Around some extensions of Casas-Alvero conjecture for non-polynomial functions

A. Cima, A. Gasull, F. Mañosas

Departament de Matemàtiques, Universitat Autònoma de Barcelona
Barcelona, Spain

cima@mat.uab.cat, gasull@mat.uab.cat, manyosas@mat.uab.cat

Received April 21, 2020
Accepted May 2, 2020

Abstract: We show that two natural extensions of the real Casas-Alvero conjecture in the non-polynomial setting do not hold.

Key words: polynomial, Casas-Alvero conjecture, zeroes of functions.


1. Introduction

The Casas-Alvero conjecture affirms that if a complex polynomial $P$ of degree $n > 1$ shares roots with all its derivatives, $P^{(k)}$, $k = 1, 2, \ldots, n - 1$, then there exist two complex numbers, $a$ and $b \neq 0$, such that $P(z) = b(z - a)^n$. Notice that, in principle, the common root between $P$ and each $P^{(k)}$ might depend on $k$. Casas-Alvero arrived to this problem at the turn of this century, when he was working in his paper [1] trying to obtain an irreducibility criterion for two variable power series with complex coefficients. See [2] for an explanation of the problem in his own words.

Although several authors have got partial answers, to the best of our knowledge the conjecture remains open. For $n \leq 4$ the conjecture is a simple consequence of the wonderful Gauss-Lucas Theorem ([6]). In 2006 it was proved in [3], by using Maple, that it is true for $n \leq 8$. Afterwards in [6, 7] it was proved that it holds when $n$ is $p^m$, $2p^m$, $3p^m$ or $4p^m$, for some prime number $p$ and $m \in \mathbb{N}$. The first cases left open are those where $n = 24$, $28$ or $30$. See again [6] for a very interesting survey or [3, 8] for some recent contributions on this question.

Adding the hypotheses that $P$ is a real polynomial and all its $n$ roots, taking into account their multiplicities, are real, the conjecture has a real
counterpart, that also remains open. It says that $P(x) = b(x - a)^n$ for some real numbers $a$ and $b \neq 0$. For this real case, the conjecture can be proved easily for $n \leq 4$, simply by using Rolle’s Theorem. This tool does not suffice for $n \geq 5$, see for instance [4] for more details, or next section.

Also in the real case, in [6] it is proved that if the condition for one of the derivatives of $P$ is removed, then there exist polynomials satisfying the remaining $n - 2$ conditions, different from $b(x - a)^n$. The construction of some of these polynomials presented in that paper is very nice and is a consequence of the Brouwer’s fixed point Theorem in a suitable context.

Finally, it is known that if the conjecture holds in $\mathbb{C}$, then it is true over all fields of characteristic 0. On the other hand, it is not true over all fields of characteristic $p$, see again [7]. For instance, consider $P(x) = x^2(x^2 + 1)$ in characteristic 5 with roots 0, 0, 2 and 3. Then $P'(x) = 2x(2x^2 + 1)$, $P''(x) = 12x^2 + 2 = 2(x^2 + 1)$ and $P'''(x) = 4x$ and all them share roots with $P$.

The aim of this note is to present two natural extensions of the real Casas-Alvero conjecture to smooth functions and show that none of them holds.

**Question 1.** Fix $1 < n \in \mathbb{N}$. Let $F$ be a class $C^n$ real function such that $F^{(n)}(x) \neq 0$ for all $x \in \mathbb{R}$, and has $n$ real zeroes, taking into account their multiplicities. Assume that $F$ shares zeroes with all its derivatives, $F^{(k)}$, $k = 1, 2, \ldots, n - 1$. Is it true that $F(x) = b(f(x))^n$ for some $0 \neq b \in \mathbb{R}$ and some $f$, a class $C^n$ real function, that has exactly one simple zero?

Notice that one of the hypotheses of the real Casas-Alvero conjecture can be reformulated as follows: The polynomial $F$ shares roots with all its derivatives but one, precisely the one corresponding to its degree. Trivially, this is so, because all the derivatives of order higher than $n$ are identically zero. The second question that we consider is:

**Question 2.** Fix $1 < n \in \mathbb{N}$. Let $F$ be a real analytic function that shares zeroes with all its derivatives but one, say $F^{(n)}$. Is it true that $F(x) = b(f(x))^n$ for some $0 \neq b \in \mathbb{R}$ and some real analytic function $f$, that has exactly one simple zero?

**Theorem A.** (i) The answer to the Question 1 is “yes” for $n \leq 4$ and “no” for $n = 5$.

(ii) The answer to the Question 2 is already “no” for $n = 2$.

Our result reinforces the intuitive idea that Casas-Alvero conjecture is mainly a question related with the rigid structure of the polynomials.
2. Proof of Theorem \[A\]

(i) The answer to Question \([1]\) is “yes” for \(n = 2, 3, 4\) because the proof of the real Casas-Alvero conjecture for the same values of \(n\), based on the Rolle’s Theorem and given in \([3]\), does not uses at all that \(P\) is a polynomial. Let us adapt it to our setting. Since \(F^{(n)}\) does not vanish we know that \(F\) has exactly \(n\) real zeroes, taking into account their multiplicities. Moreover we know that \(F\) has to have at least a double zero, that without loss of generality can be taken as 0. Next we can do a case by case study to discard all situations except that \(F\) has only a zero and it is of multiplicity \(n\). For the sake of brevity, we give all the details only in the most difficult case, \(n = 4\).

Assume, to arrive to a contradiction, that \(n = 4\), \(F\) is under the hypotheses of Question \([1]\) and \(x = 0\) is not a zero of multiplicity four. Notice that by Rolle’s theorem, for \(k = 1, 2, 3\), each \(F^{(k)}\) has exactly \(4-k\) zeroes, taking into account their multiplicities. Moreover, the only zero of \(F^{(4)}\) must be one of the zeroes of \(F\). Moreover, the only zero of \(F^{(4)}\) must be one of the zeroes of \(F\).

If \(F^{(2)}(0) = 0\) and \(F^{(4)}(0) \neq 0\) then \(F\) has only another zero at \(x = a\) and, without loss of generality, we can assume that \(a > 0\). Applying three times Rolle’s theorem we get that \(F^{(4)}(b) = 0\) for some \(b \in (0, a)\) which is a contradiction with the hypotheses. If \(F^{(2)}(0) \neq 0\) then \(F\) has two more zeroes counting multiplicities. There are three possibilities. The first one is that there is \(a > 0\) such that \(F(a) = F’(a) = 0\). In this case, applying two times Rolle’s theorem we obtain that there exist \(b, c \in (0, a)\) with \(F^{(2)}(b) = F’’(c) = 0\) and they are the only zeroes of \(F^{(4)}\). This fact gives again a contradiction because none of them is a zero of \(F\). The second one is that there exist \(a_1, a_2 \in \mathbb{R}\) with \(0 \in (a_1, a_2)\) such that \(F(a_1) = F(a_2) = 0\). Also in this case, by applying two times Rolle’s theorem we obtain that there exist \(b, c \in (a_1, a_2)\) such that \(0 \in (b, c)\) and \(F^{(2)}(b) = F^{(4)}(c) = 0\) giving us the desired contradiction. Lastly, assume that the other two zeroes of \(F\) are \(a_1\) and \(a_2\), with \(0 < a_1 < a_2\). By Rolle’s Theorem the zeroes of \(F’\) are \(0, b_1\) and \(b_2\) and satisfy \(0 < b_1 < a_1 < b_2 < a_2\). Then, since \(F^{(2)}\) has to have two zeroes, say \(c_1, c_2\), and they satisfy \(0 < c_1 < b_1 < c_2 < b_2\), the hypotheses force that \(c_2 = a_1\). Hence the zero of \(F^{(4)}\) has to be between \(c_1\) and \(c_2 = a_1\), that is in particular in \((0, a_1)\), interval that contains no zero of \(F\), arriving once more to the desired contradiction.

In short, we have proved for \(n \leq 4\), that \(F(x) = x^nG(x)\), for some class \(C^n\) function \(G\), that does not vanish. Hence

\[
F(x) = \text{sign}(G(0)) \left( x \sqrt[n]{\frac{G(x)}{\text{sign}(G(0))}} \right)^n = b(f(x))^n,
\]
where \( f \) has only one zero, \( x = 0 \), that is simple, as we wanted to prove.

To find a map \( F \) for which the answer to Question 1 is “no” we consider \( n = 5 \) and a configuration of zeroes of \( F \) and its derivatives proposed in [4] as the simplest one, compatible with the hypotheses of the Casas-Alvero conjecture and Rolle’s Theorem. Specifically, we will search for a function \( F \), of class at least \( C^5 \), with the five zeroes \( 0, 0, 1, c, d \), to be fixed, satisfying

\[
0 < 1 < c < d,
\]

and moreover

\[
F'(0) = 0, \quad F''(1) = 0, \quad F'''(c) = 0, \quad F^{(4)}(1) = 0,
\]

(2.1)

and such that \( F^{(5)} \) does not vanish. Notice that \( F'(0) = 0 \) is not a new restriction.

We start assuming that \( F^{(5)}(x) = r - \sin(x) \), for some \( r > 1 \) to be determined. By imposing that conditions (2.1) hold, together with \( F(0) = 0 \), we get that

\[
F(x) = \int_0^x \int_0^u \int_1^v \int_1^w \int_1^y (r - \sin(t)) \, dt \, dy \, dz \, dw \, dx.
\]

Some straightforward computations give that

\[
F(x) = \frac{r}{120} x^5 - \frac{r + \cos(1)}{12} x^4 + \frac{2rc - 2\sin(c) + 2\cos(1)c - rc^2}{12} x^3
\]

\[
+ \frac{6\sin(c) + 2r + 9\cos(1) - 6rc + 3rc^2 - 6\cos(1)c}{12} x^2 - 1 + \cos(x).
\]

Imposing now that \( F(1) = 0 \) we obtain that

\[
r = \frac{5 \left( 8\cos(1)c - 41\cos(1) - 8\sin(c) + 24 \right)}{4(5c^2 - 10c + 4)} = R(c).
\]

Next we have to impose that \( F(c) = 0 \). By replacing the above expression of \( r \) in \( F \) we obtain that

\[
F(c) = \frac{G(c)}{96(5c^2 - 10c + 4)},
\]

where

\[
G(c) = -c^3 \left( 12c^4 - 369c^3 + 1437c^2 - 1708c + 532 \right) \cos(1)
\]

\[
- 8c^3 (c - 1) (c - 2)^2 \sin(c) + (480c^2 - 960c + 384) \cos(c)
\]

\[
- 24 (c - 1) (9c^4 - 36c^3 + 24c^2 + 24c - 16).
\]
A carefully study shows that $G$ has exactly one real zero $c_1 \in (17/10, 19/10) = I$, with $c_1 \approx 1.79343096$. To prove its existence it suffices to show that

$$G\left(\frac{17}{10}\right) = -\frac{99211099}{500000} \cos\left(\frac{17}{10}\right) - \frac{18207}{12500} \sin\left(\frac{17}{10}\right) + \frac{696}{5} \cos\left(\frac{17}{10}\right) + \frac{1583211}{12500} > 0,$$

$$G\left(\frac{19}{10}\right) = -\frac{180110481}{500000} \cos\left(\frac{19}{10}\right) - \frac{3249}{12500} \sin\left(\frac{19}{10}\right) + \frac{1464}{5} \cos\left(\frac{19}{10}\right) + \frac{3616677}{12500} < 0.$$

By using Taylor’s formula we know that for any $c > 0$, $S^-(c) < \sin(c) < S^+(c)$ and $C^-(c) < \cos(c) < C^+(c)$ where

$$S^\pm(c) = c - \frac{c^3}{3!} + \frac{c^5}{5!} - \frac{c^7}{7!} + \frac{c^9}{9!} \pm \frac{c^{11}}{11!},$$

and

$$C^\pm(c) = 1 - \frac{c^2}{2!} + \frac{c^4}{4!} - \frac{c^6}{6!} + \frac{c^8}{8!} \pm \frac{c^{10}}{10!}.$$

Hence we can replace the values of the trigonometric functions in $G$ by rational numbers to have upper or lower bounds of this function evaluated at $1, 17/10$ or $19/10$. For instance,

$$0.5403023 \approx \frac{1960649}{3628800} = C^-(1) < \cos(1) < C^+(1) = \frac{280093}{518400} \approx 0.5403028.$$

We obtain that

$$G\left(\frac{17}{10}\right) > -\frac{99211099}{500000} C^+\left(\frac{17}{10}\right) - \frac{18207}{12500} S^+\left(\frac{17}{10}\right) + \frac{696}{5} C^-\left(\frac{17}{10}\right),$$

and

$$G\left(\frac{19}{10}\right) < -\frac{180110481}{500000} C^-\left(\frac{19}{10}\right) - \frac{3249}{12500} S^-\left(\frac{19}{10}\right) + \frac{1464}{5} C^+\left(\frac{19}{10}\right).$$
To show the uniqueness of the zero in $I$, we will prove that $G$ is strictly decreasing in this interval. It holds that

$$G'(c) = T(c) \cos(1) + U(c) \sin(1) + V(c \cos(c)) + W(c),$$

with

$$T(c) = -c \left(72c^4 - 1845c^3 + 5748c^2 - 5124c + 1064\right),$$
$$U(c) = -8 \left(5c^2 - 10c + 4\right) \left(c^2 - 2c + 12\right),$$
$$V(c) = -8 \left(c - 1\right) \left(c^4 - 4c^3 + 4c^2 - 120\right),$$
$$W(c) = -120 \left(9c^4 - 36c^3 + 36c^2 - 8\right).$$

By computing the Sturm sequences of $T$, $U$ and $V$ we can prove that $T(c) < 0$, $U(c) < 0$ and $V(c) > 0$ for all $c \in I$. Hence, for $c \in I$,

$$G'(c) < T(c)C^{-}(c) + U(c)S^{-}(c) + V(c)C^{+}(c) + W(c) = Q(c),$$

where

$$Q(c) = \frac{72469}{64800}c - \frac{669211}{43200}c^2 + \frac{18852329}{302400}c^3 - \frac{8854991}{80640}c^4$$
$$\quad + \frac{4732471}{50400}c^5 - \frac{532}{15}c^6 + \frac{8}{7}c^7 + \frac{191}{70}c^8$$
$$\quad - \frac{797}{1890}c^9 - \frac{34}{405}c^{10} + \frac{1651}{103950}c^{11} + \frac{3533}{2494800}c^{12}$$
$$\quad - \frac{193}{623700}c^{13} + \frac{1}{142560}c^{14} - \frac{1}{831600}c^{15}. $$

The Sturm sequence of $Q$ shows that it has no zeroes in $I$. Moreover, it is negative in this interval, and as a consequence, $G'$ is also negative, as we wanted to prove.

We fix $c = c_1$. Then, $r = R(c_1)$ and $F$ is also totally fixed. Moreover, by using the same techniques we get that $r = R(c_1) > R(19/10) > 1$ and as a consequence $F^{(5)}$ does not vanish. In fact, $r = R(c_1) \approx 1.04591089$. Finally, $F$ has one more real zero $d \in (33/10, 34/10)$. In fact, $d \approx 3.32178369$. This $F$ gives our desired example, see Figure 1.

(ii) Consider $F(x) = 4x^2 + \pi^2(\cos(x) - 1)$ that has a double zero at 0 and also vanishes at $\pm \pi/2$. Moreover, $F'(x) = 8x - \pi^2 \sin(x)$ vanishes at $x = 0$, $F''(x) = 8 - \pi^2 \cos(x)$ has no common zeroes with $F$ and, for any $k > 1$, 
\[ |F^{(2k)}(x)| = \pi^2 |\cos(x)| \text{ vanishes at } x = \pi/2 \text{ and } |F^{(2k-1)}(x)| = \pi^2 |\sin(x)| \text{ vanishes at } x = 0. \]

A similar example for \( n = 3 \) is \( F(x) = 4x^3 - 6\pi x^2 + \pi^3 (1 - \cos(x)), \) that vanishes at 0, \( \pi \) (double zeroes) and \( \pi/2. \)

![Plot of a map F for which the answer to Question 1 for n = 5 is “no”.](image)

**Acknowledgements**

The authors are supported by Ministerio de Ciencia, Innovación y Universidades of the Spanish Government through grants MTM2016-77278-P (MINECO/AEI/FEDER, UE, first and second authors) and MTM2017-86795-C3-1-P (third author). The three authors are also supported by the grant 2017-SGR-1617 from AGAUR, Generalitat de Catalunya.

**References**


