Granular Poiseuille flow

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Outline

- Gravity-driven Poiseuille flow for conventional gases.
- Newtonian description.
- Gravity-driven Poiseuille flow for heated granular gases.
- Kinetic theory description through second order in gravity.
- Results.
- Conclusions.
Jean-Louis Marie Poiseuille (1797-1869)

Poiseuille's law

From Wikipedia, the free encyclopedia.

The Poiseuille's law (or the Hagen-Poiseuille law) also named after Gotthilf Heinrich Ludwig Hagen (1797-1884) for his experiments in 1839 is the physical law concerning the voluminal laminar stationary flow \( \Phi_V \) of incompressible uniform viscous liquid (so called Newtonian fluid) through a cylindrical tube with the constant circular cross-section, experimentally derived in 1838, formulated and published in 1841 and 1846 by Jean Louis Marie Poiseuille (1797-1869), and defined by:

\[
\Phi_V = \frac{dV}{dt} = v_s \pi r^2 = \frac{\pi r^4}{8\eta} \left( \frac{dp}{dz} \right) = \frac{\pi r^4}{8\eta} \frac{\Delta p}{l},
\]

where \( V \) is a volume of the liquid, poured in the time unit \( t \), \( v_s \) median fluid velocity along the axial cylindrical coordinate \( z \), \( r \) internal radius of the tube, \( \Delta p \) the pressure drop at the two ends, \( \eta \) dynamic fluid viscosity and \( l \) characteristic length along \( z \), a linear dimension in a cross-section (in non-cylindrical tube).
Planar Poiseuille flow generated by a gravity field in a conventional gas

Conservation equations for momentum and energy

\[
\frac{\partial P_{yy}}{\partial y} = 0
\]

\[
\frac{\partial P_{yz}}{\partial y} = -\rho g
\]

\[
P_{yz} \frac{\partial u_z}{\partial y} + \frac{\partial q_y}{\partial y} = 0
\]
Navier-Stokes (Newtonian) description

\[ P_{xx} = P_{yy} = P_{zz} = p \]

Equal normal stresses

\[ P_{yz} = -\eta \frac{\partial u_z}{\partial y} \]

Newton’s law

\[ q_y = -\kappa \frac{\partial T}{\partial y} \]

Fourier’s law

\[ q_z = 0 \]

No longitudinal heat flux

\[ p(y) = p_0 = \text{const} \]

\[ u_z(y) = u_0 + \frac{\rho_0 g}{2\eta_0} y^2 + \mathcal{O}(g^3) \]

Temperature is maximal at the central layer \((y=0)\)
Do NS predictions agree with computer simulations?

but ...

On the validity of hydrodynamics in plane Poiseuille flows

M. Malek Mansour*, F. Baras*, Alejandro L. Garcia1,2

In the slab $y < |y_{\text{max}}|$, 

$$\text{sgn } q_y = \text{sgn } \frac{\partial T}{\partial y}$$

Heat flows from the colder to the hotter layers!!
Other Non-Newtonian properties

Non-uniform pressure

Normal stress differences

Longitudinal component of the heat flux (but no longitudinal thermal gradient!)
These Non-Newtonian effects are well accounted for by kinetic theory tools:

- Grad’s method applied to the Boltzman equation for hard spheres (S. Hess, M. Malek Mansour, D. Risso, P. Cordero).
Is the gravity-driven Poiseuille flow relevant to real gases?

\[
\frac{T_{\text{max}} - T_0}{T_0} \gtrsim 10^{-2} \Rightarrow g \frac{\lambda}{v_{th}^2} \gtrsim 2 \times 10^{-2}
\]

\(\lambda\): mean free path; \(v_{th}\): thermal velocity

Argon at room conditions:

\[
\begin{align*}
g &= 9.8 \text{ m/s}^2 \\
\lambda &\sim 700 \text{ Å} \\
v_{th} &\sim 400 \text{ m/s}
\end{align*}
\]

\[g\lambda/v_{th}^2 \sim 10^{-12} \text{ !!}\]
They are *mesoscopic* particles \((\sigma \sim 1 \text{ mm})\)

Some typical values

\[
\begin{align*}
g &= 9.8 \text{ m/s}^2 \\
\lambda &\approx 1 \text{ mm-1cm} \\
v_{\text{th}} &\geq 1 \text{ m/s}
\end{align*}
\]

\[
g\lambda/v_{\text{th}}^2 \sim 10^{-3}-10^{-1}
\]

The dimensionless parameter \(g\lambda/v_{\text{th}}^2\) measures the strength of gravity between collisions. It can be:

- Large enough as to produce measurable effects.
- Small enough as to allow for a perturbative treatment.
Our main goal is:

• Call attention to the fact that non-Newtonian properties in the gravity-driven Poiseuille flow can be observable on granular gases under laboratory conditions.

• Assess the influence of inelasticity on the hydrodynamic fields and their fluxes. E.g., is \((T_{\text{max}} - T_0)/T_0\) enhanced or inhibited by inelasticity?
A gas of (smooth) *inelastic* hard spheres

\[ \alpha: \text{coefficient of (normal) restitution} \]

\[
\begin{align*}
  v_1' &= v_1 - \frac{1 + \alpha}{2} (v_{12} \cdot \hat{\sigma}) \hat{\sigma}, \\
  v_2' &= v_2 + \frac{1 + \alpha}{2} (v_{12} \cdot \hat{\sigma}) \hat{\sigma}
\end{align*}
\]

\[
\begin{align*}
  v_1'' &= v_1 - \frac{1 + \alpha}{2 \alpha} (v_{12} \cdot \hat{\sigma}) \hat{\sigma}, \\
  v_2'' &= v_2 + \frac{1 + \alpha}{2 \alpha} (v_{12} \cdot \hat{\sigma}) \hat{\sigma}
\end{align*}
\]

(After T.P.C. van Noije & M.H. Ernst)
Boltzmann equation

\[
\left( \partial_t + \mathbf{v} \cdot \nabla + \mathbf{g} \cdot \frac{\partial}{\partial \mathbf{v}} + \mathcal{F} \right) f = J[f, f]
\]

\[
J[f, f] = \sigma^2 \int d\mathbf{v}_1 \int d\hat{\sigma} \Theta((\mathbf{v} - \mathbf{v}_1) \cdot \hat{\sigma})[(\mathbf{v} - \mathbf{v}_1) \cdot \hat{\sigma}] \left[ \alpha^{-2} f(\mathbf{v}''') f(\mathbf{v}_1''') - f(\mathbf{v}) f(\mathbf{v}_1) \right]
\]
Collisional “cooling”

\[ \frac{m}{3} \int d\mathbf{v} \, V^2 \, J[f, f] = -\zeta nT \]

Cooling rate

External “heating” (e.g., vibrations)

\[ \frac{m}{3} \int d\mathbf{v} \, V^2 \, \mathcal{F} f(\mathbf{v}) = -\gamma nT \]

Heating rate

\[ \mathbf{V} \equiv \mathbf{v} - \mathbf{u} \quad \text{(peculiar velocity)} \]

Gaussian approximation

\[ \zeta \simeq \nu \frac{5}{12} (1 - \alpha^2) \]

Effective collision frequency

\[ \nu = \frac{16}{5} n \sigma^2 \left( \frac{\pi T}{m} \right)^{1/2} \]
White noise driving

It is a bulk heating mechanism that intends to mimic the effect of boundary driving (e.g., vibrations).

Each particle is subjected to the action of a stochastic force with white noise properties:

$$
\langle F_{\text{wn}}(t) \rangle = 0, \quad \langle F_{\alpha}^{\text{wn}}(t) F_{\beta}^{\text{wn}}(t') \rangle = m^2 \xi^2 \delta_{\alpha \beta} \delta(t - t')
$$

During a small time step $\Delta t$, each particle receives a “kick,” so its velocity is incremented by a random amount $\Delta v$

$$
\Delta t \Rightarrow |\Delta v| \sim \xi \sqrt{\Delta t}
$$

Diffusion in velocity space:

$$
F = -\frac{\xi^2}{2} \left( \frac{\partial}{\partial v} \right)^2 \Rightarrow \gamma = \frac{m \xi^2}{T}
$$

Heating rate
Our choice: The white noise compensates *locally* for the collisional cooling.

\[
\gamma = \zeta \Rightarrow \frac{|\Delta v|}{v_{th}} \sim \sqrt{\nu \Delta t (1 - \alpha^2)}
\]

The *relative* magnitude of the kick scales with (the square root of) the (local) probability of a collision.

**Associated NS transport coefficients:**

(\text{Garzó & Montanero, 2002})

\[
\eta \simeq \frac{p}{\nu} \frac{4}{(1 + \alpha) (3 - \alpha)}, \quad \kappa \simeq \frac{5p}{2m\nu} \frac{48}{(1 + \alpha) (49 - 33\alpha)}
\]

- Increases with inelasticity
- Decreases with inelasticity ($\alpha \gtrsim 0.4$)
- Increases with inelasticity ($\alpha \lesssim 0.4$)
\begin{center}
\includegraphics[width=\textwidth]{figure.png}
\end{center}
Stationary Boltzmann equation

\[
\left( -\frac{\zeta T}{2m} \frac{\partial^2}{\partial v^2} - g \frac{\partial}{\partial v_z} + v_y \frac{\partial}{\partial y} \right) f = J[f, f].
\]

White noise heating  Gravity  Inelastic collisions

BGK-like kinetic model:
(Brey, Dufty, A.S.)

\[
J[f, f] \rightarrow -\beta(\alpha)\nu(f - f_\ell) + \frac{\zeta}{2} \frac{\partial}{\partial v} \cdot [(v - u) f]
\]

Modified collision frequency  Effective drag force: mimics cooling

\[
f_\ell(r, v; t) = n(r, t) \left[ \frac{m}{2\pi T(r, t)} \right]^{3/2} \exp \left[ -\frac{m(v - u(r, t))^2}{2T(r, t)} \right]
\]

Local Gaussian distribution

Modelling and numerics of kinetic dissipative systems (Lipari, May 31 - June 4 2004)
Digression: How reliable is the BGK-like model?

Journal of Statistical Physics, Vol. 103, Nos. 5/6, 2001

Nonlinear Couette Flow in a Low Density Granular Gas

M. Tölö,1 E. E. Tahiri,2 J. M. Montanero,3 V. Garzó,4 A. Santos,5 and J. W. Dufty1

Steady uniform shear flow in a low density granular gas

J. J. Brov, M. J. Ruiz-Montero, and F. Moreno

Modelling and numerics of kinetic dissipative systems (Lipari, May 31 - June 4 2004)
Perturbation expansion

\[ f(y, \mathbf{V}) = f_0(y, \mathbf{V}) \left[ 1 + \Phi^{(1)}(y, \mathbf{V})g + \Phi^{(2)}(y, \mathbf{V})g^2 + \mathcal{O}(g^3) \right] \]

\[ p(y) = p_0 + p^{(2)}(y)g^2 + \mathcal{O}(g^4) \]

\[ u_z(y) = u_0 + u^{(1)}(y)g + \mathcal{O}(g^3) \]

\[ T(y) = T_0 + T^{(2)}(y)g^2 + \mathcal{O}(g^4) \]

Velocity distribution function

Hydrodynamic profiles

Structure of the solution through second order:

\[ \Phi^{(1)}(y, \mathbf{V}) = V_z(a_0 + a_1 V_y^2 + a_2 V_y y) \]

\[ \Phi^{(2)}(y, \mathbf{V}) = b_0 + b_1 V_y^2 + b_2 V_y y + b_3 y^2 + b_4 V_y^4 + b_5 V_y^3 y + b_6 V_y^2 y^2 + b_7 V_y y^3 \]

\[ + (c_0 + c_1 V_y^2 + c_2 V_y y + c_3 y^2 + c_4 V_y^4 + c_5 V_y^3 y + c_6 V_y^2 y^2) V_z^2 \]

\[ + (d_0 + d_1 V_y^2 + d_2 V_y y + d_3 y^2 + d_4 V_y^4 + d_5 V_y^3 y + d_6 V_y^2 y^2 + d_7 V_y y^3) V^2 \]
Hydrodynamic profiles

\[ p(y) = p_0 \left[ 1 + \frac{6}{5} \left( \frac{mg}{T_0} \right)^2 y^2 \right] + \mathcal{O}(g^4) \]

\[ u_y(y) = u_0 + \frac{\rho_0 g}{2\eta_0} y^2 + \mathcal{O}(g^3) \]

\[ T(y) = T_0 \left[ 1 - \frac{\rho_0^2 g^2}{12\eta_0\kappa_0 T_0} y^4 + \frac{1}{25} \frac{38 + 43\zeta_0^* + 17\zeta_0^{*2}}{(1 + \zeta_0^*)(2 + \zeta_0^*)} \left( \frac{mg}{T_0} \right)^2 y^2 \right] + \mathcal{O}(g^4) \]

\[ \zeta_0^* = \frac{\frac{5}{12} (1 - \alpha^2)}{\beta(\alpha) + \frac{5}{12} (1 - \alpha^2)} \]
Non-monotonic temperature profile

\[ T = T_0 \left[ 1 - A_4(\alpha) \left( \frac{g\lambda_0}{v_0^2} \right)^2 \left( \frac{y}{\lambda_0} \right)^4 + A_2(\alpha) \left( \frac{g\lambda_0}{v_0^2} \right)^2 \left( \frac{y}{\lambda_0} \right)^2 \right] + O(g^4) \]

\[ \eta, \kappa = \text{Boltzmann} \Rightarrow A_4(\alpha) = \frac{4}{1125\pi} (1 + \alpha)^2 (3 - \alpha) (49 - 33\alpha) \]

\[ \beta(\alpha) = (1 + \alpha) \frac{2 + \alpha}{6} \Rightarrow A_2(\alpha) = \frac{4}{25} \frac{2719 - 2741\alpha + 706\alpha^2}{(7 - 4\alpha)(23 - 11\alpha)} \]

\[ y_{\text{max}} = \pm \lambda_0 \sqrt{ \frac{A_2(\alpha)}{2A_4(\alpha)} } \]

\[ \frac{T_{\text{max}} - T_0}{T_0} = \frac{A_2^2(\alpha)}{4A_4(\alpha)} \left( \frac{g\lambda_0}{v_0^2} \right)^2 + O(g^4) \]
If \( \alpha \geq 0.4 \), the bi-modal shape of \( T(y) \) becomes (slightly) less pronounced as inelasticity increases.

However, the opposite behavior takes place if \( \alpha \leq 0.4 \).
Fluxes

\[ P_{yy} = p_0 \left[ 1 - \frac{12}{25} \frac{102 \zeta_0 + 87 \zeta_0^* + 13 \zeta_0^*^2}{(1 + \zeta_0^*)(2 + \zeta_0^*)^2} \frac{\rho_0 \eta_0^2 g^2}{p_0^3} \right] + O(g^4) \]

\[ P_{zz}(y) = p_0 \left[ 1 + \frac{16}{25} \frac{82 + 67 \zeta_0^* + 8 \zeta_0^*^2}{(1 + \zeta_0^*)(2 + \zeta_0^*)^2} \frac{\rho_0 \eta_0^2 g^2}{p_0^3} + \frac{14}{5} \left( \frac{mg}{T_0} \right)^2 y^2 \right] + O(g^4) \]

Normal stress differences

\[ q_y(y) = \frac{\rho_0^2 g^2}{3 \eta_0} y^3 + O(g^4) \]

Longitudinal heat flux

\[ q_z = \frac{2}{5} m \kappa_0 g + O(g^3) \]

\[ q_y = -\kappa \frac{\partial}{\partial y} \left( T + \frac{y_{\text{max}}^2}{6} \nabla^2 T \right) + O(g^4) \]

Super-Burnett
\[ g \lambda_0 / v_0^2 = 0.05 \]

\[ P_{yy} < P_{xx} < p < P_{zz} \]

\[ |q_y| < q_z \]
Conclusions (I)

- Gravity-driven Poiseuille flow exhibits interesting (and even counter-intuitive) non-Newtonian properties which are accessible to granular gases.
- Non-uniform hydrostatic pressure.
- Non-isotropic normal stresses.
- Heat flux component normal to the thermal gradient.
Conclusions (II)

- Bi-modal shape of the temperature profile:
  \[ |y_{\text{max}}| \approx 3 \text{ mfp}, \frac{(T_{\text{max}} - T_0)}{T_0} \approx 10 \left(\frac{g\lambda}{v_{\text{th}}^2}\right)^2. \]

- For moderate or small inelasticity (\(\alpha \gtrsim 0.4\)), the larger the inelasticity, the more pronounced the bi-modal temperature profile.
  The reverse is true for large inelasticity (\(\alpha \lesssim 0.4\)).

- A similar influence of \(\alpha\) on normal stress differences.

- Computer simulations (DSMC or MD) would be very welcome!
THANKS!