On angular localization of spectra of perturbed operators

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Abstract: Let $A$ and $\tilde{A}$ be bounded operators in a Hilbert space. We consider the following problem: let the spectrum of $A$ lie in some angular sector. In what sector the spectrum of $\tilde{A}$ lies if $A$ and $\tilde{A}$ are “close”? Applications of the obtained results to integral operators are also discussed.

Key words: Operators, spectrum, angular location, perturbations, integral operator.


1. Introduction and preliminaries

Let $H$ be a complex separable Hilbert space with a scalar product $(\ldots, \ldots)$, the norm $\| \cdot \| = \sqrt{(\ldots, \ldots)}$ and unit operator $I$. By $B(H)$ we denote the set of bounded operators in $H$. For an $A \in B(H)$, $A^*$ is the adjoint operator, $\| A \|$ is the operator norm and $\sigma(A)$ is the spectrum.

We consider the following problem: let $A$ and $\tilde{A}$ be “close” operators and $\sigma(A)$ lie in some angular sector. In what sector $\sigma(\tilde{A})$ lies?

Not too much works are devoted to the angular localizations of spectra. The papers [5, 6, 7, 8] should be mentioned. In particular, in the papers by E.I. Jury, N.K. Bose and B.D.O. Anderson [5, 6] it is shown that the test to determine whether all eigenvalues of a complex matrix of order $n$ lie in a certain sector can be replaced by an equivalent test to find whether all eigenvalues of a real matrix of order $4n$ lie in the left half plane. The results from [5] have been applied by G.H. Hostetter [4] to obtain an improved test for the zeros of a polynomial in a sector. In [7] M.G. Krein announces two theorems concerning the angular localization of the spectrum of a multiplicative operator integral. In the paper [8] G.V. Rozenblyum studies the asymptotic behavior of the distribution functions of eigenvalues that appear in a fixed angular region of the complex plane for operators that are close to normal. As applications, he calculates the asymptotic behavior of the spectrum of two classes of oper-
operators: elliptic pseudo-differential operators acting on the sections of a vector bundle over a manifold with a boundary, and operators of elliptic boundary value problems for pseudo-differential operators. It should be noted that in the just pointed papers the perturbations of an operator whose spectrum lie in a given sector are not considered. Below we give bounds for the spectral sector of a perturbed operator.

Without loss of the generality it is assumed that

\[ \beta(A) := \inf \Re \sigma(A) > 0. \]  

(1.1)

If this condition does not hold, instead of \( A \) we can consider perturbations of the operator \( A_1 = A + Ic \) with a constant \( c > |\beta(A)| \).

For a \( Y \in \mathcal{B}(\mathcal{H}) \) we write \( Y > 0 \) if \( Y \) is positive definite, i.e., \( \inf_{x \in \mathcal{H}, \|x\|=1} (Yx, x) > 0 \). Let \( Y > 0 \). Define the angular \( Y \)-characteristic \( \tau(A,Y) \) of \( A \) by

\[ \cos \tau(A,Y) := \inf_{x \in \mathcal{H}, \|x\|=1} \frac{\Re(YAx,x)}{|(YAx,x)|}. \]

The set

\[ S(A,Y) := \{ z \in \mathbb{C} : |\arg z| \leq \tau(A,Y) \} \]

will be called the \( Y \)-spectral sector of \( A \).

**Lemma 1.1.** For an \( A \in \mathcal{B}(\mathcal{H}) \), let condition [1.1] hold and \( Y \) be a positive definite operator, such that \( (YA)^*+YA > 0 \). Then \( \sigma(A) \) lies in the \( Y \)-spectral sector of \( A \).

**Proof.** Take a ray \( z = re^{it} \) \( (0 < r < \infty) \) intersecting \( \sigma(A) \), and take the point \( z_0 = r_0e^{it} \) on it with the maximum modulus. By the theorem on the boundary point of the spectrum [1] Section I.4.3, p. 28 there exists a normed sequence \( \{x_n\} \), such that \( Ax_n - z_0x_n \to 0 \), \( (n \to \infty) \). Hence,

\[ \frac{\Re(YAx_n,x_n)}{|(YAx_n,x_n)|} = \frac{\Re r_0e^{it}(Yx_n,x_n)}{r_0|(Yx_n,x_n)|} + \epsilon_n = \cos t + \epsilon_n \]

with \( \epsilon_n \to 0 \) as \( n \to \infty \). So \( z_0 \) is in \( S(A,Y) \). This proves the lemma. \( \blacksquare \)

**Example 1.2.** Let \( A = A^* > 0 \). Then condition [1.1] holds. For any \( Y > 0 \) commuting with \( A \) (for example \( Y = I \)) we have \( (YA)^*+YA = 2YA \) and \( \Re(YAx, x) = |(YAx, x)| \). Thus \( \cos \tau(A,Y) = 1 \) and \( S(A,Y) = \{ z \in \mathbb{C} : \arg z = 0 \} \).
So Lemma 1.1 is sharp.

Remark 1.3. Suppose $A$ has a bounded inverse. Recall that the quantity $\text{dev}(A)$ defined by

$$\cos \text{dev}(A) := \inf_{x \in H, x \neq 0} \frac{\Re(Ax, x)}{\|Ax\| \|x\|}$$

is called the angular deviation of $A$, cf. [1, Chapter 1, Exercise 32]. For example, for a positive definite operator $A$ one has

$$\cos \text{dev}(A) = \frac{2\sqrt{\lambda_M \lambda_m}}{\lambda_M + \lambda_m},$$

where $\lambda_m$ and $\lambda_M$ are the minimum and maximum of the spectrum of $A$, respectively (see [1, Chapter 1, Exercise 33]). Besides, in Exercise 32 it is pointed that the spectrum of $A$ lies in the sector $|\arg z| \leq \text{dev}(A)$. Since $|(Ax, x)| \leq \|Ax\| \|x\|$, Lemma 1.1 refines the just pointed assertion.

2. The main result

Let $A$ be a bounded linear operator in $H$, whose spectrum lies in the open right half-plane. Then by the Lyapunov theorem, cf. [1, Theorem I.5.1], there exists a positive definite operator $X \in B(H)$ solving the Lyapunov equation

$$2 \Re(AX) = XA + A^*X = 2I. \quad (2.1)$$

So $\Re(XAx, x) = ((XA + A^*X)x, x)/2 = (x, x)$ ($x \in H$) and

$$\cos \tau(A, X) = \inf_{x \in H, \|x\| = 1} \frac{(x, x)}{|(XA - \tilde{A})x, x|} = \frac{1}{\sup_{x \in H, \|x\| = 1} |(XA - \tilde{A})x, x|} \geq \frac{1}{\|AX\|}.$$

Put

$$J(A) = 2 \int_0^\infty \|e^{-At}\|^2 dt.$$

Now we are in a position to formulate our main result.

Theorem 2.1. Let $A, \tilde{A} \in B(H)$, condition (1.1) hold and $X$ be a solution of (2.1). Then with the notation $q = \|A - \tilde{A}\|$ one has

$$\cos \tau(\tilde{A}, X) \geq \cos \tau(A, X) \frac{(1 - qJ(A))}{(1 + qJ(A))},$$

provided

$$qJ(A) < 1.$$
The proof of this theorem is based on the following lemma.

**Lemma 2.2.** Let $A, \tilde{A} \in \mathcal{B}(\mathcal{H})$, condition (1.1) hold and $X$ be a solution of (2.1). If, in addition,

$$q\|X\| < 1,$$

(2.2)

then

$$\cos \tau(\tilde{A}, X) \geq \cos \tau(A, X) \frac{(1 - \|X\|q)}{(1 + \|X\|q)}.$$

**Proof.** Put $E = \tilde{A} - A$. Then $q = \|E\|$ and due to (2.1), with $x \in \mathcal{H}, \|x\| = 1$, we obtain

$$\text{Re}(X(A + E)x, x) \geq \text{Re}(XAx, x) - |(XE, x)|$$

$$= (x, x) - |(XE, x)|$$

$$\geq (x, x) - \|X\|\|E\|\|x\|^2 = 1 - \|X\|q. \quad (2.3)$$

In addition,

$$|(X(A + E)x, x)| \leq |(XA, x)| + \|X\|\|E\|\|x\|^2$$

$$= |(XA, x)| \left(1 + \frac{\|X\|q}{|(XA, x)|}\right) \quad (\|x\| = 1).$$

But

$$|(XA, x)| \geq |\text{Re}(XA, x)| = \text{Re}(XA, x) = (x, x) = 1.$$

Hence

$$|(X(A + E)x, x)| \leq |(XA, x)| \left(1 + \frac{\|X\|q}{\text{Re}(XA, x)}\right) \leq |(XA, x)|(1 + \|X\|q).$$

Now (2.3) yields.

$$\frac{\text{Re}(X\tilde{A}x, x)}{|(XA, x)|} \geq \frac{(1 - \|X\|q)}{|(XA, x)|(1 + \|X\|q)} \quad (\|x\| = 1),$$

provided (2.2) holds. Since

$$\cos \tau(\tilde{A}, X) = \inf_{x \in \mathcal{B}, \|x\| = 1} \frac{\text{Re}(X\tilde{A}x, x)}{|(XA, x)|},$$

we arrive at the required result. \(\blacksquare\)
Proof of Theorem 2.1: Note that $X$ is representable as

$$X = 2 \int_0^\infty e^{-At} e^{-At} dt$$

Hence, we easily have $\|X\| \leq J(A)$. Now the latter lemma proves the theorem.

3. Operators with Hilbert-Schmidt Hermitian components

In this section we obtain an estimate for $J(A)$ ($A \in \mathcal{B}(\mathcal{H})$) assuming that $A \in \mathcal{B}(\mathcal{H})$ and

$$A_I := (A - A^*)/i$$

is a Hilbert-Schmidt operator, (3.1)

i.e., $N_2(A_I) := (\text{trace}(A_I^2))^{1/2} < \infty$. Numerous integral operators satisfy this condition. Introduce the quantity (the departure from normality)

$$g_I(A) := \left(2N_2^2(A_I) - 2 \sum_{k=1}^\infty |\text{Im} \lambda_k(A)|^2 \right)^{1/2} \leq \sqrt{2}N_2(A_I),$$

where $\lambda_k(A)$ ($k = 1, 2, \ldots$) are the eigenvalues of $A$ taken with their multiplicities and ordered as $|\text{Im} \lambda_{k+1}(A)| \leq |\text{Im} \lambda_k(A)|$. If $A$ is normal, then $g_I(A) = 0$, cf. [2, Lemma 9.3].

Lemma 3.1. Let conditions (1.1) and (3.1) hold. Then $J(A) \leq \hat{J}(A)$, where

$$\hat{J}(A) := \sum_{j,k=0}^\infty \frac{g_{j+k}^j(A)(k+j)!}{2^{j+k} \beta_{j+k+1}(A)(j! k!)^{3/2}}.$$

Proof. By [2, Theorem 10.1] we have

$$\|e^{-At}\| \leq \exp \left[ -\beta(A)t \right] \sum_{k=0}^\infty \frac{g_k^j(A)t^k}{(k!)^{3/2}} \quad (t \geq 0).$$
Then
\[
J(A) \leq 2 \int_0^\infty \exp[-2\beta(A)t] \left( \sum_{k=0}^\infty \frac{g_k^k(A)t^k}{(k!)^{3/2}} \right)^2 dt
= 2 \int_0^\infty \exp[-2\beta(A)t] \left( \sum_{j,k=0}^\infty \frac{g^{k+j}(A)t^{k+j}}{(j!k!)^{3/2}} \right) dt
= \sum_{j,k=0}^\infty \frac{2(k+j)!g^{j+k}(A)}{(2\beta(A))^{j+k+1}(j!k!)^{3/2}},
\]
as claimed.

If \( A \) is normal, then \( g_l(A) = 0 \) and with \( 0^0 = 1 \) we have \( \hat{J}(A) = \frac{1}{\beta(A)} \).

The latter lemma and Theorem 2.1 imply

**Corollary 3.2.** Let \( A, \tilde{A} \in \mathcal{B}(\mathcal{H}) \) and let the conditions (1.1), (3.1) and \( q\hat{J}(A) < 1 \) hold. Then

\[
\cos \tau(\tilde{A},X) \geq \frac{(1 - q\hat{J}(A))}{(1 + q\hat{J}(A))} \cos \tau(A,X).
\]

**4. Integral operators**

As usually \( L^2 = L^2(0,1) \) is the space of scalar-valued functions \( h \) defined on \([0,1]\) and equipped with the norm

\[
\|h\| = \left[ \int_0^1 |h(x)|^2 dx \right]^{1/2}.
\]

Consider in \( L^2(0,1) \) the operator \( \tilde{A} \) defined by

\[
(\tilde{A}h)(x) = a(x)h(x) + \int_0^1 k(x,s)h(s)ds \quad (h \in L^2, x \in [0,1]),
\]

where \( a(x) \) is a real bounded measurable function with

\[
a_0 := \inf a(x) > 0,
\]

and \( k(x,s) \) is a scalar kernel defined on \( 0 \leq x, s \leq 1 \), and

\[
\int_0^1 \int_0^1 |k(x,s)|^2 ds dx < \infty.
\]
So the Volterra operator \( V \) defined by
\[
(Vh)(x) = \int_x^1 k(x, s) h(s) \, ds \quad (h \in L^2, x \in [0, 1]),
\]
is a Hilbert-Schmidt one. Define operator \( A \) by
\[
(Ah)(x) = a(x) h(x) + \int_x^1 k(x, s) h(s) \, ds \quad (h \in L^2, x \in [0, 1]).
\]
Then \( A = D + V \), where \( D \) is defined by \((Dh)(x) = a(x) h(x)\). Due to Lemma 7.1 and Corollary 3.5 from [3] we have \( \sigma(A) = \sigma(D) \). So \( \sigma(A) \) is real and \( \beta(A) = a_0 \). Moreover,
\[
N_2(A) = N_2(V) \leq \sqrt{2} N_2(V)
\]
Here \( V_I = (V - V^*)/2i \). Thus,
\[
g_I(A) \leq g_V := \sqrt{2} N_2(V)
\]
and
\[
\|A - \tilde{A}\| \leq q_0 := \left[ \int_0^1 \int_x^1 |k(x, s)|^2 \, ds \, dx \right]^{1/2}.
\]
Simple calculations show that under consideration
\[
\hat{J}(A) \leq \hat{J}_0 := \sum_{j,k=0}^{\infty} \frac{g_j^j k^k \, (k + j)!}{2^{2j+k} a_0^{j+k+1} (j! k!)^{3/2}}.
\]
Making use of Corollary 3.2 and taking into account that in the considered case \( \cos \tau(A, X) = 1 \), we arrive at the following result.

**Corollary 4.1.** Let \( \tilde{A} \) be defined by (4.1) and the conditions (4.2) and (4.3) hold. If, in addition, \( q_0 \hat{J}_0 < 1 \), then \( \sigma(\tilde{A}) \) lies in the angular sector
\[
\left\{ \, z \in \mathbb{C} \, : \, |\arg z| \leq \arccos \left( \frac{1 - q_0 \hat{J}_0}{1 + q_0 \hat{J}_0} \right) \, \right\}.
\]

**Example 4.2.** To estimate the sharpness of our results consider in \( L^2(0,1) \) the operators
\[
(Ah)(x) = 2h(x) \quad \text{and} \quad (\tilde{A}h)(x) = (2 + i) h(x) \quad (h \in L^2, x \in [0, 1]).
\]
$\sigma(A)$ consists of the unique point $\lambda = 2$ and so $\cos(A, X) = \cos \arg \lambda = 1$. We have

$$J(A) = 2 \int_0^\infty e^{-4t}dt = 1/2 \quad \text{and} \quad q = 1.$$ 

By Corollary 3.2

$$\cos \tau(\tilde{A}, X) \geq \frac{1 - 1/2}{1 + 1/2} = 1/3.$$ 

Compare this inequality with the sharp result: $\sigma(\tilde{A})$ consists of the unique point $\tilde{\lambda} = 2 + i$. So $\tan(\arg \tilde{\lambda}) = 1/2$, and therefore $\cos(\arg \tilde{\lambda}) = 2/(\sqrt{5})$.

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References


