On Nicely Smooth Banach Spaces

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1. Introduction

We work with real Banach spaces. We will denote by \( B(X) \), \( S(X) \) and \( B[x, r] \) respectively the closed unit ball, the unit sphere and the closed ball of radius \( r > 0 \) around \( x \in X \). We will identify any element \( x \in X \) with its canonical image in \( X^{**} \). All subspaces we usually consider are norm closed.

Definition 1.1. (a) We say \( A \subseteq B(X^*) \) is a norming set for \( X \) if \( \|x\| = \sup\{x^*(x) : x^* \in A\} \), for all \( x \in X \). A closed subspace \( F \subseteq X^* \) is a norming subspace if \( B(F) \) is a norming set for \( X \).

(b) A Banach space \( X \) is

(i) nicely smooth if \( X^* \) contains no proper norming subspace;

(ii) has the Ball Generated Property (BGP) if every closed bounded convex set in \( X \) is ball-generated, i.e., intersection of finite union of balls;

(iii) has Property (II) if every closed bounded convex set in \( X \) is the intersection of closed convex hulls of finite union of balls, or equivalently, w*-points of continuity (w*-PCs) of \( B(X^*) \) are norm dense in \( S(X^*) \) [5];

(iv) has the Mazur Intersection Property (MIP) (or, Property (I)) if every closed bounded convex set in \( X \) is intersection of balls, or equivalently, w*-denting points of \( B(X^*) \) are norm dense in \( S(X^*) \) [8].
Clearly, the MIP implies both Property (II) and the BGP. In this work, we obtain a sufficient condition for the BGP, and show that Property (II) implies the BGP, which, in turn, implies nice smoothness.

Notice that if a separable space is nicely smooth then it has separable dual. (The converse is false, since a dual space is nicely smooth if and only if it is reflexive.) It follows that if nice smoothness was inherited by subspaces, a nicely smooth space would necessarily be Asplund. However, recent work of Jiménez Sevilla and Moreno [15] shows that this is not true. More precisely, they showed that any Banach space can be isomorphically embedded in a Banach space with the MIP, a property stronger than nice smoothness.

In this work, we obtain some necessary and/or sufficient conditions for a space to be nicely smooth, and show that they are all equivalent for separable or Asplund spaces. These sharpen known results. We observe that every equivalent renorming of a space is nicely smooth if and only if it is reflexive. We also show that if $X$ is nicely smooth, $X \subseteq E \subseteq X^{**}$ and $E$ has the finite-infinite intersection property ($\infty$.f.IP) (Definition 2.15), then $E = X^{**}$. In particular, $X$ is nicely smooth with $\infty$.f.IP if and only if $X$ is reflexive.

Coming to stability results, we show that the class of nicely smooth spaces is stable under $c_0$ and $\ell_p$ sums ($1 < p < \infty$) and also under finite $\ell_1$ sums. We show that while nice smoothness is not a three space property, existence of a nicely smooth renorming is a three space property. We show that the Bochner $L_p$ spaces ($1 < p < \infty$) are nicely smooth if and only if $X$ is both nicely smooth and Asplund. And for a compact Hausdorff space $K$, $C(K, X)$ is nicely smooth if and only if $K$ is finite and $X$ is nicely smooth. We also study nice smoothness of certain operator and tensor product spaces.

2. Main Results

We record the following corollary of the proof of the Hahn-Banach Theorem (see e.g., [18, Section 48]) for future use.

**Lemma 2.1.** Let $X$ be a normed linear space and let $Y$ be a subspace. Suppose $x_0 \notin Y$ and $y^* \in S(Y^*)$. Then

$$\sup\{y^*(y) - \|x_0 - y\| : y \in Y\} \leq \inf\{y^*(y) + \|x_0 - y\| : y \in Y\}$$

and $\alpha$ lies between these two numbers if and only if there exists a Hahn-Banach (i.e., norm preserving) extension $x^*$ of $y^*$ with $x^*(x_0) = \alpha$. 

For a Banach space $X$, let us define

$$\mathcal{O}(X) = \{ x^{**} \in X^{**} : \|x^{**} + x\| \geq \|x\| \text{ for all } x \in X \}.$$ 

And we have the following characterization:

**Proposition 2.2.** For a Banach space $X$, the following are equivalent:

(a) $X$ is nicely smooth.

(b) For all $x^{**} \in X^{**}$,

$$\bigcap_{x \in X} B[x, \|x^{**} - x\|] = \{x^{**}\}.$$ 

(c) $\mathcal{O}(X) = \{0\}$.

(d) For all nonzero $x^{**} \in X^{**}$, there exists $x^* \in S(X^*)$ such that every Hahn-Banach extension of $x^*$ to $X^{**}$ takes nonzero value at $x^{**}$.

(e) Every norming set $A \subseteq B(X^*)$ separates points of $X^{**}$.

**Proof.** Equivalence of (a) and (b) is [12, Lemma 2.4].

(a) $\iff$ (c) [10, Lemma I.1] shows that $F$ is a norming subspace of $X^*$ if and only if $F^\perp \subseteq \mathcal{O}(X)$. Hence the result.

(b) $\iff$ (d) Each of the statements below is clearly equivalent to the next:

(i) $x^{**} \in \mathcal{O}(X)$,

(ii) $\|x\| \leq \|x^{**} - x\|$ for all $x \in X$,

(iii) for every $x^* \in S(X^*)$,

$$\sup\{x^*(x) - \|x^{**} - x\| : x \in X\} \leq 0 \leq \inf\{x^*(x) + \|x^{**} - x\| : x \in X\},$$

(iv) every $x^* \in S(X^*)$ has a Hahn-Banach extension $x^{***}$ with $x^{***}(x^{**}) = 0$ (Lemma 2.1).

Hence the result.

(a) $\iff$ (e) Since any norming set spans a norming subspace, this is clear.

We note a characterization of the BGP.
Theorem 2.3. A Banach space $X$ has the BGP if and only if for every $x^* \in X^*$ and $\varepsilon > 0$, there exists $w^*$-slices $S_1, S_2, \ldots, S_n$ of $B(X^*)$ such that for any $(x^*_1, x^*_2, \ldots, x^*_n) \in \prod_{i=1}^n S_i$, there are scalars $a_1, a_2, \ldots, a_n$ such that $\|x^* - \sum_{i=1}^n a_i x^*_i\| \leq \varepsilon$.

Proof. Observe that $X$ has the BGP if and only if every $x^* \in X^*$ is ball-continuous on $B(X)$ [11, Theorem 8.3]. Now the result follows from the characterization of such functionals obtained in [5, Theorem 2].

This leads to the following, more tractable sufficient condition for the BGP.

Definition 2.4. A point $x^*_0$ in a convex set $K \subseteq X^*$ is called a $w^*$-small combination of slices ($w^*$-SCS) point of $K$, if for every $\varepsilon > 0$, there exist $w^*$-slices $S_1, S_2, \ldots, S_n$ of $K$, and a convex combination $S = \sum_{i=1}^n \lambda_i S_i$ such that $x^*_0 \in S$ and $\text{diam}(S) < \varepsilon$.

Proposition 2.5. If $X^*$ is the closed linear span of the $w^*$-SCS points of $B(X^*)$, then $X$ has the BGP.

Proof. Let $x^* \in X^*$ and $\varepsilon > 0$. Since the set of $w^*$-SCS points of $B(X^*)$ is symmetric and spans $X^*$, there exist $w^*$-SCS points $x^*_1, x^*_2, \ldots, x^*_n$ of $B(X^*)$, and positive scalars $a_1, a_2, \ldots, a_n$ such that $\|x^* - \sum_{i=1}^n a_i x^*_i\| \leq \varepsilon/2$. By definition of $w^*$-SCS points, for each $i = 1, 2, \ldots, n$, there exist $w^*$-slices $S_1, S_2, \ldots, S_{n_i}$ of $B(X^*)$, and a convex combination $S_i = \sum_{k=1}^{n_i} \lambda_{ik} S_{ik}$ such that $x^*_i \in S_i$ and $\text{diam}(S_i) < \varepsilon/(2 \sum_{i=1}^n a_i)$. Now, for any $(x^*_{ik}) \in \prod_{i=1}^n \prod_{k=1}^{n_i} S_{ik}$,

\[
\left\| x^* - \sum_{i=1}^n \sum_{k=1}^{n_i} a_i \lambda_{ik} x^*_{ik} \right\| \leq \left\| x^* - \sum_{i=1}^n a_i x^*_i \right\| + \sum_{i=1}^n a_i \left\| x^*_i - \sum_{k=1}^{n_i} \lambda_{ik} x^*_{ik} \right\|
\leq \varepsilon/2 + \sum_{i=1}^n a_i \text{diam}(S_i) \leq \varepsilon.
\]

Hence by Theorem 2.3, $X$ has the BGP.

Corollary 2.6. Property (II) implies the BGP, which, in turn, implies nicely smooth.

Proof. Since $X$ has Property (II), $w^*$-PCs of $B(X^*)$ are norm dense in $S(X^*)$, and a $w^*$-PC is necessarily a $w^*$-SCS point (this follows from Bourgain’s Lemma, see e.g., [17, Lemma 1.5]). Thus, Property (II) implies the BGP.
That the BGP implies nicely smooth is proved in [11, Theorem 8.3]. But here is an elementary proof.

Let $F$ be a norming subspace of $X^*$. Then $B(X)$ is $\sigma(X, F)$-closed, so that every ball-generated set is also $\sigma(X, F)$-closed. But if every closed bounded convex set is $\sigma(X, F)$-closed, then $F = X^*$. ■

We now obtain a localization of [5, Theorem 3.1] and [9, Lemma 6].

**Proposition 2.7.** Let $X$ be a Banach space. Let $x_0^* \in S(X^*)$ and $x_0^{**} \in X^{**}$. The following are equivalent:

(a) every Hahn-Banach extension of $x_0^*$ to $X^{**}$ takes the same value at $x_0^{**}$;

(b) $x_0^{**}$, considered as a function on $B(X^*, w^*)$, is continuous at $x_0^*$;

(c) $\sup \{x_0^*(x) - \|x_0^{**} - x\| : x \in X\} = \inf \{x_0^*(x) + \|x_0^{**} - x\| : x \in X\}$;

(d) for any $\alpha \in \mathbb{R}$, if $x_0^{**}(x_0^*) \neq \alpha$, then there exists a ball $B^{**}$ in $X^{**}$ with centre in $X$ such that $x_0^{**} \in B^{**}$ and $B^{**}$ and $x_0^*$ lies in the same open half space determined by $x_0^*$ and $\alpha$.

**Proof.** (a) $\Leftrightarrow$ (b) This is a natural localization of the proof of [13, Lemma III.2.14]. We omit the details.

(a) $\Leftrightarrow$ (c) Follows from Lemma 2.1.

(c) $\Rightarrow$ (d) Suppose $x_0^{**}(x_0^*) > \alpha$. By Lemma 2.1, $\inf \{x_0^*(x) + \|x_0^{**} - x\| : x \in X\} > \alpha$. By (c), it follows that $\sup \{x_0^*(x) - \|x_0^{**} - x\| : x \in X\} > \alpha$. So, there exists $x \in X$ such that $x_0^*(x) - \|x_0^{**} - x\| > \alpha$. Put $B^{**} = B^{**}[x, \|x_0^{**} - x\|]$. Clearly, $x_0^{**} \in B^{**}$, and $\inf x_0^*(B^{**}) = x_0^*(x) - \|x_0^{**} - x\| > \alpha$. Similarly for $x_0^{**}(x_0^*) < \alpha$.

(d) $\Rightarrow$ (c) Suppose $x_0^{**}(x_0^*) > \alpha$. By (d), there exists a ball $B^{**} = B^{**}[x, r]$ in $X^{**}$ such that $x_0^{**} \in B^{**}$ and $\inf x_0^*(B^{**}) > \alpha$. This implies $\|x_0^{**} - x\| \leq r$ and $\inf x_0^*(B^{**}) = x_0^*(x) - r > \alpha$. It follows that $x_0^*(x) - \|x_0^{**} - x\| > \alpha$, whence $\sup \{x_0^*(x) - \|x_0^{**} - x\| : x \in X\} > \alpha$. Since $\alpha$ was arbitrary, it follows that $\sup \{x_0^*(x) - \|x_0^{**} - x\| : x \in X\} \leq x_0^{**}(x_0^*)$. Similarly, $\inf x_0^*(x) + \|x_0^{**} - x\| : x \in X\} \leq x_0^{**}(x_0^*)$. The result now follows from Lemma 2.1. ■

**Corollary 2.8.** [5, Theorem 3.1] For a Banach space $X$ and $f_0 \in S_{X^*}$, the following are equivalent:

(i) $f_0$ is a $w^*$-$w$ PC of $B_{X^*}$;

(ii) $f_0$ has a unique Hahn-Banach extension in $X^{**}$;
(iii) for any $x_0^{**} \in X^{**}$ and $\alpha \in \mathbb{R}$, if $f_0(x_0^{**}) \neq \alpha$, then there exists a ball $B^{**}$ in $X^{**}$ with centre in $X$ such that $x_0^{**} \in B^{**}$ and $B^{**}$ and $x_0^{**}$ lies in the same open half space determined by $f_0$ and $\alpha$.

We now identify some necessary and some sufficient conditions for a space to be nicely smooth.

**Definition 2.9.** For $x \in S(X)$, let $D(x) = \{ f \in S(X^*) : f(x) = 1 \}$. The set valued map $D$ is called the duality map and any selection of $D$ is called a support mapping.

**Theorem 2.10.** For a Banach space $X$, consider the following statements:

(a) $X^*$ is the closed linear span of the w*-weak PCs of $B(X^*)$.

(b) Any two distinct points in $X^{**}$ are separated by disjoint closed balls having centre in $X$.

(b$_1$) For every $x^{**} \in X^{**}$, the points of w*-continuity of $x^{**}$ in $S(X^*)$ separates points of $X^{**}$.

(b$_2$) For every nonzero $x^{**} \in X^{**}$, there is a point of w*-continuity $x^* \in S(X^*)$ of $x^{**}$ such that $x^{**}(x^*) \neq 0$.

(c) $X$ is nicely smooth.

(d) For every norm dense set $A \subseteq S(X)$ and every support mapping $\phi$, the set $\phi(A)$ separates points of $X^{**}$.

Then (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) and (a) $\Rightarrow$ (b$_1$) $\Rightarrow$ (b$_2$) $\Rightarrow$ (c) $\Rightarrow$ (d). Moreover, if $X$ is an Asplund space (or, separable), all the above conditions are equivalent, and equivalent to each of the following:

(e) $X^*$ is the closed linear span of the w*-strongly exposed points of $B(X^*)$.

(f) $X^*$ is the closed linear span of the w*-denting points of $B(X^*)$.

(g) $X^*$ is the closed linear span of the w*-SCS points of $B(X^*)$.

(h) $X$ has the BGP.

**Proof.** (a) $\Rightarrow$ (b) Let $x_0^{**} \neq y_0^{**}$. By (a), there exists a w*-w PC $x_0^* \in B_{X^*}$, such that $(x_0^{**} - y_0^{**})(x_0^*) > 0$. Let $\alpha \in \mathbb{R}$ be such that

$$x_0^{**}(x_0^*) > \alpha > y_0^{**}(x_0^*).$$
Now applying Corollary 2.8, it follows that there exists a ball $B^*_1$ with centre in $X$ with $x_0^* \in B^*_1$ and $\inf x_0^*(B^*_1) > \alpha$. And there exists a ball $B^*_2$ with centre in $X$ such that $y_0^* \in B^*_2$ and $\sup y_0^*(B^*_2) < \alpha$. Clearly, $B^*_1 \cap B^*_2 = \emptyset$.

**(b) ⇒ (c)** Clearly, (b) implies condition (b) of Proposition 2.2.

**(a) ⇒ (b) ⇒ (b)** follows from definitions.

**(b) ⇒ (c)** By (b), for every nonzero $x^* \in X^*$, there is a point of w*-continuity $x^* \in S(X^*)$ of $x^*$ such that $x^*(x^*) \neq 0$. By Proposition 2.7, every Hahn-Banach extension of $x^*$ to $X^*$ takes the same value at $x^*$. The result now follows from Proposition 2.2(d).

**(c) ⇒ (d)** We simply observe that $\phi(A)$ is a norming set for $X$.

Clearly, even without $X$ being Asplund, $(e) \Rightarrow (f) \Rightarrow (g) \Rightarrow (h) \Rightarrow (c)$, and $(f) \Rightarrow (a)$. Now if $X$ is Asplund (if $X$ is separable, (d) implies $X^*$ is separable), then for $A = \{x \in S(X) : \text{the norm is Fréchet differentiable at } x\}$, and any support mapping $\phi$, $\phi(A) = \{w^*-strongly exposed points of $B(X^*)\}$. Hence, $(d) \Rightarrow (e)$.

**Remark 2.11.** (a) If in the first part, we simply assume that the w*-weak PCs of $B(X^*)$ form a norming set, then (a) – (c) are equivalent. And under the stronger assumption that the set

$$\{x \in S(X) : D(x) \text{ intersects the } w^*-\text{weak PCs of } B(X^*)\}$$

is dense in $S(X)$, (a) – (d) are equivalent. This happens if $X$ is Asplund.

Can any of the implications be reversed in general?

**(b)** In [9, Lemma 5], Godefroy proves $(a) \Rightarrow (c)$ by actually showing $(b_1) \Rightarrow (c)$. (b) is a weaker sufficient condition.

**(c)** In [4, Theorem 7], it is observed that $(f) \Rightarrow (h)$. (g) is a weaker sufficient condition.

**Proposition 2.12.** If every separable subspace of $X$ is nicely smooth, then $X$ has the BGP, and hence, is nicely smooth.

**Proof.** Since for separable spaces nice smoothness is equivalent to the BGP, the result follows from the characterization of ball-continuous functionals obtained in [11, Theorem 2.4 and 2.5].

**Theorem 2.13.** A Banach space $X$ is reflexive if and only if every equivalent renorming is nicely smooth.
Proof. The converse being trivial, suppose $X$ is not reflexive. Let $x^{**} \in X^{**} \setminus X$ and let $F = \{x^* \in X^* : x^{**}(x^*) = 0\}$. Define a new norm on $X$ by

$$
\|x\|_1 = \sup \{x^*(x) : x^* \in B(F)\} \quad \text{for } x \in X.
$$

It follows from the proof of [11, Theorem 8.2] that $\|\cdot\|_1$ is an equivalent norm on $X$ with $F$ as a proper norming subspace. 

Remark 2.14. (a) In [14] the authors showed that $X$ is reflexive if and only if for any equivalent norm, $X$ is Hahn-Banach smooth and has ANP-III. This was strengthened in [3, Corollary 2.5] to just Hahn-Banach smooth. The above is an even stronger result with even easier proof.

(b) In [11, Theorem 8.1 and 8.2], it is shown that a subset $W \subseteq X$ is ball-generated for every equivalent renorming if and only if it is weakly compact. The global analogue of this local result would read: every equivalent renorming of $X$ has the BGP if and only if $X$ is reflexive. Thus in the global version, our result is somewhat stronger, except for separable spaces, though, as the proof shows, it is implicit in [11, Theorem 8.2].

Definition 2.15. A Banach space $X$ is said to have the finite-infinite intersection property (\(\infty.f.IP\)) if every family of closed balls in $X$ with empty intersection contains a finite subfamily with empty intersection. It is well known that all dual spaces and their 1-complemented subspaces have \(\infty.f.IP\).

Theorem 2.16. $X$ is nicely smooth with \(\infty.f.IP\) if and only if $X$ is reflexive.

Proof. Sufficiency is obvious.

For necessity, recall from [11, Theorem 2.8] that $X$ has \(\infty.f.IP\) if and only if $X^{**} = X + O(X)$. Since $X$ is nicely smooth, $O(X) = \{0\}$ and consequently, $X$ is reflexive.

Remark 2.17. Since Hahn-Banach smooth spaces (respectively, spaces with Property (II)) are nicely smooth, [3, Theorem 2.11] (respectively, [3, Theorem 3.4]) follows as an immediate corollary with a simpler proof. Indeed, we have stronger results.
Theorem 2.18. If $X$ is nicely smooth, $X \subseteq E \subseteq X^{**}$ and $E$ has the $\infty.f.IP$, then $E = X^{**}$.

Proof. Using the Principle of Local Reflexivity and w*-compactness of dual balls, it is easy to see that $E$ has $\infty.f.IP$ if and only if any family of closed balls with centres in $E$ that intersects in $E^{**}$ also intersects in $E$.

Now if $X \subseteq E \subset X^{**}$, let $x_0^* \in X^{**} \setminus E$. Consider the family

$$B = \{B[x, \|x_0^* - x\|] : x \in X\}.$$  

Clearly, they intersect at $x_0^* \in X^{**} \subseteq E^{**}$. And since $X$ is nicely smooth, by Proposition 2.2 (b), the intersection in $X^{**}$ is singleton $\{x_0^*\}$. But then, this family cannot intersect in $E$.

Remark 2.19. Can one strengthen this to show that if $X$ is nicely smooth, $E$ has $\infty.f.IP$ and $X \subseteq E$, then $X^{**} \subseteq E$? Recall that $c_0$ is nicely smooth, and it is known [16] that if $c_0 \subseteq E$ and $E$ is 1-complemented in a dual space, then $\ell_{\infty}$ is isomorphic to a subspace of $E$. Our question is isometric.

In the special case of $X = c_0$, we can prove:

Proposition 2.20. If $c_0 \subseteq E$ and $E$ has $\infty.f.IP$, then $\ell_{\infty}$ is a quotient of $E$.

Proof. Since $c_0 \subseteq E$, $\ell_{\infty} \subseteq E^{**}$. Let $F = E \cap \ell_{\infty}$. Since $\ell_{\infty}$ is “injective”, let $T : E \rightarrow \ell_{\infty}$ be the norm preserving extension of the inclusion map from $F$ to $\ell_{\infty}$. Note that $T$ is identity on $c_0$. We will prove that $T$ is onto.

Let $x_0^{**} \in \ell_{\infty}$. Let

$$B = \{B[x, \|x_0^{**} - x\|] : x \in c_0\}.$$  

As before, this family intersects in $E^{**}$, and hence in $E$. And since $\|T\| = 1$, if $e \in E$ belongs to this intersection, so does $Te$. But the intersection in $\ell_{\infty}$ is singleton $\{x_0^{**}\}$, since $c_0$ is nicely smooth. Thus, $Te = x_0^{**}$.

And in general, we have the result under stronger assumptions on both $X$ and $E$:

Proposition 2.21. If $X$ has Property (II), $E$ is a dual space and $X \subseteq E$, then $X^{**} \subseteq E$.
Proof. Let $E = Y^*$. Let $i : X \to Y^*$ be the inclusion map. Then $i^* : Y^{**} \to X^*$ is onto. We will show that $i^*|_Y : Y \to X^*$ is onto. It will follow that $X^*$ is a quotient of $Y$ and hence, $X^{**}$ is a subspace of $Y^* = E$.

To show that $i^*|_Y : Y \to X^*$ is onto, it suffices to check that $i^*(B(Y)) = B(X^*)$. Since $X$ has Property (II), it suffices to check that any w*-PC $x^* \in B(X^*)$ belongs to $i^*(B(Y))$. Let $\Lambda \in Y^{**}$ be a norm preserving extension of $x^*$ to $Y^*$. Let $\{y_\alpha\} \subseteq B(Y)$ such that $y_\alpha \to \Lambda$ in the w*-topology of $B(Y^{**})$. Then $y_\alpha|_X \to x^*$ in the w*-topology of $B(X^*)$. Since $x^*$ is a w*-PC, $y_\alpha|_X \to x^*$ in norm, i.e., $x^* \in i^*(B(Y))$. 

3. Stability Results

**Theorem 3.1.** Let $\{X_\alpha\}_{\alpha \in \Gamma}$ be a family of Banach spaces. Then $X = \bigoplus_{\ell_p} X_\alpha \ (1 < p < \infty)$ is nicely smooth if and only if for each $\alpha \in \Gamma$, $X_\alpha$ is nicely smooth.

**Proof.** We show that $\mathcal{O}(X) = \{0\}$ if and only if for every $\alpha \in \Gamma$, $\mathcal{O}(X_\alpha) = \{0\}$.

Now, $X = \bigoplus_{\ell_p} X_\alpha$ implies $X^{**} = \bigoplus_{\ell_p} X_\alpha^{**}$, and $x^{**} \in \mathcal{O}(X)$ if and only if

\[
\|x^{**} + x\|_p \geq \|x\|_p \text{ for all } x \in X
\]

\[
\iff \sum_{\alpha \in \Gamma} \|x^{**}_{\alpha} + x_{\alpha}\|_p \geq \sum_{\alpha \in \Gamma} \|x_{\alpha}\|_p \text{ for all } x \in X.
\]

It is immediate that if for every $\alpha \in \Gamma$, $x^{**}_{\alpha} \in \mathcal{O}(X_\alpha)$, then $x^{**} \in \mathcal{O}(X)$. And hence, $\mathcal{O}(X) = \{0\}$ implies for every $\alpha \in \Gamma$, $\mathcal{O}(X_\alpha) = \{0\}$.

Conversely, suppose for every $\alpha \in \Gamma$, $\mathcal{O}(X_\alpha) = \{0\}$. Let $x^{**} \in X^{**} \setminus \{0\}$. Let $\alpha_0 \in \Gamma$ be such that $x^{**}_{\alpha_0} \neq 0$. Then $x^{**}_{\alpha_0} \notin \mathcal{O}(X_{\alpha_0})$. Hence, there exists $x_{\alpha_0} \in X_{\alpha_0}$ such that $\|x^{**}_{\alpha_0} + x_{\alpha_0}\| < \|x_{\alpha_0}\|$. Choose $\varepsilon > 0$ such that $\|x^{**}_{\alpha_0} + x_{\alpha_0}\| + \varepsilon < \|x_{\alpha_0}\|$. Then there exists a finite $\Gamma_0 \subseteq \{\alpha \in \Gamma : x^{**}_{\alpha} \neq 0\}$ such that $\alpha_0 \in \Gamma_0$ and $\sum_{\alpha \notin \Gamma_0} \|x^{**}_{\alpha}\| < \varepsilon$. If $\alpha \in \Gamma_0$, then $x^{**}_{\alpha} \notin \mathcal{O}(X_\alpha)$. Hence, there exists $x_{\alpha} \in X_{\alpha}$ such that $\|x^{**}_{\alpha} + x_{\alpha}\| < \|x_{\alpha}\|$. Define $y \in X$ by

\[
y_{\alpha} = \begin{cases} x_{\alpha} & \text{if } \alpha \in \Gamma_0 \\ 0 & \text{otherwise}. \end{cases}
\]
Then we have,
\[
\|x^{**} + y\|_p^p = \sum_{\alpha \in \Gamma} \|x_{\alpha}^{**} + y_{\alpha}\|_p^p \\
= \sum_{\alpha \in \Gamma_0} \|x_{\alpha}^{**} + x_{\alpha_0}\|_p^p + \sum_{\alpha \in \Gamma_0} \|x_{\alpha}^{**}\|_p^p \\
< \sum_{\alpha \in \Gamma_0} \|x_{\alpha}\|_p^p + \|x_{\alpha_0}^{**} + x_{\alpha_0}\|_p^p + \varepsilon \\
< \sum_{\alpha \in \Gamma_0} \|x_{\alpha}\|_p^p = \|y\|_p^p
\]
which shows that \(x^{**} \notin \mathcal{O}(X)\). 

Remark 3.2. The above argument also works for finite \(\ell_1\) (or \(\ell_\infty\)) sums and shows that if \(X\) is the \(\ell_1\) (or \(\ell_\infty\)) sum of \(X_1, X_2, \ldots, X_n\), then \(X\) is nicely smooth if and only if for every coordinate space \(X_i\) is so. However, if \(\Gamma\) is infinite, \(X = \bigoplus \ell_1 X_\alpha\) is never nicely smooth as \(\bigoplus c_0 X_\alpha^*\) is a proper norming subspace of \(X^* = \bigoplus \ell_\infty X_\alpha^*\).

A similar argument also shows that nice smoothness is not stable under infinite \(\ell_\infty\) sums.

We now show that nice smoothness is stable under \(c_0\) sums.

**Theorem 3.3.** Let \(\{X_\alpha\}_{\alpha \in \Gamma}\) be a family of Banach spaces. Then \(X = \bigoplus c_0 X_\alpha\) is nicely smooth if and only if for each \(\alpha \in \Gamma\), \(X_\alpha\) is nicely smooth.

**Proof.** As before, we will show that \(\mathcal{O}(X) = \{0\}\) if and only if for every \(\alpha \in \Gamma\), \(\mathcal{O}(X_\alpha) = \{0\}\).

Necessity is similar to that in Theorem 3.1.

Conversely, suppose for every \(\alpha \in \Gamma\), \(\mathcal{O}(X_\alpha) = \{0\}\). And let \(x^{**} \in X^{**} \setminus \{0\}\). Let \(\alpha_0 \in \Gamma\) be such that \(x_{\alpha_0}^{**} \neq 0\). Then \(x_{\alpha_0}^{**} \notin \mathcal{O}(X_{\alpha_0})\). Hence, there exists \(x_{\alpha_0} \in X_{\alpha_0}\) such that \(\|x_{\alpha_0}^{**} + x_{\alpha_0}\| < \|x_{\alpha_0}\|\). The triangle inequality shows that for any \(\lambda \geq 1\), \(\|x_{\alpha_0}^{**} + \lambda x_{\alpha_0}\| < \|\lambda x_{\alpha_0}\|\). Thus, replacing \(x_{\alpha_0}\) by \(\lambda x_{\alpha_0}\) for some \(\lambda \geq 1\), if necessary, we may assume \(\|x_{\alpha_0}\| > \|x^{**}\|_\infty\).

Define \(y \in X\) by
\[
y_\alpha = \begin{cases} 
  x_{\alpha_0} & \text{if } \alpha = \alpha_0 \\
  0 & \text{otherwise}
\end{cases}
\]
Then, 
\[ \|x^{**} + y\|_\infty = \max\{\sup\{\|x^{**}_\alpha\|_{\alpha \neq \alpha_0}\}, \|x^{**}_{\alpha_0} + x_{\alpha_0}\}\} < \|x_{\alpha_0}\| = \|y\|_\infty \]
whence \( x^{**} \notin \mathcal{O}(X) \). □

**Corollary 3.4.** Nice smoothness is not a three space property.

**Proof.** Let \( X = c \), the space of all convergent sequences with the sup norm. Recall that \( c^* = \ell_1 \) and that \( \ell_1 \) acts on \( c \) as

\[ \langle a, x \rangle = a_0 \lim x_n + \sum_{n=0}^{\infty} a_{n+1} x_n, \quad a = \{a_n\}_{n=0}^{\infty} \in \ell_1, \quad x = \{x_n\}_{n=0}^{\infty} \in c. \]

It follows that \( \{a \in \ell_1 : a_0 = 0\} \) is a proper norming subspace for \( c \).

Put \( Y = c_0 \). Then, by Theorem 3.3, \( Y \) is nicely smooth and \( \dim(X/Y) = 1 \), so that \( X/Y \) is also nicely smooth. But, by above, \( X \) is not nicely smooth. □

**Remark 3.5.** Since nice smoothness is not an isomorphic property, perhaps a more pertinent question here would be whether having a nicely smooth renorming is a three space property. The answer in this case is yes.

**Theorem 3.6.** Let \( X \) be a Banach space. Let \( Y \) be a subspace of \( X \). If both \( Y \) and \( X/Y \) have a nicely smooth renorming, then so does \( X \).

**Proof.** Observe that a Banach space \( X \) has a nicely smooth renorming if and only if for any proper subspace \( M \subseteq X^* \), the norm interior of the w*-closure of its unit ball is empty.

Now suppose \( Y \) and \( X/Y \) both have a nicely smooth renorming. Suppose there exists a subspace \( M \subseteq X^* \) such that \( \overline{B(M)}^{w^*} \) has nonempty interior, i.e., there exists \( m \in M \) and \( \varepsilon > 0 \) such that \( B(m, \varepsilon) \subseteq \overline{B(M)}^{w^*} \). We will show that \( M \) is not proper.

Consider the inclusion map \( i : Y \to X \). Then \( i^* : X^* \to Y^* \) is the natural quotient map and is w*-continuous. It follows that \( i^*(B(m, \varepsilon)) \subseteq \overline{B(i^*(M))}^{w^*} \). Therefore, \( i^*(M) \subseteq Y^* \) is a subspace the w*-closure of whose unit ball has nonempty interior. Because of our assumption on \( Y \), we have \( i^*(M) = Y^* \), and hence, \( X^* = M + Y^\perp \).

Put \( N = M \cap Y^\perp \). The natural isomorphism between \( X^*/M \) and \( Y^\perp/N \) shows that the relative norm interior of \( \overline{B(N)}^{w^*} \) is also nonempty. Since
$Y^\perp = (X/Y)^*$, our assumption on $X/Y$ implies that $M \cap Y^\perp = Y^\perp$, i.e., $M$ contains $Y^\perp$.

Thus, $X^* = M + Y^\perp = M$.  

**Remark 3.7.** Since finite $\ell_1$ sums of infinite dimensional Banach spaces fail Property (II) [3, Proposition 3.7], the space $c_0 \oplus \ell_1 c_0$ produces an example of a nicely smooth space, which being Asplund, also has the BGP, but lacks Property (II).

Recall that a closed subspace $M \subseteq X$ is said to be an $M$-summand if there is a projection $P$ on $X$ with range $M$ such that $\|x\| = \max\{\|Px\|, \|x - Px\|\}$ for all $x \in X$.

**Proposition 3.8.** The BGP is inherited by $M$-summands.

**Proof.** We follow the arguments of [1, Proposition 2].

Let $Y$ be an $M$-summand in $X$ with $P$ the corresponding projection. Let $K$ be a closed bounded convex set in $Y$. Since $X$ has the BGP,

$$K = \bigcap_{i \in I} \bigcup_{k=1}^{n_i} B[x_{ik}, r_{ik}],$$

where for each $i$ and $k$, $K \cap B[x_{ik}, r_{ik}] \neq \emptyset$.

Given $i$ and $k$, let $x \in K \cap B[x_{ik}, r_{ik}] \subseteq Y$, then $\|x - x_{ik}\| \leq r_{ik}$, so that $\|x_{ik} - Px_{ik}\| = \|(x - x_{ik}) - P(x - x_{ik})\| \leq \|x - x_{ik}\| \leq r_{ik}$.

**Claim:** $K = \bigcap_{i \in I} \bigcup_{k=1}^{n_i} B[Y_{x_{ik}}, r_{ik}]$.  

Since $\|P\| = 1$, we have

$$K = P(K) \subseteq P(\bigcap_{i \in I} \bigcup_{k=1}^{n_i} B[x_{ik}, r_{ik}]) \subseteq \bigcap_{i \in I} \bigcup_{k=1}^{n_i} B[Y_{x_{ik}}, r_{ik}].$$

Conversely, if $x$ is in the RHS of $(*)$, for all $i \in I$, there exists $k$ such that $\|x - x_{ik}\| = \max\{\|x - Px_{ik}\|, \|x_{ik} - Px_{ik}\|\} \leq r_{ik}$, as $\|x_{ik} - Px_{ik}\| \leq r_{ik}$.

Thus, $x \in \bigcap_{i \in I} \bigcup_{k=1}^{n_i} B[x_{ik}, r_{ik}] = K$.  

**Theorem 3.9.** Let $X$ be a Banach space, $\mu$ denote the Lebesgue measure on $[0,1]$ and $1 < p < \infty$. The following are equivalent:
(a) $L_p(\mu, X)$ has BGP.
(b) $L_p(\mu, X)$ is nicely smooth.
(c) $X$ is nicely smooth and Asplund.

**Proof.** Clearly (a) $\Rightarrow$ (b).

(b) $\Rightarrow$ (c) Since $L^q(\mu, X^*)$ is always a norming subspace of $L^p(\mu, X)^*$, $\frac{1}{p} + \frac{1}{q} = 1$, and they coincide if and only if $X^*$ has the RNP with respect to $\mu$ [7, Chapter IV], (b) implies $X^*$ has the RNP, or, $X$ is Asplund. Also for any norming subspace $F \subseteq X^*$, $L^q(\mu, F)$ is a norming subspace of $L^p(\mu, X)^*$.

Hence, (b) also implies $X$ is nicely smooth.

(c) $\Rightarrow$ (a) If $X$ is nicely smooth and Asplund, by Theorem 2.10, $X^*$ is the closed linear span of the w*-denting points of $B(X^*)$. And it suffices to show that $L_p(\mu, X)^* = L^q(\mu, X^*)$ is the closed linear span of the w*-denting points of $B(L^q(\mu, X^*))$.

Let $F = \sum_{i=1}^{n} \alpha_i x_i^* \chi_{A_i}$ with $x_i^* \in S(X^*)$ for all $i = 1, 2, \ldots, n$ be a simple function in $S(L^q(\mu, X^*))$. Let $\varepsilon > 0$. Now, for each $i = 1, 2, \ldots, n$, there exists $\lambda_{ik} \in \mathbb{R}$, and $x_{ik}^*$, w*-denting points of $B(X^*)$, $k = 1, 2, \ldots, N$, such that $\|x_i^* - \sum_{k=1}^{N} \lambda_{ik} x_{ik}^*\| < \varepsilon$. For $k = 1, 2, \ldots, N$. Define

$$F_k = \sum_{i=1}^{n} \alpha_i \lambda_{ik} x_{ik}^* \chi_{A_i}.$$ 

Since each $x_{ik}^*$ is a w*-denting point of $B(X^*)$, for each $k$, $F_k/\|F_k\|$ is a w*-denting point of $B(L^q(\mu, X^*))$ [1, Lemma 10]. And,

$$\left\| F - \sum_{k=1}^{N} F_k \right\|_q^q = \left\| \sum_{i=1}^{n} \alpha_i x_i^* \chi_{A_i} - \sum_{k=1}^{N} \sum_{i=1}^{n} \alpha_i \lambda_{ik} x_{ik}^* \chi_{A_i} \right\|_q^q$$

$$= \sum_{i=1}^{n} |\alpha_i|^q \|x_i^* - \sum_{k=1}^{N} \lambda_{ik} x_{ik}^*\|_q^q \mu(A_i)$$

$$< \sum_{i=1}^{n} \varepsilon^q |\alpha_i|^q \mu(A_i) \leq \varepsilon^q \|F\|_q^q \leq \varepsilon.$$ 

The analogues of the following results for Property (II) were obtained in [3].

**Proposition 3.10.** Let $K$ be a compact Hausdorff space, then $C(K, X)$ is nicely smooth if and only if $K$ is finite and $X$ is nicely smooth.
Proof. For a compact Hausdorff space $K$ and a Banach space $X$, the set

$$A = \{ \delta(k) \otimes x^* : k \in K, x^* \in S(X^*) \} \subseteq B(C(K, X)^*)$$

is a norming set for $C(K, X)$. So, if $C(K, X)$ is nicely smooth, $C(K, X)^* = \text{span}(A)$. It follows that $K$ admits no nonatomic measure, whence $K$ is scattered. Now, let $K_1$ denote the set of isolated points of $K$. Then $K_1$ is dense in $K$, so, the set

$$A_1 = \{ \delta(k) \otimes x^* : k \in K_1, x^* \in S(X^*) \}$$

is also norming. Thus, $C(K, X)^* = \text{span}(A_1)$. But if $k \in K \setminus K_1$, then for any $x^* \in S(X^*)$, $\delta(k) \otimes x^* \notin \text{span}(A_1)$. Hence, $K = K_1$, whence $K$ must be finite. And if $k_0 \in K_1$, $x \to \chi(k_0)x$ is an isometric embedding of $X$ into $C(K, X)$ as an $M$-summand. Thus, $X$ is nicely smooth.

The converse is immediate from Theorem 3.3. 

Remark 3.11. (a) It is immediate that for $C(K)$ spaces Property (II), the BGP and nice smoothness (indeed, any of the conditions of Theorem 2.10) are equivalent, and are equivalent to reflexivity.

(b) It follows from the above and [3, Theorem 3.9] that $C(K, X)$ has Property (II) if and only if $K$ is finite and $X$ has Property (II). Only the special case of $C(K)$ is noted in [3].

Proposition 3.12. Let $X$ be a Banach space such that there exists a bounded net $\{K_\alpha\}$ of compact operators such that $K_\alpha \to Id$ in the weak operator topology. If $\mathcal{L}(X)$ is nicely smooth, then $X$ is finite dimensional.

Proof. For $x \in X$, $x^* \in X^*$, let $x \otimes x^*$ denote the functional defined on $\mathcal{L}(X)$ by $(x \otimes x^*)(T) = x^*(T(x))$. Then $\|x \otimes x^*\| = \|x\|\|x^*\|$. And, since $\|T\| = \sup\{\|x^*(T(x))\| : \|x\| = 1, \|x^*\| = 1\} = \sup\{(x \otimes x^*)(T) : \|x\| = 1, \|x^*\| = 1\}$, it follows that $A = \{x \otimes x^* : \|x\| = 1, \|x^*\| = 1\}$ is a norming set, and hence, $\mathcal{L}(X)^* = \text{span}(A)$.

Claim: $K_\alpha \to Id$ weakly.

Since $\{K_\alpha\}$ is bounded, it suffices to check that $K_\alpha \to Id$ on $A$, i.e., to check $x^*(K_\alpha(x)) \to x^*(x)$ for all $\|x\| = 1, \|x^*\| = 1$. But, $K_\alpha(x) \to x$ weakly, hence the claim.

Thus, $Id$ is a compact operator, so that $X$ is finite dimensional. 

Proposition 3.13. For a compact Hausdorff space $K$, $\mathcal{L}(X, C(K))$ is nicely smooth if and only if $\mathcal{K}(X, C(K))$ is nicely smooth if and only if $X$ is reflexive and $K$ is finite.
**Proof.** Suppose $\mathcal{L}(X, C(K))$ is nicely smooth. By definition of the norm, $A = \{\delta(k) \otimes x : x \in B(X), k \in K\}$ is a norming set for $\mathcal{L}(X, C(K))$, and hence, $\mathcal{L}(X, C(K))^* = \text{span}(A)$. It follows that $\mathcal{L}(X, C(K)) = K(X, C(K))$ and that $K(X, C(K))$ is nicely smooth.

Now, from the easily established identification, $K(X, C(K)) = C(K, X^*)$ and Proposition 3.10, it follows that $K(X, C(K))$ is nicely smooth if and only if $K$ is finite and $X^*$ is nicely smooth, which, in turn, is equivalent to $K$ is finite and $X$ is reflexive.

Also, if $K$ is finite, $C(K)$ is finite dimensional, so that $\mathcal{L}(X, C(K)) = K(X, C(K))$. This completes the proof.

Coming to general tensor product spaces, the proof of [12, Theorem 5.2] combined with Theorem 2.10 actually shows that:

**Theorem 3.14.** If $X$, $Y$ are nicely smooth Asplund spaces, then $X \otimes_{\varepsilon} Y$ is nicely smooth.

We prove the converse for general Banach spaces.

**Theorem 3.15.** Let $X$, $Y$ be Banach spaces such that $X \otimes_{\varepsilon} Y$ is nicely smooth. Then both $X$ and $Y$ are nicely smooth.

**Proof.** Let $M$ and $N$ be norming subspaces of $X^*$ and $Y^*$ respectively. Then $B(M)$ and $B(N)$ are norming sets for $X$ and $Y$ respectively. Hence, $\overline{co}^w(B(M)) = B(X^*)$ and $\overline{co}^w(B(N)) = B(Y^*)$. Thus

$$B(X^*) \otimes B(Y^*) = \overline{co}^w(B(M)) \otimes \overline{co}^w(B(N)).$$

By definition of the injective norm, $B(X^*) \otimes B(Y^*)$ is a norming set for $X \otimes_{\varepsilon} Y$. Thus it follows that $co(B(M)) \otimes co(B(N))$ is a norming set for $X \otimes_{\varepsilon} Y$. Since $co(B(M) \otimes B(N)) \supseteq co(B(M)) \otimes co(B(N))$, it follows that $co(B(M) \otimes B(N))$ and hence $B(M) \otimes B(N)$ is a norming set for $X \otimes_{\varepsilon} Y$. And since this space is nicely smooth,

$$(X \otimes_{\varepsilon} Y)^* = \text{span}(B(M) \otimes B(N)).$$

Suppose $x^* \in X^*$, then for any $y^* \in S(Y^*)$ and $\varepsilon > 0$, there exist $f_i \in B(M)$, $e_i \in B(N)$ and $\lambda_i \in \mathbb{R}$ such that

$$\|x^* \otimes y^* - \sum_{i=1}^{n} \lambda_i f_i \otimes e_i\| < \varepsilon.$$
Applying to elementary tensors, this implies
\[
\left| (x^* \otimes y^* - \sum_{i=1}^{n} \lambda_i f_i \otimes e_i)(x \otimes y) \right| < \varepsilon \|x\| \|y\| \text{ for all } x \in X, y \in Y
\]
\[
\implies \left| x^*(x)y^*(y) - \sum_{i=1}^{n} \lambda_i f_i(x)e_i(y) \right| < \varepsilon \|x\| \|y\| \text{ for all } x \in X, y \in Y
\]
\[
\implies \left| x^*(x)y^* - \sum_{i=1}^{n} \lambda_i f_i(x)e_i \right| < \varepsilon \|x\| \text{ for all } x \in X.
\]

Let \( x \in E = \bigcap \ker f_i \). Then, \( \|x^*(x)y^*\| < \varepsilon \|x\| \), i.e., \( |x^*(x)| < \varepsilon \|x\| \). That is, \( \|x^*|_E\| < \varepsilon \). This implies \( d(x^*, \text{span}\{f_i\}) < \varepsilon \).

It follows that \( x^* \in M \) and hence \( X \) is nicely smooth. Similarly for \( Y \). □

It seems difficult to obtain analogues of Theorems 3.14 and 3.15 for the projective tensor product. However, we have the following

**Proposition 3.16.** Suppose \( X, Y \) are Banach spaces such that \( X^* \) has the approximation property and \( \mathcal{L}(X,Y^*) = \mathcal{K}(X,Y^*) \), i.e., any bounded linear operator from \( X \) to \( Y^* \) is compact. Then the following are equivalent:

(a) \( \mathcal{K}(X,Y^*) \) is nicely smooth.

(b) \( X, Y \) are reflexive (and hence nicely smooth).

(c) \( X \otimes_{\pi} Y \) is reflexive (and hence nicely smooth).

**Proof.** (a) \( \Rightarrow \) (b) Since \( X^* \) has the approximation property,
\[
\mathcal{K}(X,Y^*) = X^* \otimes_{\varepsilon} Y^*
\]
and it follows from Theorem 3.15 that \( X^* \) and \( Y^* \) are nicely smooth, and therefore, \( X \) and \( Y \) are reflexive.

(b) \( \Rightarrow \) (c) This is a well-known result of Holub (see [7]).

(c) \( \Rightarrow \) (a) \( X \) and \( Y \) being closed subspaces of the reflexive space \( X \otimes_{\pi} Y \) are themselves reflexive and from
\[
\mathcal{K}(X,Y^*)^* = (X \otimes_{\pi} Y)^{**} = X \otimes_{\pi} Y
\]

it follows that \( \mathcal{K}(X,Y^*) \) is reflexive, and hence, nicely smooth. □
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