a bounded operator on a Hilbert space* $\mathcal{H}$. *Then for every* $\alpha > 2$, *there exists* $\Gamma(\alpha) > 0$ *such that the operator* T *belongs to* $\mathcal{C}_{(\rho_n)}$, *where* ρ_n *is a sequence given by* $\rho_n = \Gamma(\alpha).\|T^n\|(1 + n^\alpha)$.

Proof. Let $\Gamma > 0$ and $\rho_\alpha(z) = \sum_{n\geq 1} \frac{z^n}{\Gamma.\|T^n\|(1+n^\alpha)}$ for all $|z| \leq 1$. Then

$$\rho_\alpha(z\mathbf{T}) = \sum_{n\geq 1} \frac{z^n \mathbf{T}^n}{\Gamma.\|T^n\|(1 + n^\alpha)} \text{ for all } |z| \leq 1.$$

For every vector h in $\mathcal{H}$, we set

$$A(z) = \langle \rho_\alpha(zT)h; h\rangle$$

$$
\begin{aligned}
|A(z)| &= |\sum_{n\geq 0}\langle\frac{z^n}{\Gamma.\|T^n\|(1+n^\alpha)}T^n h;h\rangle| \\
&\leq \sum_{n\geq 0}|\frac{\langle T^n h;h\rangle}{\Gamma.\|T^n\|(1+n^\alpha)}z^n|.
\end{aligned}
$$

Setting $a_n = \frac{\langle T^n h;h\rangle}{\Gamma.\|T^n\|(1+n^\alpha)}$, we have

$$
|a_n| \leq \frac{\|T^n\|\|h\|^2}{\Gamma.\|T^n\|(1+n^\alpha)} = \frac{\|h\|^2}{\Gamma.(1+n^\alpha)} < \infty.
$$

We conclude that $A(z)$ is holomorphic in the unit disc and continuous on the boundary. Since the maximun is attained of the circle $|z| = 1$, we obtain

$$
\begin{aligned}
|A(z)| &= |\sum_{n\geq 0} a_n z^n| \\
&\leq \sum_{n\geq 0}|a_n||z|^n = \sum_{n\geq 1}|a_n| \\
&\leq \sum_{n\geq 0}\frac{\|h\|^2}{\Gamma.(1+n^\alpha)}
\end{aligned}
$$

Now, since $\sum_{n\geq 0}\frac{1}{1+(n)^\alpha}$ is a convergent sequence ($\alpha > 2$), then choosing $\Gamma = 2\sum_{n\geq 0}\frac{1}{1+n^\alpha}$ will leads to

$$
|A(z)| \leq \frac{1}{2}\|h\|^2,
$$

and then

$$
\|h\|^2 + 2\mathrm{Re}\ \langle\rho_\alpha(zT)h;h\rangle \geq 0 \text{ for all } h\in\mathcal{H} \text{ and } z\in D.
$$

Finally , Inequality (6) is satisfied and the operator T belongs to the class $\mathcal{C}_{(\rho_n)}$. ∎

3. The Bergmann shift

We devote this section to the membership of the Bergmann shift to the class $\mathcal{C}_{(\rho_n)}$ for some suitable sequence ρ_n. Let $\mathcal{H}$ be a Hilbert space and $(e_i)_{i\in\mathbb{N}^*}$ be an orthonormal basis of $\mathcal{H}$. Recall that for a given sequence $(\omega_n)_{n\in\mathbb{N}}$ of non negative numbers; the weighted shift S_ω associated with ω_n is defined on the basis by $S_\omega(e_n) = \omega_n e_{n+1}$. A detailed study on weighted shifts can be found in the survey [9]. On the other hand; the membership of weighted shifts to the class C_ρ is investigated in [3].

The Bergman shift is the weighted shift defined on the basis by the expression $Te_n = w_n e_{n+1}$, where

$$w_n = \frac{n+1}{n} \text{ for all integer } n \in \mathbb{N}^*.$$

It is easy to see that,

- $\|\mathbf{T}\| = \sup(w_n)_{n\in\mathbb{N}^*} = 2.$
- The weight $(w_n)_{n\in\mathbb{N}^*}$ is decreasing and then

$$\|\mathbf{T}^n\| = \prod_{i=1}^{n} w_i = n+1.$$

In particular T is not power bounded and hence does not belong to the class $\mathcal{C}_\rho$ for any $\rho > 0$.

We have

PROPOSITION 3.1. *Let T be the Bergmann shift and ρ_n be the sequence given by $\rho_n = n^\alpha$ for some $\alpha > 0$. Then for every $\alpha > 2$, there exists $\Gamma(\alpha)$ such that $T \in \mathcal{C}_{\Gamma(\alpha)\rho_n}$.*

Proof. Let $\Gamma > 0$ and $\rho_\alpha(z) = \sum_{n\geq 1} \frac{z^n}{\Gamma n^\alpha}$ for all $|z| \leq 1$ in that

$$\rho(z\mathbf{T}) = \sum_{n\geq 1} \frac{z^n \mathbf{T}^n}{\Gamma n^\alpha} \text{ for all } |z| \leq 1.$$

We set, $\mathcal{S} = \rho(z\mathbf{T})$ and let x be a vector in $\mathcal{H}$. Therefore

$$\mathcal{S}(x) = \rho(z\mathbf{T})(x) = \sum_{i\geq 1} \frac{z^n \mathbf{T}^n x}{\Gamma n^\alpha}.$$

Writing $x = \sum_{i\geq 1} x_i e_i$, we get

$$\mathcal{S}(x) = \sum_{i\geq 1} \langle \mathcal{S}(x); e_i\rangle e_i = \sum_{i\geq 1} (\sum_{j\geq 1} x_j \langle \mathcal{S}e_j; e_i\rangle) e_i,$$

and

$$\langle \mathcal{S}e_j; e_i\rangle = \langle \sum_{n\geq 1} \frac{z^n \mathbf{T}^n}{\Gamma n^\alpha}(e_j); e_i\rangle = \sum_{n\geq 1} \frac{z^n}{\Gamma n^\alpha}\langle \mathbf{T}^n(e_j); e_i\rangle$$

$$= \sum_{n\geq 1} \frac{z^n}{\Gamma n^\alpha}\langle (\prod_{p=j}^{j+n-1} w_p)e_{j+n}; e_i\rangle.$$

It follows that

$$\langle \mathcal{S}e_j; e_i\rangle = \frac{z^{i-j}}{\Gamma}(i-j)^\alpha \prod_{p=j}^{i-1} w_p,$$

and then

$$\mathcal{S}(x) = \sum_{i\geq 2}(\sum_{j=1}^{i-1}(\prod_{p=j}^{i-1} w_p)x_j \frac{z^{i-j}}{\Gamma(i-j)^\alpha})e_i.$$

For the Bergman shift, we have $\prod_{p=j}^{i-1} w_p = \frac{i}{j}$ and thus

$$\rho(z\mathbf{T})(x) = \sum_{i\geq 2}(\sum_{j=1}^{i-1} \frac{i}{\Gamma j(i-j)^\alpha} x_j z^{i-j})e_i.$$

Finally, we conclude that the inequality (6) is equivalent to

$$\sum_{i\geq 1} |x_i|^2 + 2\mathrm{Re}\,(\sum_{i\geq 2}\sum_{j=1}^{i-1} \frac{i}{\Gamma j(i-j)^\alpha} x_i x_j z^{i-j}) \geq 0.$$

If we consider the function

$$A(z) = 2\sum_{i\geq 2}\sum_{j=1}^{i-1} \frac{i}{\Gamma j(i-j)^\alpha} x_i x_j z^{i-j}$$

and we write $n = i - j$, we will obtain,

$$A(z) = 2\sum_{i\geq 2}\sum_{n=1}^{i-1} \frac{i}{\Gamma(i-n)\Gamma n^\alpha} x_i x_{i-n} z^n = 2\sum_{n\geq 1}\sum_{i\geq n+1} \frac{i}{\Gamma(i-n)n^\alpha} x_i x_{i-n} z^n.$$

We denote by $(\hat{A}(n))_n = (a_n)_{n\in\mathbb{N}^*}$ the sequence of coefficients of A,

$$a_n = \frac{1}{2}\sum_{i\geq n+1} \frac{i}{\Gamma n^\alpha (i-n)} x_i x_{i-n}$$

Since $\frac{i}{n(i-n)} = \frac{1}{n} + \frac{1}{i-n} \leq 2$ for every $i \geq n+1$, we obtain

$$|a_n| = \Big|\frac{1}{2}\sum_{i\geq n+1} \frac{i}{\Gamma n^\alpha (i-n)} x_i x_{i-n}\Big| \leq \frac{1}{\Gamma n^{\alpha-1}} \sum_{i\geq n+1} |x_i x_{i-n}| \leq \frac{1}{\Gamma n^{\alpha-1}} \sum_{i\geq n+1} |x_i x_{i-n}|;$$

and by the Cauchy-Schwartz inequality, it follows,

$$|a_n| \leq \frac{1}{\Gamma n^{\alpha-1}} \|x\|^2 \leq \infty.$$

We deduce that $A(z)$ is holomorphic in the open unit disc and continuous on the closed unit disc. As the maximum is attained on the circle $|z| = 1$, we have

$$|A(z)| = \Big|\frac{1}{2}\sum_{n\geq 1}(\sum_{i\geq n+1} \frac{i}{\frac{1}{\Gamma n^\alpha}(i-n)} x_i x_{i-n}) z^n\Big| \leq \sum_{n\geq 1} |a_n||z|^n = \sum_{n\geq 1} |a_n|.$$

Now, since $\sum_{n\geq 1} \frac{1}{(n)^{\alpha-1}}$ is a convergente sequence ($\alpha \geq 2$), choosing $\Gamma = \sum_{n\geq 1} \frac{1}{(n)^{\alpha-1}}$ would lead us to

$$|A(z)| \leq \|x\|^2 = \sum_{i\geq 1} |x_i|^2.$$

We derive that,

$$|\text{Re } (A(z))| \leq |A(z)| \ \leq \|x\|^2 = \sum_{i\geq 1} |x_i|^2,$$

and hence

$$\Big|\frac{1}{2}\text{Re } (\sum_{i\geq 2}\sum_{j=1}^{i-1} \frac{i}{j\Gamma(i-j)^\alpha} x_i x_j z^{i-j})\Big| \leq \sum_{i\geq 1} |x_i|^2.$$

Therefore for all $x \in \mathcal{H}$ and a complex z such that $|z| \leq 1$ we have

$$\sum_{i\geq 1} |x_i|^2 + 2\text{Re } (\sum_{i\geq 2}\sum_{j=1}^{i-1} \frac{i}{j\Gamma(i-j)^\alpha} x_i x_j z^{i-j}) \geq 0.$$

We conclude that the weighted shift $\{w_n\}$ is a ρ_n-contraction with $\rho_n = \Gamma n^\alpha$. ∎

Remark 3.2. We claim that for every $\alpha \geq 1$ the Bergmann shift belongs to a class $\mathcal{C}_{\infty,n^\alpha}$. A proof is not available for this claim; however it is motivated by the incomplete computations below.

Let us set, for exemple, $\rho_n = 4.n$ for all integer $n \geq 1$, Let $\mathcal{H}$ be a Hilbert space and $(e_i)_{i\in\mathbb{N}^*}$ be a an orthonormal basis for the Hilbert space $\mathcal{H}$. Consider the Bergmann shift defined on the basis by $\mathbf{T}e_n = \frac{n+1}{n}e_{n+1}$ for all $n \in \mathbb{N}^*$. Then as in the proof of the previous proposition, we show that inequality (6) is equivalent to the next

$$\sum_{i\geq 1} |x_i|^2 + \mathrm{Re}\,(\sum_{i\geq 2}\sum_{j=1}^{i-1} \frac{i}{2j(i-j)} x_i x_j z^{i-j}) \geq 0. \tag{7}$$

We write

$$\sum_{i\geq 1} |x_i|^2 + \mathrm{Re}\,(\sum_{i\geq 2}\sum_{j=1}^{i-1} \frac{i}{2j(i-j)} x_i x_j z^{i-j}) \geq \sum_{i\geq 1} |x_i|^2 - \sum_{i\geq 2}\sum_{j=1}^{i-1} \frac{i}{2j(i-j)} |x_i||x_j|,$$

and

$$\sum_{i\geq 1} |x_i|^2 + \sum_{i\geq 2}\sum_{j=1}^{i-1} \frac{i}{2j(i-j)} |x_i||x_j| = \sum_{i,j\geq 1} a_{i,j}|x_i||x_j|,$$

with

$$\begin{cases} a_{i;i} = 1 & \text{for all } i \geq 1 \\ a_{i;j} = \frac{i}{4j|(i-j)|} & \text{for all } j \neq i \end{cases}$$

Then to show inequality (7), it suffices to prove that the infinite symmetric matrix with the real entries $M = [a_{i;j}]$ is nonnegative. To this aim, we compute the determinant of the first $n \times n$-corner, to check if it is nonnegative. An attempt on classical softwares allow to show this fact for $n \leq 150$. It is hence reasonable to conjecture that the Bergman shift belongs to $\mathcal{C}_{\infty,n}$.

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