A simple model kinetic equation for inelastic Maxwell particles

Andrés Santos
Departamento de Física, Universidad de Extremadura, Badajoz (Spain)
Outline

• The Boltzmann equation for Inelastic Maxwell Particles
• Model kinetic equation
• Results
• Conclusions
Model of Inelastic Hard Spheres (HCS)

- Smooth inelastic hard spheres (mass $m$, diameter $\sigma$, coefficient of normal restitution $\alpha$)
- Post-collisional velocities:

$$v' = v - \frac{1 + \alpha}{2} (g \cdot \hat{\sigma}) \hat{\sigma}$$

$$v_1' = v_1 + \frac{1 + \alpha}{2} (g \cdot \hat{\sigma}) \hat{\sigma}$$
Boltzmann equation for IHS

- Dilute granular gas
- Absence of velocity correlations before collision

\[ \partial_t f + \mathbf{v} \cdot \nabla f = J[\mathbf{v}|f] \]

\[ J[\mathbf{v}|f] = \sigma^2 \int d\mathbf{v}_1 \int d\hat{\sigma} \Theta(\mathbf{g} \cdot \hat{\sigma})(\mathbf{g} \cdot \hat{\sigma}) \]

\[ \times \left[ \alpha^{-2} f(\mathbf{v}^\prime) f(\mathbf{v}_1^\prime) - f(\mathbf{v}) f(\mathbf{v}_1) \right] \]

pre-collisional
Model of Inelastic Maxwell Particles (IMP)

- Bobylev, Carrillo, Gamba, Cercignani (2000)

\[ \mathbf{g} \cdot \mathbf{\hat{\sigma}} \rightarrow \text{const} \sqrt{2T/m} \mathbf{\hat{g}} \cdot \mathbf{\hat{\sigma}} \]

- Ben-Naim, Krapivsky, Ernst, Brito (2002)

\[ \mathbf{g} \cdot \mathbf{\hat{\sigma}} \rightarrow \text{const} \sqrt{2T/m} \]
Boltzmann equation for IHS

\[ \partial_t f + \mathbf{v} \cdot \nabla f = J[\mathbf{v} | f] \]

\[ J[\mathbf{v} | f] = \frac{5}{8\pi n} \int dv_1 \int d\hat{\sigma} \Theta (g \cdot \hat{\sigma})(g \cdot \hat{\sigma}) \times \left[ \alpha^{-2} f(\mathbf{v}'') f(\mathbf{v}_1'') - f(\mathbf{v}) f(\mathbf{v}_1) \right] \]

\[ \nu_0 = \frac{16}{5} n\sigma^2 \sqrt{T/m\pi} \text{: collision frequency} \]
Boltzmann equation for IMP

• The IMP model is interesting by itself since it allows the derivation of some exact results.
• Those results show unambiguously the strong influence of inelasticity on the nonequilibrium properties of the gas.
• The model is useful to gain a broader perspective on the peculiar properties of dissipative gases.
Basic properties of the Boltzmann equation for IMP

- Cooling rate: \( \frac{m}{3n} \int dv \ V^2 J[f] = -\zeta(\alpha)T \)

- Collisional rates of change:

\[
m \int dv \left( V_i V_j - \frac{1}{3} V^2 \delta_{ij} \right) J[f] = -\nu_\eta(\alpha) \left( P_{ij} - p \delta_{ij} \right)
\]

\[
\frac{m}{2} \int dv \ V^2 V J[f] = -\nu_\kappa(\alpha) q
\]

\[
n^{-1} \int dv \ V^4 J[f] = -\nu_2(\alpha)\langle V^4 \rangle + \lambda(\alpha)\nu_0 (2T/m)^2
\]
Basic properties of the Boltzmann equation for IMP

• Uniform, free cooling state
• Scaling solution: homogeneous cooling state (HCS):

\[ \partial_t f(v) = J[v|f], \quad \partial_t T = -\zeta T \]

\[ f(v,t) = n \left[ \frac{m}{2T(t)} \right]^{3/2} f_{hcs}^*(c), \quad c = \frac{v}{\sqrt{2T(t)/m}} \]

\[ \frac{\zeta^*}{2} \partial_c \cdot c f_{hcs}^*(c) = J^*[c|f_{hcs}^*] \]
Basic properties of the Boltzmann equation for IMP

• Homogeneous cooling state (HCS):
  – Approach to the HCS (Bobylev, Cercignani, Toscani, 2003)
  – Fourth cumulant (kurtosis):
    \[ a_2(\alpha) = \frac{4}{15} \langle c^4 \rangle_{\text{hcs}} - 1 \]
  – Algebraic high-energy tail:
    \[ c \gg 1 \Rightarrow f_{\text{hcs}}^*(c) \sim c^{-3-s(\alpha)} \]
Basic properties of the Boltzmann equation for IMP

- Navier-Stokes transport coefficients:

\[ P_{ij} = p\delta_{ij} - \eta(\alpha) \left( \nabla_i u_j + \nabla_j u_i - \frac{2}{3} \nabla \cdot \mathbf{u} \delta_{ij} \right) \]

\[ q = -\kappa(\alpha) \nabla T - \mu(\alpha) \nabla n \]

\( \kappa(\alpha) \) and \( \mu(\alpha) \) are negative for \( \alpha < 1/9 \)
Why a model kinetic equation for IMP?

- The Boltzmann equation for IMP is more manageable than for IHS and some important properties are accessible in an exact way.
- However, its explicit solution $f(v)$ is not known, even for the HCS.
- Is it possible to construct a simple (and accurate) generalization of the well-known BGK model kinetic equation to the case of IMP?
Model kinetic equation for IMP

\[ J[v|f] \rightarrow \tilde{J}[v|f] \equiv -\beta(\alpha)\nu_0 [f(v) - f_0(v)] \]
\[ + \gamma(\alpha)\nu_0 \partial_v \cdot V f(v) \]

\[ f_0(v) = n \left( \frac{m}{2\pi \theta(\alpha) T} \right)^{3/2} e^{-mV^2/2\theta(\alpha)T} \]

- Effective collision frequency
- Friction coefficient
- Effective reference temperature
### Expressions for the main quantities

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Boltzmann equation</th>
<th>Kinetic model</th>
</tr>
</thead>
<tbody>
<tr>
<td>ζ* ≡ ζ/ν₀</td>
<td>$\frac{5}{12}(1 - \alpha^2)$</td>
<td>$\beta(1 - \theta) + 2\gamma$</td>
</tr>
<tr>
<td>η* ≡ η/ν₀</td>
<td>$\frac{1}{4}(1 + \alpha)^2 + \zeta^*$</td>
<td>$\beta \theta + \zeta^*$</td>
</tr>
<tr>
<td>κ* ≡ κ/ν₀</td>
<td>$\frac{1}{6}(1 + \alpha)^2 + \frac{3}{2}\zeta^*$</td>
<td>$\frac{1}{2}\beta (3\theta - 1) + \frac{3}{2}\zeta^*$</td>
</tr>
<tr>
<td>ν₂* ≡ ν₂/ν₀</td>
<td>$\frac{1}{48}(1 + \alpha)^2 (5 + 6\alpha - 3\alpha^2) + 2\zeta^*$</td>
<td>$\beta(2\theta - 1) + 2\zeta^*$</td>
</tr>
<tr>
<td>λ</td>
<td>$\frac{5}{64}(1 + \alpha)^2 (11 - 6\alpha + 3\alpha^2)$</td>
<td>$\frac{15}{4}\beta \theta^2$</td>
</tr>
<tr>
<td>a₂</td>
<td>$6(1 - \alpha)^2/(5 + 6\alpha - 3\alpha^2)$</td>
<td>$\frac{(1 - \theta)^2}{(2\theta - 1)}$</td>
</tr>
<tr>
<td>s</td>
<td>Transcendental eqn.</td>
<td>$2/(1 - \theta)$</td>
</tr>
</tbody>
</table>

$\beta(\alpha)$, $\theta(\alpha)$, and $\gamma(\alpha)$ are determined by requiring the kinetic model to reproduce the correct $\zeta(\alpha)$, $\nu_\eta(\alpha)$, and $a_2(\alpha)$.
RESULTS

Parameters of the model

- Parameters: $\alpha$, $1-\gamma$, $\beta$, $\theta$

Graph showing the relationship between $\alpha$ and the parameters $1-\gamma$, $\beta$, and $\theta$. The graph indicates how these parameters vary with the value of $\alpha$. 

---

25th INTERNATIONAL SYMPOSIUM ON RAREFIED GAS DYNAMICS
July 21-28, 2006, Saint-Petersburg, Russia
RESULTS

Transport coefficients

\[ \frac{\kappa(\alpha)}{\kappa(1)} \]

\[ \frac{\mu(\alpha) n}{T \kappa(1)} \]
RESULTS

Homogeneous cooling state

\[
f_{hcs}(c) = \frac{(1 - \theta)^{-1}}{\left(\pi \theta \theta\right)^{3/2}} \left(\frac{\theta}{c^2}\right)^{3/2} + (1 - \theta)^{-1} \int_0^{c^2/\theta} dx x^{3/2} + (1 - \theta)^{-1} e^{-x}\]

\[
f_{hcs}(c) \sim c^{-3-s(\alpha)}
\]
RESULTS

Homogeneous cooling state

\[ f_{\text{hcs}}^*(c) = \frac{(1 - \theta)^{-1}}{(\pi \theta)^{3/2}} \left( \frac{\theta}{c^2} \right)^{3/2} + (1 - \theta)^{-1} \int_{c^2/\theta}^{\infty} dx \, x^{3/2} + (1 - \theta)^{-1} e^{-x} \]
RESULTS

Approach to the HCS

\[ \delta f^*(c, \tau) = e^{-\beta[1+3(1-\theta)/2]} \delta f^* \left( e^{-\beta(1-\theta)\tau/2} c, 0 \right) \]
Conclusions

• The proposed kinetic model is a simple extension of the BGK model.
• The effect of the inelastic collisions is played by (i) a relaxation term toward a reference Maxwellian distribution plus (ii) a term representing the action of a friction force.
• The model contains three free parameters: a factor $\beta(\alpha)$ modifying the collision frequency, a factor $\theta(\alpha)$ modifying the temperature of the reference Maxwellian, and a friction coefficient $\gamma(\alpha)$.
• The parameters are determined by fitting the cooling rate, kurtosis, and shear viscosity of IMP.
Conclusions

• The kinetic model can be useful to have access, at least at a semi-quantitative way, to relevant information (such as the velocity distribution function itself) not directly available from the Boltzmann equation for IMP.

• The same philosophy can be applied to extensions of the ellipsoidal statistical kinetic model and to mixtures of IMP.
\[ J[v|f] \rightarrow \tilde{J}[v|f] = -\beta(\alpha)\nu_0 [f(v) - f_0(v)] + \gamma(\alpha)\nu_0 \partial_v \cdot V f(v) \]