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Some Aspects of $\lambda(\mathbf{P_0}, \mathbb{N})$-Nuclearity

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In the first part of this article we deal with the characterization of $\lambda(P_0;\mathbb{N})$-nuclearity of a sequence space when equipped with other 'natural' (and more general) topologies. Indeed, efforts have been made to explore conditions for the $\lambda(P_0;\mathbb{N})$-nuclearity of a sequence space when it is endowed with the $\sigma\mu$-topology of Ruckle. In an analogous way, a Grothendieck-Pietsch like criterion is obtained for the $\lambda(P_0;\mathbb{N})$-nuclearity of the class of the generalized Köthe spaces $\lambda_\mu(P)$. For $\mu = \ell^1$, this yields the well-known Grothendieck-Pietsch criterion for the $\lambda(P_0;\mathbb{N})$-nuclearity of a Köthe space $\lambda(P)$. It is observed that for a Hilbert K-space μ having a monotone normalized Schauder basis, $\lambda(P_0;\mathbb{N})$-nuclearity of the extended Köthe space $\lambda_\mu(P)$ is synonymous with the $\lambda(P_0;\mathbb{N})$-nuclearity of the Köthe space $\lambda(P)$. It is shown that for a $\lambda(P_0;\mathbb{N})$-nuclear space $(\lambda, \sigma\mu)$ (resp., λ^μ), a sequentially complete space having a fully-λ-base (resp., fully-λ^μ-base) is $\lambda(P_0;\mathbb{N})$-nuclear. In addition, there are some results which make it amply clear that the impact of the associated sequence space μ is equally significant so far as the structure of a sequentially complete space possessing a fully-λ-base (or fully-λ^μ-base) is concerned.

1. Introduction

For various terms, definitions and notations unexplained here regarding the nuclearity and sequence space we request to refer [10] and [14] in order to appreciate the subject matter of the discussions.

Throughout this article we assume $P_0 = \{(b_i^k) \colon k \geq 1\}$ to be a stable, countable nuclear power set of infinite type. For $k \geq 1$, we define the sequence space

$$\lambda(P_0;k) = \{x \in \omega\colon \sum_{i\geq 1} |x_i| b_i^k < \infty\}.$$

We say an l.c. TVS E is $\lambda(P_0;\mathbb{N})$-nuclear if it is $\lambda(P_0;k)$-nuclear for each $k \geq 1$. Equivalently, E is $\lambda(P_0;\mathbb{N})$-nuclear if and only if for each $k \geq 1$ and $u \in \mathcal{B}_E$, there exists $v \in \mathcal{B}_E$, $v < u$, with $\{b_i^k \delta_i(v,u)\} \in \ell^\infty$ (cf. [3], [6] and [15]). Well-known example of a $\lambda(P_0;\mathbb{N})$-nuclear space is provided by $\lambda(P_0)$ itself (cf. [12], [15]). At this stage let us recall from [15] (cf. [12]) that $\lambda(P_0)$ is not $\lambda(P_0)$-nuclear. This tells that there does exists a $\lambda(P_0;\mathbb{N})$-nuclear space which fails to be $\lambda(P_0)$-nuclear. The details concerning this aspect of investigations can be had from [3], [6], [12] and [15].

2. Criteria for $\lambda(P_0;\mathbb{N})$-nuclearity

Given a Köthe set P and a sequence space μ the generalized Köthe space (or the extended Köthe space) $\lambda_\mu(P)$ is defined by

$$\lambda_\mu(P) = \{x \in \omega\colon xa \in \mu,\ \forall a \in P\}.$$

We equip $\lambda_\mu(P)$ with its natural locally convex topology, generated by the family $\{p_{a,y}\colon a \in P, y \in \mu^x\}$ of semi-norms where

$$p_{a,y}(x) = p_y(xa) = \sum_{i\geq 1} |x_i y_i| a_i \quad (x \in \lambda_\mu(P)).$$

Clearly, for $\mu = \ell^1$, $\lambda_\mu(P)$ coincides with the Köthe space $\lambda(P)$ set theoretically as well as topologically.

The Grothendieck-Pietsch like criterion for the $\lambda(P_0;\mathbb{N})$-nuclearity of $\lambda_\mu(P)$ is provided by the following

Proposition 2.1. *$\lambda_\mu(P)$ is $\lambda(P_0;\mathbb{N})$-nuclear iff to each $j \geq 1$, $a \in P$ and $y \in \mu^x$, there correspond $b \in P$ and $z \in \mu^x$ such that the sequence $\{a_n y_n / b_n z_n\}$ can be re-arranged into a member of $\lambda(P_0;j)$.*

Proof. Assume that $\lambda_\mu(P)$ is $\lambda(P_0;\mathbb{N})$-nuclear and let $j \in \mathbb{N}$, $a \in P$ and $y \in \mu^x$. By [9, p. 32] there exists $k \in \mathbb{N}$ such that $\lambda(P_0;k)$-nuclearity implies $\lambda(P_0;j)$-type. By definition, there exist $b \in P$ and $z \in \mu^x$ such that the canonical map

$$\hat{K}_{(a,y)}^{(b,z)} : \hat{\lambda}_{(b,z)} \longrightarrow \hat{\lambda}_{(a,y)}$$

is $\lambda(P_0; j)$-type, where $\hat{\lambda}_{(a,y)}$ is the completion of the quotient space $\lambda_{a,y} = \lambda_\mu(P)/\ker p_{a,y}$. The mapping $\psi_{a,y} : \lambda_{a,y} \to \ell_{a,y}$ where $\psi_{a,y}(\hat{x}) = \{a_n x_n y_n\}$, $\hat{x} \in \lambda_{a,y}$, can be uniquely extended to an isometric isomorphism $\hat{\psi}_{a,y} : \hat{\lambda}_{(a,y)} \to \ell_{a,y}$. Here

$$\ell_a = \{a \in \ell^1 \colon x_n = 0, \ \forall\, n \text{ where } a_n = 0\}.$$

But

$$D^{(b,z)}_{(a,y)} = \hat{\psi}_{a,y} \circ \hat{K}^{(b,z)}_{(a,y)} \circ \hat{\psi}^{-1}_{b,z}$$

is a diagonal transformation determined by the sequence $\{a_n y_n / b_n z_n\}$. So $D^{(b,z)}_{(a,y)}$ is of $\lambda(P_0; j)$-type. Thus, by [11, p. 158], the decreasing rearrangement of $\{a_n y_n / b_n z_n\}$ belongs to $\lambda(P_0; j)$.

Conversely, if the given condition is satisfied, it follows that $\lambda_\mu(P)$ is nuclear such that the canonical maps are of $\lambda(P_0; j)$-type on Hilbert spaces for each given $j \in \mathbb{N}$. Then $\lambda(P_0; \mathbb{N})$-nuclearity of $\lambda_\mu(P)$ now follows by applying Lemma 3.5(i) of [12] to these canonical mappings. ∎

Remark 2.2. (i) For $\mu = \ell^1$, this reduces to the famous Grothendieck-Pietsch criterion for the $\lambda(P_0; \mathbb{N})$-nuclearity of the Köthe space $\lambda(P)$ (cf. [15, Proposition 2.2.1]).

(ii) For a $\lambda(P_0; \mathbb{N})$-nuclear space $(\mu, \eta(\mu, \mu^x))$, $\lambda_\mu(P)$ is $\lambda(P_0; \mathbb{N})$-nuclear.

Following Ruckle [13], we have a generalization of the traditional normal topology, namely, $\sigma\mu$-topology on a sequence space λ, corresponding to a sequence space μ; defined by the family $\{p_{y,z} \colon y \in \lambda^\mu, z \in \mu^x\}$ of semi-norms where

$$\lambda^\mu = \{y \in \omega \colon xy \in \mu, \ \forall\, x \in \lambda\}$$

and

$$p_{y,z}(x) = \sum_{i \geq 1} |x_i y_i z_i|, \quad (x \in \lambda).$$

Observe that this μ-dual λ^μ includes the well-known duals like α-dual (or cross dual), β-dual and γ-dual (cf. [13], [14]). We say that λ is μ-perfect if $\lambda = \lambda^{\mu\mu} = (\lambda^\mu)^\mu$ where

$$(\lambda^\mu)^\mu = \{z \in \omega \colon zy \in \mu, \ \forall\, y \in \lambda^\mu\}.$$

For $\mu = \lambda^1$, obviously this gives the perfectness of λ. Analogously, the $\sigma^*\mu$-topology on λ^μ is obtained by the collection $\{p_{y,z} \colon y \in \lambda, z \in \mu^x\}$ of semi-norms where

$$p_{y,z}(x) = \sum_{i \geq 1} |x_i y_i z_i|, \quad (x \in \lambda^\mu).$$

The details concerning the above topologies and μ-perfectness with its related aspects can be seen from [1], [2] and [7].

The Grothendieck-Pietsch like criterion for the $\lambda(P_0;\mathbb{N})$-nuclearity of $(\lambda, \sigma\mu)$ is contained in

Theorem 2.3. *Let λ be a μ-perfect sequence space for a perfect sequence space μ. Then λ is $\lambda(P_0;\mathbb{N})$-nuclear iff to each $j \geq 1$, $y \in \lambda^\mu$ and $z \in \mu^x$, there correspond $u \in \lambda^\mu$ and $v \in \mu^x$ such that the sequence $(y_n z_n/u_n v_n)$ can be rearranged into a sequence of $\lambda(P_0;j)$.*

Remark 2.4. (i) The above result yields the $\lambda(P_0;\mathbb{N})$-nuclearity of Köthe space $\lambda(P)$ when $\mu = \ell^1$ (cf. [12], [15]).

(ii) $(\lambda, \sigma\mu)$ is $\lambda(P_0;\mathbb{N})$-nuclear, for a $\lambda(P_0;\mathbb{N})$-nuclear space μ, no matter what sequence space is choosen for λ.

Likewise, one obtains

Proposition 2.5. *The μ-dual λ^μ is $\lambda(P_0;\mathbb{N})$-nuclear iff for each $j \geq 1$, $y \in \lambda$ and $z \in \mu^x$, there exist $u \in \lambda$ and $v \in \mu^x$ such that $\{y_n z_n/u_n v_n\}$ can be re-arranged into a sequence of $\lambda(P_0;j)$.*

Remark 2.6. (i) For $\mu = \ell^1$, the above gives us the criterion for the $\lambda(P_0;\mathbb{N})$-nuclearity of $(\lambda^x, \eta(\lambda^x, \lambda))$.

(ii) λ^μ is $\lambda(P_0;\mathbb{N})$-nuclear provided μ is $\lambda(P_0;\mathbb{N})$-nuclear (irrespective of the choice of λ).

In the final result of this section we assert that $\lambda(P_0;\mathbb{N})$-nuclearity of the generalized Köthe space $\lambda_\mu(P)$ is synonymous with the $\lambda(P_0;\mathbb{N})$-nuclearity of the Köthe space $\lambda(P)$, for a Hilbert space μ having a monotone normalized Schauder basis. Precisely, we have the

Theorem 2.7. *Let μ be a Hilbert K-space with a monotone normalized Schauder basis. Then $\lambda_\mu(P)$ is $\lambda(P_0;\mathbb{N})$-nuclear iff $\lambda(P)$ is $\lambda(P_0;\mathbb{N})$-nuclear.*

Proof. If $\lambda(P)$ is $\lambda(P_0;\mathbb{N})$-nuclear then, in view of Proposition 2.1, by [15, Proposition 2.2.1], $\lambda_\mu(P)$ will be always $\lambda(P_0;\mathbb{N})$-nuclear. So we prove the other part.

Let $\lambda_\mu(P)$ be $\lambda(P_0;\mathbb{N})$-nuclear. Suppose $j \in \mathbb{N}$ and $a \in P$ are choosen arbitrarily. By [9, p.32], there exist some $k \in \mathbb{N}$ such that $\lambda(P_0;k)$-nuclearity implies $\lambda(P_0;j)$-type. So $\hat{K}_a^b : \hat{\lambda}_\mu(P;b) \to \hat{\lambda}_\mu(P;a)$ is $\lambda(P_0;j)$-type. As before one can identify $\lambda_\mu(P;a) = \lambda_\mu(P)/\ker p_a$ with $\mu_a = \{x \in \mu\colon x_n = 0 \text{ for } n \text{ where } a_n = 0\}$ via the unique extension $\hat{\psi}_a(x) = \{a_n x_n\}$, $x \in \lambda_\mu(P)$. Then clearly $D_a^b = \hat{\psi}_a \circ \hat{K}_a^b \circ \hat{\psi}_b^{-1}$ is a diagonal map on μ, determined by $\{a_n/b_n\}$. But K_a^b is $\lambda(P_0;j)$-type and hence D_a^b will be of $\lambda(P_0;j)$-type. Then by modifying [8, Lemma 3.3] we can conclude that $\{a_n/b_n\}$ can be rearranged into a sequence of $\lambda(P_0;j)$, which is equivalent to the $\lambda(P_0;\mathbb{N})$-nuclearity of $\lambda(P)$ in view of [15, Proposition 2.2.1]. ∎

3. $\lambda(P_0;\mathbb{N})$-NUCLEARITY OF LOCALLY CONVEX SPACES WITH GENERALIZED BASES

We begin this section with the following

DEFINITION 3.1. Let E be a locally convex TVS and λ be a sequence space carrying the $\sigma\mu$-topology and λ^μ be equipped with $\sigma^*\mu$-topology. Then a Schauder basis $\{x_i, f_i\}$ for E is said to be a semi-λ-basis (resp., semi-λ^μ-basis) if, for each $p \in \mathcal{B}_E$, $\{f_i(x)p(x_i)\} \in \lambda$ (resp. $\{f_i(x)p(x_i) \in \lambda^\mu\}$) and it is called a fully-λ-basis (resp. fully-λ^μ-basis) provided for each $p \in \mathcal{B}_E$ the map $\psi_p : E \to \lambda$ (resp. $\psi_p : E \to \lambda^\mu$) is continuous where $\psi_p(x) = \{f_i(x)p(x_i)\}$.

The details regarding fully-λ-basis (resp. fully-λ^μ-basis) and its application can be had from [1] and [2].

The result to follow, establishes that a sequentially complete space with a fully-λ-basis can be topologically identified with a $\lambda(P_0;\mathbb{N})$-nuclear sequence space $(\lambda, \sigma\mu)$. Indeed, we have

THEOREM 3.2. *Let E be a sequentially complete space having a fully-λ-basis $\{x_i, f_i\}$. Let $y \in \lambda^\mu$ and $z \in \mu^x$ be such that $y_i \geq \epsilon > 0$ and $z_i \geq l > 0$, $\forall i$, for some epsilon and l. Then E is $\lambda(P_0;\mathbb{N})$-nuclear if $(\lambda, \sigma\mu)$ is $\lambda(P_0;\mathbb{N})$-nuclear.*

Proof. By [1, Theorem 3.1], E can be topologically identified with a Köthe space $\lambda(P_1)$ where

$$P_1 = \{p(x_i)a_i b_i \colon p \in \mathcal{B}_E,\ a \in \lambda_+^\mu,\ b \in \mu_+^x\}.$$

Thus, E is $\lambda(P_0;\mathbb{N})$-nuclear iff $\lambda(P_1)$ is $\lambda(P_0;\mathbb{N})$-nuclear. Since $(\lambda, \sigma\mu)$ is $\lambda(P_0;\mathbb{N})$-nuclear, in view of Theorem 2.3 to each $j \geq 1$, $a \in \lambda_+^\mu$ and $b \in \mu_+^x$ there correspond $c \in \lambda_+^\mu$, $d \in \mu_+^x$ and a permutation π such that

$$\left\{\frac{a_{\pi(i)} b_{\pi(i)}}{c_{\pi(i)} d_{\pi(i)}}\right\} \in \lambda(P_0;j).$$

Consequently, E is $\lambda(P_0;\mathbb{N})$-nuclear by the famous Grothendieck-Pietsch criteria (cf. [15]) because, for any $j \geq 1$, $p \in \mathcal{B}_E$, $a \in \lambda^\mu_+$ and $b \in \mu^x_+$, we have

$$\left\{\frac{p(x_{\pi(i)})a_{\pi(i)}b_{\pi(i)}}{p(x_{\pi(i)})c_{\pi(i)}d_{\pi(i)}}\right\} \in \lambda(P_0;j).$$ ∎

Note. For $\mu = \ell^1$, this yields that a sequentially complete space with a fully-λ-basis is $\lambda(P_0;\mathbb{N})$-nuclear, provided $(\lambda, \eta(\lambda,\lambda^x))$ is a $\lambda(P_0;\mathbb{N})$-nuclear space with k-property. So what we find easily is that a sequentially complete space with a fully-$\lambda(P)$-basis is $\lambda(P_0;\mathbb{N})$-nuclear provided $\lambda(P)$ is a $\lambda(P_0;\mathbb{N})$-nuclear G_∞-space. Hence a sequentially complete space with a fully-$\lambda(P_0)$-basis is $\lambda(P_0;\mathbb{N})$-nuclear (cf. [15]).

In view of Remark 2.4 (ii), we have the

COROLLARY 3.3. *Let E be a sequentially complete space with a fully-λ-basis. Suppose that there exist $y \in \lambda^\mu$ and $z \in \mu^x$ with $y_i \geq \epsilon > 0$ and $z_i \geq l > 0$, for all i, for some ϵ and l. If $(\mu, \eta(\mu,\mu^x))$ is $\lambda(P_0;\mathbb{N})$-nuclear then E is $\lambda(P_0;\mathbb{N})$-nuclear.*

A review of the analysis involved in the proof of Theorem 3.2, suggest that the following holds

PROPOSITION 3.4. *Let E be a sequentially complete space with a fully-λ-basis such that for some $a \in \lambda^\mu$ and $b \in \mu^x$ we have $a_i \geq \epsilon > 0$, $b_i \geq l > 0$, $\forall i \geq 1$, for some ϵ and l. Suppose that given $j \geq 1$, $y \in \lambda^\mu_+$ there exists $z \in \lambda^\mu_+$ such that $\{y_i/z_i\}$ can be rearranged into a sequence of $\lambda(P_0;j)$. Then E is $\lambda(P_0;\mathbb{N})$-nuclear.*

A cursory glance at the proof of Theorem 3.2 also reveals that the following is true

THEOREM 3.5. *Let E be a sequentially complete space having a fully-λ^μ-basis $\{x_i, f_i\}$ such that for some $a \in \lambda$ and $b \in \mu^x$, $a_i \geq \epsilon > 0$ and $b_i \geq l > 0$, for all i, for some ϵ and l. If μ is $\lambda(P_0;\mathbb{N})$-nuclear then E is $\lambda(P_0;\mathbb{N})$-nuclear.*

Proof. Invoking [1, Proposition 3.3], we can identify E topologically with a Köthe space $\lambda(P)$ where

$$P = \{p(x_i)a_ib_i \colon p \in \mathcal{B}_E, a \in \lambda_+, b \in \mu^x_+\}.$$

The rest of the proof is analogous to the proof of Theorem 3.2; of course, in this case we make use of Proposition 2.5. ∎

COROLLARY 3.6. *Let E be a sequentially complete space having a fully-λ^μ-basis such that for some $a \in \lambda$ and $b \in \mu^x$, $a_i \geq \epsilon > 0$, $b_i \geq l > 0$, for all i, for some ϵ and l. If μ is $\lambda(P_0; \mathbb{N})$-nuclear then E is $\lambda(P_0; \mathbb{N})$-nuclear.*

Proof. This follows from Theorem 3.5 in view of Remark 2.6 (ii). ∎

Analogous to Proposition 3.4 we have

PROPOSITION 3.7. *Let (E, T) be a sequentially complete space possessing a fully-λ^μ-basis where for some $a \in \lambda$, $b \in \mu^x$, $a_i \geq \epsilon > 0$ and $b_i \geq l > 0$, for all i and for some ϵ and l. Suppose that for each $j \geq 1$ and $y \in \lambda_+$ there corresponds $z \in \lambda_+$ such that $\{y_i/z_i\}$ can be rearranged into a sequence of $\lambda(P_0; j)$. Then E is $\lambda(P_0; \mathbb{N})$-nuclear.*

Note. For $\mu = \ell^1$, this says that a sequentially complete space with a fully-λ^x-basis is $\lambda(P_0; \mathbb{N})$-nuclear provided $\{\lambda^x, \eta(\lambda^x, \lambda)\}$ is $\lambda(P_0; \mathbb{N})$-nuclear and there is some $y \in \lambda$ with $y_i \geq \epsilon > 0$, $\forall i$, for some $\epsilon > 0$. So a sequentially complete space with a fully-$\Lambda_1(\alpha)^x$-basis is $\lambda(P_0; \mathbb{N})$-nuclear, if $\Lambda_1(\alpha)$ is $\lambda(P_0; \mathbb{N})$-nuclear.

The following results bear the testimony of the importance of the weak sequential completeness of the dual E^* in obtaining the $\lambda(P_0; \mathbb{N})$-nuclearity of E from the presence of a semi-λ-basis or a semi-λ^μ-basis.

THEOREM 3.8. *Suppose E is a sequentially complete space whose dual E^* is weakly sequentially complete. Let $\{x_i, f_i\}$ be an equicontinuous semi-λ-basis for E where λ is μ-perfect for a perfect sequence space μ such that for some $y \in \lambda^\mu$ and $z \in \mu^x$, $y_i \geq \epsilon > 0$ and $z_i \geq l > 0$, for all i, for some ϵ and l. If λ is $\lambda(P_0; \mathbb{N})$-nuclear, then E is $\lambda(P_0; \mathbb{N})$-nuclear.*

Proof. Since $\{x_i, f_i\}$ is a semi-λ-basis, for each $p \in \mathcal{B}_E$, $a \in \lambda^\mu$ and $b \in \mu^x$ we have

$$\sum |f_i(x)| p(x_i) |a_i b_i| < \infty. \tag{*}$$

Now one can identify E with the sequence space $\Delta = \{(f_i(x)) \colon x \in E\}$. Then modifying the proof of [5, Proposition 2.3], E^* can be identified with

$$\Delta^\beta = \{(\alpha_i) \colon \sum \alpha_i u_i \text{ converges for all } u \in \Delta\}$$

wherein the identification is given by

$$f \in E^* \longleftrightarrow \{f(x_i)\} \in \Delta^\beta.$$

Now $(*)$ means that $\{p(x_i)a_ib_i\} \in \Delta^\beta$. Thus, what we have proved is, for all $p \in \mathcal{B}_E$, $a \in \lambda^\mu$ and $b \in \mu^x$ there exists $f \in E^*$ with $f(x_i) = p(x_i)a_ib_i$. Due to the continuity of f we get some $q \in \mathbb{B}_E$ and $k > 0$ such that

$$p(x_i)|a_ib_i| \leq kq(x_i). \tag{+}$$

Since $(\lambda, \sigma\mu)$ is $\lambda(P_0; \mathbb{N})$-nuclear, in particular it is nuclear, so for each $a \in \lambda^\mu_+$ and $b \in \mu^x_+$, by [15, Proposition 1.1] there correspond $c \in \lambda^\mu_+$ and $d \in \mu^x_+$ with $\{a_ib_i/c_id_i\} \in l^1$. Consequently, by $(+)$ we get some $k > 0$ and $q \in \mathcal{B}_E$, with

$$p(x_i)|c_id_i| \leq kq(x_i), \quad \forall i.$$

Thus, we have the inequality

$$\sum_i |f_i(x)|p(x_i)|a_ib_i| \leq k \sup\{|f_i(x)|p(x_i)\} \cdot \sum \frac{a_ib_i}{c_id_i}.$$

From this inequality it follows that $\{x_i, f_i\}$ is a fully-λ-basis for E as the basis is equicontinuous and λ is μ-perfect. Now the desired conclusion follows by applying Theorem 3.2. ∎

Note. This above result tells us in particular that a sequentially complete space with an equicontinuous semi-λ-basis $\{x_i, f_i\}$ is $\lambda(P_0; \mathbb{N})$-nuclear provided E^* is weakly sequentially complete and $(\lambda, \eta(\lambda, \lambda^x))$ is a $\lambda(P_0; \mathbb{N})$-nuclear space with k-property. Hence, a sequentially complete space with an equicontinuous semi-$\lambda(\mathbb{R})$-basis is $\lambda(P_0; \mathbb{N})$-nuclear, provided E^* is weakly sequentially complete and $\lambda(\mathbb{R})$ is a $\lambda(P_0; \mathbb{N})$-nuclear G_∞-space. Thus a sequentially complete space with an equicontinuous semi-$\lambda(P_0)$-basis is $\lambda(P_0; \mathbb{N})$-nuclear provided E^* is weakly sequentially complete.

Since, for a $\lambda(P_0; \mathbb{N})$-nuclear space $(\mu, \eta(\mu, \mu^x))$, $(\lambda, \sigma\mu)$ is always $\lambda(P_0; \mathbb{N})$-nuclear, we obtain

Corollary 3.9. *Let E be a sequentially complete space whose dual E^* is weakly sequentially complete. Suppose $\{x_i, f_i\}$ is an equicontinuous semi-λ-basis for E where λ is μ-perfect for a perfect space μ such that for some $y \in \lambda^\mu$ and $z \in \mu^x$, $y_i \geq \epsilon > 0$, $z_i \geq l > 0$, for all i, for some ϵ and l. If $(\mu, \eta(\mu, \mu^x))$ is $\lambda(P_0; \mathbb{N})$-nuclear [or if for each $j \geq 1$, $y \in \lambda^\mu_+$ there exist $z \in \lambda^\mu_+$ and a permutation π with $\{y_{\pi(i)}/z_{\pi(i)}\} \in \lambda(P_0; j)$], then E is $\lambda(P_0; \mathbb{N})$-nuclear.*

An inspection of the proof of Theorem 3.8 suggest that the following is true

THEOREM 3.10. *Let E be a sequentially complete space with an equicontinuous semi-λ^{μ}-basis $\{x_i, f_i\}$ such that μ is perfect and for some $a \in \lambda$ and $b \in \mu^x$, $a_i \geq \epsilon > 0$ and $b_i \geq l > 0$, $\forall i$, for some ϵ and l. If λ^{μ} is $\lambda(P_0; \mathbb{N})$-nuclear then E is $\lambda(P_0; \mathbb{N})$-nuclear provided E^* is weakly sequentially complete.*

Proof. The proof follows, mutatis mutandis on lines similar to that of Theorem 3.8. ∎

Note. From the above result it is clear that a sequentially complete space with an equicontinuous semi-λ^x-basis is $\lambda(P_0; \mathbb{N})$-nuclear, provided $(\lambda^x, \eta(\lambda^x, \lambda))$ is $\lambda(P_0; \mathbb{N})$-nuclear and for some $y \in \lambda$, $y_i \geq \epsilon > 0$, $\forall i$, and E^* is weakly sequentially complete. Consequently, a sequentially complete space having an equicontinuous semi-$\Lambda_1^x(\alpha)$-basis is $\lambda(P_0; \mathbb{N})$-nuclear provided $\Lambda_1(\alpha)$ is $\lambda(P_0; \mathbb{N})$-nuclear (cf. [4]).

We know that λ^{μ} is always $\lambda(P_0; \mathbb{N})$-nuclear for a $\lambda(P_0; \mathbb{N})$-nuclear space μ. This in turn, implies that

COROLLARY 3.11. *Let E be a sequentially complete space with an equicontinuous semi-λ^{μ}-basis $\{x_i, f_i\}$ such that μ is perfect and for some $a \in \lambda$ and $b \in \mu^x$, $a_i \geq \epsilon > 0$ and $b_i \geq l > 0$, $\forall i$, for some ϵ and l. Suppose E^* is weakly sequentially complete and if μ is $\lambda(P_0; \mathbb{N})$-nuclear [or for each $j \geq 1$, $y \in \lambda$ there correspond $z \in \lambda$ and a permutation π such that $\{y_{\pi(i)/z_{\pi(i)}}\} \in \lambda(P_0; j)$], then E is $\lambda(P_0; \mathbb{N})$-nuclear.*

The present article ends with

PROPOSITION 3.12. *Let E be a sequentially complete space with a fully-λ^x-basis $\{x_i, f_i\}$. Suppose further that $\{x_i, f_i\}$ is also a fully-μ-basis or $\{e_i, e_i\}$ is a fully-μ-basis for λ^x, where μ is perfect. Then E is $\lambda(P_0; \mathbb{N})$-nuclear provided λ^{μ} is $\lambda(P_0; \mathbb{N})$-nuclear and for some $a \in \lambda$ and $b \in \mu^x$, $a_i \geq \epsilon > 0$ and $b_i \geq l > 0$, $\forall i$, for some ϵ and l.*

Proof. It follows from Theorem 3.5, as the basis turns out to be a fully-λ^{μ}-basis. ∎

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