EXTRACTA MATHEMATICAE Vol. **16**, Núm. 1, 63 – 77 (2001)

Modular Annihilator Jordan Pairs

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(Research paper presented by A. Rodríguez Palacios)

AMS *Subject Class.* (2000): 46Hxx, 17Axx *Received July* 12, 1999

Introduction

Barnes proved, in [2] and [3], that a complex semiprimitive associative Banach algebra A is modular annihilator if and only if 0 is the only possible accumulation point of the spectrum of x for each $x \in A$. A complex Jordan Banach algebra J which satisfies the above spectral property is called inessential. Fernández proved in [9] that inessential complex semiprimitive Jordan Banach algebras are modular annihilator. Benslimane and Rodriguez proved in [5] that the converse also holds.

The purpose of this paper is to give a characterization of modular annihilator Jordan pairs generalizing, by the way, the characterization of Jordan algebras given by Fernández in [10]. The main ingredients of the proof are the theory of primitive Jordan pairs and the use of local algebras. In the same spirit we prove that a complex semiprimitive Banach Jordan pair is modular annihilator if and only if it is inessential. The use of local algebras is in fact an alternative to the proof given by Hessenberger in [14], who obtained the same characterization of semiprimitive Banach Jordan pairs by means of more analytical methods. The strategy of reducing questions on Jordan pairs to questions on Jordan algebras, via local algebras, was used by Zelmanov as a minor part of his brilliant classification of strongly prime Jordan systems [20], and more recently by D'Amour and McCrimmon [8], and by Anquela and Cortes [1].

As an immediate consequence of our results we establish the reciprocal local-to-global inheritance of modular annihilator property asserting that a complex semiprimitive Banach Jordan pair is modular annihilator if and only if all its local algebras are so. We finish by studying the complex compact Banach Jordan pairs as an example of modular annihilator Banach Jordan pairs.

1. Preliminaries

In this paper we shall deal with Jordan pairs and Jordan algebras over a ring of scalars Φ. Nevertheless, we shall be mainly interested in the linear case ($\frac{1}{2} \in \Phi$), and very specially in the case that Φ is the complex field. The reader is referred to [15] for notations, conventions and basic results. In particular, the numbering of identities JP_n refers to [15], however, we shall record in this section some of those notations and results.

Given a Jordan pair $V = (V^+, V^-)$, we write $Q_\sigma : V^\sigma \to Hom_\Phi(V^{-\sigma}, V^\sigma)$, $\sigma \in \{+, -\}$, to denote the quadratic maps of V. The multiplication $Q_x y$ is quadratic in x and linear in y and has linearizations

$$\{x, y, z\} = Q(x, z)y = D(x, y)z = Q_{x+z}y - Q_x y - Q_z y.$$

Note that $\{x, y, x\} = 2Q_x y$, so we only need to consider the triple product in the linear case. For every $(x, y) \in V$ we define the Bergmann operator as follows

$$B_{(x,y)} = Id_{V^\sigma} - D_{(x,y)} + Q_x Q_y.$$

A typical example of Jordan pair is given by taking

$$V^+ = Hom_\Delta(X, Y), \quad V^- = Hom_\Delta(Y, X),$$

linear maps between right vector spaces X and Y over a division associative Φ-algebra Δ, with $Q_a b = aba$. Any associative, alternative or Jordan algebra A gives rise to a Jordan pair (A, A) with quadratic multiplication xyx or $U_x y$, with U denoting the usual U-operator of the Jordan algebra.

In the opposite direction, given a Jordan pair $V = (V^+, V^-)$ and an element $y \in V^{-\sigma}$ we can define a Jordan algebra on V^σ by

$$U_a^{(y)} = Q_a Q_y, \quad \text{and} \quad a^{(2,y)} = Q_a y.$$

This Jordan algebra, denoted by $V^{\sigma(y)}$, is called the y-homotope of V. If V is a linear Jordan pair, we just need to define the linear product in $V^{\sigma(y)}$ as follows

$$x \cdot z = \frac{1}{2}\{x, y, z\}.$$

Local algebras of a Jordan pair. Let V be a Jordan pair and $y \in V^{-\sigma}$. By [8, 1.2.2] the set

$$Ker(y) = \{x \in V^\sigma : Q_y x = Q_y Q_x y = 0\}$$

turns out to be an ideal of $V^{\sigma(y)}$ and the quotient $V^{\sigma(y)}/Ker(y)$ is a Jordan algebra called the local algebra of V at y which we denote by V_y. As pointed out in [8, 1.2.4 (ii)] the condition $Q_yQ_xy = 0$ is superfluous if V is linear. A Jordan pair is non-degenerate if it has no nonzero absolute zero divisors, i.e., $Q_x = 0$ implies $x = 0$.

(1.1) *If V is non-degenerate then so are all its local algebras by JP_3.*

Annihilators. Following [15, p. 104] the annihilator of a subset X of V^σ is the inner ideal $ann_V X \subset V^{-\sigma}$ ($ann_V X$ when V need not be specified) of all $a \in V^{-\sigma}$ satisfying

$$Q_aX = Q_Xa = 0, \quad Q_aQ_X = D_{(a,X)} = 0, \quad \text{and} \quad Q_XQ_a = D_{(X,a)} = 0.$$

In the linear case, (see [11, Lemma 1]), only two conditions are required:

$$D_{(a,X)} = 0 \quad \text{and} \quad D_{(X,a)} = 0.$$

Note that if $X \subseteq Y \subseteq V^\sigma$ then $ann_V Y \subseteq ann_V X$.

Let $I = (I^+, I^-)$ be an ideal of a Jordan pair V. We write $ann_V I$ to denote the annihilator ideal $(ann_V I^-, ann_V I^+)$. By [18], the annihilator ideal $ann_V I$ of any ideal I of a non-degenerate Jordan pair V, has an easy expression

$$ann_V I^\sigma = \left\{z \in V^{-\sigma} : Q_z I^\sigma = o\right\},$$

and it is orthogonal to I

$$I^\sigma \cap ann_V I^{-\sigma} = 0.$$

In particular, the annihilator of an ideal I of a non-degenerate Jordan algebra J is given by

$$ann_J I = \{x \in J : U_x I = 0\}$$

Radical and socle. Write $Rad(V) = (Rad(V^+), Rad(V^-))$ to denote the Jacobson radical of a Jordan pair, where $Rad(V^\sigma)$ is the set of properly quasi-invertible elements of V^σ. We say that V is semiprimitive if $Rad(V) = 0$. For a non-degenerate Jordan pair V, the socle of V is $Soc(V) = (Soc\, V^+, Soc\, V^-)$, that is $Soc\, V^\sigma$ is the sum of all minimal inner ideals of V^σ. Equivalently, the elements of the socle are those of the form

$$s_1 + s_2 + \cdots + s_n$$

where the s_i are simple elements in the sense that the inner ideals (s_i) generated by s_i are minimal. The reader is referred to [16] for basic results on socle theory for Jordan pairs. In particular, we note that $Soc\,V$ is an ideal of V. The following local characterization of the socle can be found in [19, 0.6 (b)].

(1.2) *Let V be a non-degenerate Jordan pair and $x \in V^\sigma$. Then $x \in Soc\,V^\sigma$ if and only if V_x has finite capacity. Furthermore,*

(1.3) *for each $x \in V^\sigma$, V_x is unital if and only if x is Von Neumann regular.*

It follows from Innerness Correspond Proposition [8, 2.4] that if $\pi : V^{-\sigma(x)} \to V_x$ denote the canonical map then

(1.4) $\pi(Soc\,V^{-\sigma}) = Soc\,(V_x)$.

Full subpairs. A subpair $P = (P^+, P^-)$ of a Jordan pair V is called *full* if

$$Q_x V^{-\sigma} = Q_x P^{-\sigma}, \text{ for all } x \in P^\sigma.$$

For example the Peirce space$V_2(e) = Q_e(V) = (Q_{e^+}V^-, Q_{e^-}V^+)$, in the Peirce decomposition of V with respect to the idempotent $e = (e^+, e^-)$ $(Q_{e^\sigma} e^{-\sigma} = e^\sigma$, $\sigma \in \{+,-\})$, is a full subpair of V.

Primitive Jordan pairs. A Jordan pair $V = (V^+, V^-)$ is called *primitive* at $b \in V^{-\sigma}$ if there exists a proper inner ideal K of V^σ such that

(i) K is c-modular at b for some $c \in V^\sigma$, i.e.,

(a) $B_{(c,b)}V^\sigma \subseteq K$,
(b) $c - Q_c b \in K$,
(c) $D_{(c,b)}K \subseteq K$,
(d) $(D_{(x,b)} - D_{(c,Q_b x)}) \cdot K \subseteq K$ for any $x \in V^\sigma$.

Equivalently, if K is a c-modular inner ideal of the homotope $V^{\sigma(b)}$.

(ii) $I^\sigma + K = V^\sigma$ for any ideal $I = (I^+, I^-)$ of V such that $I^\sigma \neq 0$.

(iii) $Q_{V^\sigma} z = Q_{V^\sigma} Q_z V^\sigma = 0$, $z \in V^{-\sigma}$, implies $z = 0$.

An ideal P of a Jordan system (algebra or pair) V is called primitive if the factor system (algebra or pair) V/P is primitive.

Anquela and Cortes proved in [1] the following result

(1.5) *V is primitive at $b \in V^{-\sigma}$ if and only if V_b is a primitive Jordan algebra and V is strongly prime.*

Further results on primitive Jordan pairs can be founded in [1], and [8].

The following result, proved in [7, Theorem 1] for the case of a Jordan triple system, is included here for completeness.

LEMMA 1.6. *Let V be a non-degenerate Jordan pair.*

(i) *If M is a simple ideal of V containing a nonzero idempotent, then $ann_V M$ is a primitive ideal. In particular,*

(ii) *if V has nonzero socle, for any simple component M of the socle of V the annihilator ideal $ann_V M$ is primitive, and*

(iii) *strongly prime Jordan pairs containing minimal inner ideals are primitive.*

Proof. (i) Let $e = (e^+, e^-)$ be a nonzero idempotent of M. By replacing V by the quotient pair $V/ann_V M$, we may suppose that V is a strongly prime Jordan pair with a simple ideal M which contains a nonzero idempotent $e = (e^+, e^-)$. Then we must prove that V is primitive. Let us see that $B_{(e^+,e^-)}V^+$ is a primitizor of V. By Peirce relations [15, p. 44], the inner ideal $B_{(e^+,e^-)}V^+ = V_0(e^+)$ is clearly e^+-modular at e^- [1, (3.1)]. Moreover, if I is a nonzero ideal of V, M is contained in I by simplicity of M and primeness of V, and again by Peirce relations,

$$V^+ = V_2(e)^+ + V_1(e)^+ + V_0(e)^+ = M^+ + B_{(e^+,e^-)}V^+ = I^+ + B_{(e^+,e^-)}V^+.$$

The third condition required in the definition of primitivity automatically holds by non-degeneracy of V.

(ii) By socle theory [16], every simple component of the socle contains a division idempotent, so (i) applies.

(iii) It is a direct consequence of (ii) and the fact that every minimal inner ideal generates a simple component of the socle. ∎

Remarks. The fact that the annihilator of a simple component of the socle is a primitive ideal was proved by Fernández López and Rodríguez Palacios in [12, Proposition 11] for non-degenerate noncommutative Jordan algebras.

An alternative proof of (iii) can be given by using the local characterization of primitivity (1.5). Let V be a strongly prime Jordan pair containing a minimal inner ideal. Then V contains a rank one element, say $b \in V^-$, equivalently, the local algebra V_b of V at b is a division Jordan algebra, therefore primitive. Hence the whole pair is primitive by (1.5).

2. Modular annihilator Jordan pairs

Throughout this section Φ will be a fixed unital commutative ring of scalars containing $\frac{1}{2}$.

Definition 2.1. A Jordan system V (algebra or pair) is called modular annihilator if V is non-degenerate and $V/Soc\,V$ is radical.

Zelmanov showed in [20] that the Jacobson radical of a Jordan pair is the intersection of its primitive ideals. Consequently, for a Jordan pair V and an ideal K of V such that V/K is radical, if there exists a primitive ideal I of V so that $K \subseteq I$, then I/K is a primitive ideal of V/K, which is a contradiction. Conversely if V/K is not radical then there exists a primitive ideal L/K of V/K such that L is a primitive ideal of V. This result imply:

Lemma 2.1. *Let K be an ideal of a Jordan pair V. Then V/K is radical if and only if for every primitive ideal I, $K \nsubseteq I$.*

The following result extends to Jordan pairs some of the characterizations of modular annihilator Jordan algebras in [10].

Theorem 2.1. *For a non-degenerate Jordan pair V the following conditions are equivalent.*

(i) *V is modular annihilator.*

(ii) *No primitive ideal contains $Soc\,V$.*

(iii) *The primitive ideals of V are precisely the annihilators of the simple components of the socle.*

(iv) *$Rad(V) = ann_V(Soc\,V)$, and $ann_V P \neq 0$ for any primitive ideal P of V.*

Proof. (i)⇒(ii) It follows from Lemma 2.1.

(ii)⇒(iii) By Lemma 1.6(ii), we just need to see that any primitive ideal P of V is of the form $ann_V M$ for a simple component of the socle of V. By socle theory [16, Theorem 2], given P ideal primitive of V there exists a simple component M of $Soc\,V$ which is not contained in P. Then $P \cap M = 0$ by simplicity of M, equivalently, $P \subset ann_V M$. Conversely, since primitive ideals are prime ideals (1.5),

$$Q_{ann_V M} M \subset M \cap ann_V M = 0 \subset P \Rightarrow ann_V M \subset P,$$

which proves the equality ($M \subset P \subset ann_V M$ would lead to contradiction).

(iii)⇒(iv) Since the Jacobson radical of a Jordan pair is the intersection of the primitive ideals,

$$Rad(V) = \cap P = \cap ann_V M,$$

where M ranges over all simple components of $Soc\,V$. Hence

$$Rad(V) = ann_V Soc\,V.$$

Now if P is a primitive ideal of V, $P = ann_V M$ for a simple ideal M implies

$$0 \neq M \subset ann_V(ann_V M) = ann_V P.$$

(iv)⇒(i) By Lemma 2.1 again, if $V/Soc\,V$ is not radical, there exists a primitive ideal P of V such that $Soc\,V \subset P$. Then

$$ann_V P \subset ann_V(Soc\,V) = Rad(V) \subset P$$

implies $ann_V P = 0$, which is a contradiction. ∎

We finish this section by listing some properties of modular annihilator Jordan pairs.

Proposition 2.1. *Let V be a modular annihilator Jordan pair.*

(i) *Every local algebra of V is modular annihilator.*

(ii) *Every ideal I of V is a modular annihilator Jordan pair.*

(iii) *If I is an ideal of V such that V/I is non-degenerate (annihilator ideals enjoy this property), then V/I is modular annihilator.*

(iv) *Every Von Neumann regular element of V lies in the socle. Hence the socle of V coincides with the set of all Von Neumann regular elements of V.*

(v) *For every idempotent e of V, $V_2(e)$ has finite capacity.*

Proof. (i) Let $y \in V^{-\sigma}$. Denote by $x \longmapsto \overline{x}$ the canonical map of V onto $V/Soc\,V$. Since V is modular annihilator, for every $x \in V^{\sigma}$, $(\overline{x}, \overline{y})$ is quasi-invertible in $V/Soc\,V$. Then $\overline{x}$ is quasi-invertible in $(V^{\sigma}/Soc\,V^{\sigma})^{\overline{y}} = V^{\sigma(y)}/Soc\,V^{\sigma}$. But

$$x + Soc\,V^{\sigma} \longmapsto x + (Soc\,V^{\sigma} + Ker(y))$$

is a homomorphism of Jordan algebras of $V^{\sigma(y)}/Soc\,V^{\sigma}$ onto $V^{\sigma(y)}/(Soc\,V^{\sigma} + Ker(y))$. Moreover, by (1.4), $V^{\sigma(y)}/(Soc\,V^{\sigma} + Ker(y))$ is isomorphic to $V_y/Soc\,V_y$. Therefore $(x + Ker(y)) + Soc\,V_y$ is quasi-invertible in $V_y/Soc\,V_y$. Taking into account the non-degeneracy of V_y inherited from that of V (1.1), the local algebra V_y is shown to be modular annihilator.

(ii) and (iii) can be proved using similar techniques to those of the case of Jordan algebras [10].

(iv) Let $x \in V^{\sigma}$ be a Von Neumann element. Denote by $\pi : V \longrightarrow V/Soc\,V$ the canonical homomorphism. Since $V/Soc\,V$ is radical, $\pi(x) = 0$ because $Rad(V/Soc\,V)$ contains no nonzero Von Neumann regular elements [15, 5.1]. Then $x \in Soc\,V^{\sigma}$. To conclude, it is known that $Soc\,V^{\sigma}$ consists of Von Neumann regular elements [16, Theorem 1].

(v) follows from (iv) and [16, Theorem 1(ii)]. ∎

3. Modular annihilator Banach Jordan pairs

A normed Jordan algebra is a (complex) Jordan algebra J endowed with a norm $\|\cdot\|$ making continuous the product of J. If the norm is complete, J is said to be a Banach Jordan algebra. Of course we can always renorm J with an equivalent norm $\|\cdot\|'$ so that $\|x \cdot y\|' \leq \|x\|'\|y\|'$ for all x, y in J.

By a normed Jordan pair we shall mean a Jordan pair $V = (V^+, V^-)$, where the vector spaces V^+ and V^- are endowed with norms making continuous the products of V. If these norms are complete, then we shall say that V is a Banach Jordan pair. It's clear that if V is a normed (Banach) Jordan pair then for each $y \in V^{-\sigma}$, the homotope $V^{\sigma(y)}$ with the same norm as V^{σ}, is a normed (Banach) Jordan algebra. Moreover by the continuity of the operator Q_y, $Ker(y) = Ker Q_y$ is a closed ideal of $V^{\sigma(y)}$. Hence for the quotient norm

(3.1) *the local algebra V_y is a normed (Banach) Jordan algebra.*

Let J be a unital Banach Jordan algebra over the complex field. Recall that the spectrum of an element x in J is defined by the set

$$Sp_J(x) = \{\lambda \in \mathbb{C} : \lambda - x \text{ is not invertible in } J\}.$$

If J is not unital then its unital hull $J' = \mathbb{C} \oplus J$ is a Banach Jordan algebra with the norm $\|\alpha + x\| = |\alpha| + \|x\|$ and the spectrum of $x \in J$ is defined to be $Sp_{J'}(x)$. Further results on spectral theory can be found in [6].

Let V be a Jordan pair over the complex field and $(x, y) \in V = (V^+, V^-)$. The spectrum of (x, y) is defined by the set

$$Sp_V(x,y) = \left\{ \lambda \in \mathbb{C} : \lambda - x \text{ is not invertible in } \mathbb{C} \oplus V^{\sigma(y)} \right\}.$$

Lemma 3.1. *Let V be a complex Jordan pair and $(x, y) \in V$ then*

$$Sp_V(x,y) = Sp_{V_y}(\bar{x}) \cup \{0\},$$

where $x \mapsto \bar{x}$ denote the canonical mapping of $V^{\sigma(y)}$ onto V_y.

Proof. First one proves that for a quasi-invertible ideal I of a unital Jordan algebra J

$$Sp_J(x) = Sp_{J/I}(\bar{x}),$$

where $\bar{x}$ is the image of x in J/I. Indeed if x is invertible in J then so is $\bar{x}$ in J/I. Conversely, if $\bar{x}$ is invertible in J/I, then there exists $y \in J$ such that $U_{\bar{x}}\bar{y} = \bar{1}$ that is $1 - U_x y = i$ for some $i \in I$. Therefore $U_x y = 1 - i$ is invertible in J and then so is x. Moreover if J is not unital then

$$Sp_{\mathbb{C}\oplus J}(x) = Sp_{\mathbb{C}\oplus(J/I)}(\bar{x}).$$

Now $Ker(y)$ is a quasi-invertible ideal in $V^{\sigma(y)}$, [15, Proposition 4.18.(2)]. ∎

The following result by Benslimane and Rodriguez [5] will be needed in the proof of the main result of this section (Theorem 3.2) together with Lemma 3.2 below.

Theorem 3.1. *Let x be an element of a complex Jordan Banach algebra J and $0 \neq \lambda$ an isolated point of $Sp_J(x)$. Then there exists an idempotent e of J such that $\lambda \notin Sp_J(x - U_e x)$.*

Let V be a complex Jordan pair. An element $x \in V^-$, is said to have *properly finite spectrum* if $Sp_V(y, x)$ is finite for all $y \in V^+$, equivalently, by Lemma 3.1, the local algebra V_x of V at x is spectrally finite.

Under some conditions, Hessenberger, basing his argument on the spectral boundness of properly finite spectrum elements, had established an elemental characterization of the socle [13], giving a positive answer to the question in [17, 3.9]. Now, by making use of local algebras, we give a simple proof of the same result. This suggests that local theory plays a crucial role in this work.

LEMMA 3.2. *Let V be a complex semiprimitive Banach Jordan pair. For $x \in V^{-\sigma}$, the following conditions are equivalent.*

(i) $x \in Soc\, V^{-\sigma}$.

(ii) *x has properly finite spectrum.*

Proof. (i)⇒(ii) This is immediate from [17, Theorem 3.6].

(ii)⇒(i) If x has properly finite spectrum then, by Lemma 3.1, the local algebra V_x is finite spectrum. Moreover since V is semiprimitive then so is V_x [7, Theorem 3.1 (ii)]. Now, in virtue of [4], V_x has a finite capacity. Finally, by (1.2), $x \in Soc\, V^{-\sigma}$. ∎

DEFINITION 3.1. A complex Banach Jordan pair V is said to be inessential if, for every $(x, y) \in V$, $Sp_V(x, y)$ has 0 as the only possible accumulation point. Equivalently, by Lemma 3.1, every local algebra of V is inessential or Riesz [9].

We next state the main result of this section.

THEOREM 3.2. *For a complex semiprimitive Banach Jordan pair V the following conditions are equivalent.*

(i) *V is modular annihilator.*

(ii) *V is inessential.*

Proof. (i)⇒(ii) By Proposition 2.1(i), for every $y \in V^{-\sigma}$, the local algebra V_y of V at y is modular annihilator. Moreover, it is semiprimitive by [8, Theorem 3.1 (ii)] and Banach by (3.1). Hence, by the main result of [5], V_y is inessential, equivalently as pointed above, $Sp_V(x, y)$ has 0 as the only possible accumulation point for every x in V^{σ}.

(ii)⇒(i) First we see that every Von Neumann regular element $u \in V^{-\sigma}$ is in the socle. Consider the local algebra V_u of V at u. As before, V_u is a semiprimitive Banach Jordan algebra which is inessential by Lemma 3.1. Since it is also unital (1.3), it follows that it is spectrum finite ($Sp_J(1 + x) = 1 + Sp_J(x)$ for every x in a unital Banach Jordan algebra J). Therefore u has properly finite spectrum, which implies by Lemma 3.2 that u is in the socle. Let us now see that $V/Soc\, V$ is radical. Given $(x, y) \in V$, we distinguish the cases $1 \notin Sp_V(x, y)$ and $1 \in Sp_V(x, y)$.

If $1 \notin Sp_V(x, y)$ then x is quasi-invertible in $V^{\sigma(y)}$. Denote by $x \mapsto \bar{x}$ the canonical map of V onto $V/Soc\, V$. Thus $\bar{x}$ is quasi-invertible in $(V^{\sigma}/Soc\, V^{\sigma})^{(\bar{y})}$

that is $(\bar{x}, \bar{y})$ is quasi-invertible in $V/Soc\,V$. If $1 \in Sp_V(x, y)$, then 1 is an isolated point of $Sp_{V^{\sigma(y)}}(x)$ since V is inessential. In virtue of Theorem 3.1 there exists an idempotent e in $V^{\sigma(y)}$ such that

$$1 \notin Sp_{V^{\sigma(y)}}(x - U_e x) = Sp_V(x - U_e x, y),$$

where $U_e = Q_e Q_y$ is the U-operator in $V^{\sigma(y)}$. Thus

$$1 \notin Sp_V(\overline{x - U_e x}, \bar{y}).$$

But $e = e^{(2,y)} = Q_e y$ is Von Neumann regular in V, so $e \in Soc\,V^\sigma$ by we have just proved. Therefore

$$1 \notin Sp_V(\bar{x}, \bar{y}),$$

that is, $(\bar{x}, \bar{y})$ is quasi-invertible in $V/Soc\,V$. ∎

The following result is a partial converse of Proposition 2.1(i).

COROLLARY 3.1. *A complex semiprimitive Banach Jordan pair is modular annihilator if and only if so are all its local algebras.*

Remark 3.1. In the implication (i)⇒(ii) of Theorem 3.2 the condition "V is semiprimitive" is superfluous. However, it turns out to be necessary in the converse. To justify this assertion it suffices to consider the following counterexample. The Banach Jordan pair $V = (A', A')^J$, associated to the associative pair (A', A') obtained by doubling the unital hull of a radical semiprime Banach associative algebra A with the product $Q_x y = xyx$ for all x, y in A', is inessential ($Sp_V(x, y) = Sp_{A'}(xy) = \{0\}$). However V is not modular annihilator. Indeed, since $Soc\,A = Soc\,A' = 0$ and $Soc\,V = (Soc\,A', Soc\,A')$, we see that $V = V/Soc\,V$ is nonradical because so is A'.

4. COMPACT BANACH JORDAN PAIRS

DEFINITION 4.1. A complex Banach Jordan pair V (algebra J) is said to be compact if Q_x (U_x) is a compact operator for all $x \in V^\sigma$ ($x \in J$).

LEMMA 4.1. *Let V be a compact Jordan pair. Then every homotope of V is a compact Jordan algebra.*

Proof. It follows immediately by means of the continuity of the operators Q_x and the following classical characterization of linear compact operators.

Given X, Y, two Banach spaces and $T \in L(X,Y)$ a continuous linear operator. T is compact if and only if, for every bounded sequence $\{x_n\}$ of X, there exists a subsequence of $\{T(x_n)\}$ which converges. ∎

LEMMA 4.2. *Let V be a complex compact non-degenerate Banach Jordan pair. Then*

(i) *Every idempotent e of V lies in the socle.*

(ii) *If $I = (I^+, I^-)$ is an ideal of V not contained in $Rad(V)$ then I contains a nonzero division idempotent.*

Proof. (i) Let $e = (e^+, e^-)$ be an idempotent in V. Then the Peirce-2-projection $E_2^\sigma = Q_{e^\sigma} Q_{e^{-\sigma}}$ is a compact operator which is the identity when restricted to $Q_{e^\sigma} V^{-\sigma}$. Hence the unit ball of $Q_{e^\sigma} V^{-\sigma}$ is compact, so $Q_{e^\sigma} V^{-\sigma}$ has finite dimension. Therefore $V_2(e)$ has *dcc* on principal inner ideals. On the other hand $V_2(e)$ is non-degenerate because so is V [15, 5.10]. Now, in virtue of [16, Corollary 1], $V_2(e) = Soc\, V_2(e)$. But $Soc\, V_2(e) = Soc\, V \cap V_2(e)$ since $V_2(e)$ is a full subpair of V [16, Proposition 3]. This implies that $e \in Soc\, V$.

(ii) Let $I = (I^+, I^-)$ be an ideal of V such that $I \nsubseteq Rad(V)$. By [15, Proposition 4.18], there exists some $y \in V^\sigma$ such that $I^\sigma \nsubseteq Rad(V^{\sigma(y)})$. Using again [10, Lemma 6.2], together with Lemma 4.1, I^σ contains a nonzero idempotent $a = a^{(2,y)}$ of $V^{\sigma(y)}$ that is $e = (a, Q_y a)$ is an idempotent in V [15, 5.2]. As in the proof of (i) we obtain:

$$V_2(e) = Soc\,(V_2(e)).$$

Thus $V_2(e)$ contains a nonzero division idempotent and lies in I. ∎

COROLLARY 4.1. *For a complex compact nonradical Banach Jordan pair V, $Soc\, V \neq 0$. Moreover if $Rad(V) = 0$ then $Soc\, V$ is essential in the sense that $Soc\, V^\sigma$ hits I^σ for all ideals $I = (I^+, I^-)$ of V such that $I^\sigma \neq 0$ and $ann_V Soc\, V = 0$.*

Proof. Since V is nonradical, there exists $0 \neq a \in V^\sigma$ such that the ideal $\langle a \rangle$ generated by a is not contained in $Rad(V)$. Thus by Lemma 4.2 $\langle a \rangle$ contains a nonzero idempotent $e \in Soc\, V$. On the other hand, if $Rad(V^\sigma) = 0$, then, by Lemma 4.2, any nonzero ideal I of V contains a nonzero division idempotent e lying in the socle. The last assertion follows from the fact that $Soc\, V$ is essential and $ann_V Soc\, V \cap Soc\, V = 0$. ∎

THEOREM 4.1. *Every complex non-degenerate compact Banach Jordan pair V is modular annihilator.*

Proof. We will show that $Rad(V) = ann_V Soc\, V$ and $ann_V P \neq 0$ for every primitive ideal P of V, which is equivalent to the modularity of V by Theorem 2.1. In general, $Rad(V) \subseteq ann_V Soc\, V$. Indeed

$$Q_{Rad(V^\sigma)} Soc\, V^{-\sigma} \subseteq Rad(V^\sigma) \cap Soc\, V^\sigma = 0,$$

since $Rad(V^\sigma)$ contains no nonzero Von Neumann regular elements [15, 5.1]. Before going on with the proof let us note that if V is compact and non-degenerate then, by the continuity of the canonical projection $\pi : V^{\sigma(y)} \to V_y$, the local algebra V_y of V at y is also compact and by (1.1) it is modular annihilator [10, Theorem 6.4]. This result is a key fact in the proof of what follows. On the other hand, for $x \in ann_V Soc\, V^{-\sigma}$, $Q_x Soc\, V^{-\sigma} = 0$. Thus for every $y \in V^{-\sigma}$, $Q_x Q_y Soc\, V^\sigma = 0$. It follows that

$$\pi(Q_x Q_y Soc\, V^\sigma) = 0,$$

equivalently, $U_{\pi(x)}\pi(Soc\, V^\sigma) = 0$, where $U_{\pi(x)}$ is the U-operator in the Jordan algebra V_y. By (1.4), $U_{\pi(x)}(Soc\, V_y) = 0$ that is $\pi(x) \in ann_{V_y} Soc\, V_y$. Since V_y is modular annihilator, we have by [10, Theorem 4.2] that $\pi(x) \in Rad(V_y)$. But $Rad(V_y) = Rad(V^{\sigma(y)})/Ker(y)$, since $Ker(y) \subseteq Rad(V^{\sigma(y)})$ [15, Proposition 4.18.(2)]. This implies that $x \in Rad(V^{\sigma(y)})$. Now $x \in Rad(V^\sigma)$, since

$$Rad(V^\sigma) = \bigcap_{y \in V^{-\sigma}} Rad(V^{\sigma(y)}) \quad [15, \text{Proposition } 4.18].$$

For a primitive ideal $P = (P^+, P^-)$ of V the Jordan pair V/P is primitive. Thus, by (1.5), there exists a $y \notin P^{-\sigma}$ such that the local algebra $(V^\sigma/P^\sigma)_{\bar{y}}$ is primitive where $\bar{y}$ is the image of y in V/P. It's easily seen that $Ker(\bar{y}) = Q_y^{-1}(P^{-\sigma})/P^\sigma$ since $P^\sigma \subseteq Q_y^{-1}(P^{-\sigma})$. Therefore

$$(V/P)_{\bar{y}} = (V^\sigma/P^\sigma)^{(\bar{y})}/Ker(\bar{y}) = (V^{\sigma(y)}/P^\sigma)/(Q_y^{-1}(P^{-\sigma})/P^\sigma).$$

But $(V^{\sigma(y)}/P^\sigma)/(Q_y^{-1}(P^{-\sigma})/P^\sigma)$ is isomorphic to $V_y/\pi(Q_y^{-1}(P^{-\sigma}))$. It follows that $\pi(Q_y^{-1}(P^{-\sigma}))$ is a primitive ideal of V_y. By [10, Theorem 4.2] again we conclude that $ann_{V_y}\pi(Q_y^{-1}(P^{-\sigma})) \neq 0$. Thus there exists $x \notin Ker(y)$ such that

$$U_{\pi(x)}\pi(Q_y^{-1}(P^{-\sigma})) = 0,$$

that is, $Q_xQ_yQ_y^{-1}(P^{-\sigma}) \subseteq Ker(y)$. Therefore, using JP_3, this leads to $Q_{Q_yx}Q_y^{-1}(P^{-\sigma}) = 0$. This implies that $Q_{Q_yx}P^{\sigma} = 0$ say $0 \neq Q_yx \in ann_V P^{\sigma}$. ∎

Acknowledgements

The authors are indebted to the referee for several valuable suggestions concerning improvements of some proofs and his careful reading of the manuscript.

References

[1] Anquela, J., Cortes, T., Primitive Jordan pairs and triple systems, *J. Algebra,* **184** (1996), 632 – 678.

[2] Barnes, B.A., A generalized Fredholm theory for certain maps in the regular representation of an algebra, *Canad. J. Math.,* **20** (1968), 495 – 504.

[3] Barnes, B.A., On the existence of minimal ideals in Banach algebras, *Trans. Amer. Math. Soc.,* **133** (1968), 511 – 517.

[4] Benslimane, M., Kaidi, A., Structure des algèbres de Jordan Banach non commutatives complexes régulières ou semi-simples à spectre fini, *J. Algebra,* **113**(1) (1988), 201 – 206.

[5] Benslimane, M., Rodríguez, A., Caractérisation spectrale des algèbres de Jordan Banach non commutatives complexes modulaires annihilatrices, *J. Algebra,* **140** (1991), 344 – 354.

[6] Bonsall, F.F., Duncan, J., "Complete Normed Algebras", Springer-Verlag, Berlin-Heidelberg-New York, 1973.

[7] Bouhya, K., Fernández López, A., Jordan-*-triples with minimal inner ideals and compact JB*-triples, *Proc. London Math. Soc.,* **68** (1994), 380 – 398.

[8] D'Amour, A., McCrimmon, K., The local algebras of Jordan systems, *J. Algebra,* **177** (1995), 199 – 239.

[9] Fernández López, A., Noncommutative Jordan Riesz algebras, *Quart. J. Math. Oxford,* **39** (1988), 67 – 80.

[10] Fernández López, A., Modular annihilator Jordan algebras, *Comm. Algebra,* **13**(12) (1985), 2597 – 2613.

[11] Fernández López, A., García Rus E., Jaa, O., Towards a Goldie theory for Jordan pairs, *Manuscripta Math.,* **95** (1998), 79 – 90.

[12] Fernández López, A., Rodríguez Palacios, A., Primitive noncommutative Jordan algebras with nonzero socle, *Pro. Amer. Math. Soc.,* **96** (1986), 199 – 206.

[13] Hessenberger, G., A spectral characterization of the socle of Banach Jordan systems, *Math. Z.,* **223** (1996), 561 – 568.

[14] Hessenberger, G., Inessential and Riesz elements in a Banach Jordan systems, *Quart. J. Math. Oxford,* **47**(2) (1996), 337 – 347.

[15] Loos, O., "Jordan Pairs", Lecture Notes in Mathematics, Vol. 460, Springer-Verlag, New York, 1975.

[16] Loos, O., On the socle of a Jordan pair, *Collect. Math.,* **40** (1989), 109 – 125.

[17] LOOS, O., Properly algebraic and spectrum-finite ideals in Jordan systems, *Math. Proc. Camb. Phil. Soc.,* **114** (1993), 149 – 161.

[18] MCCRIMMON, K., Strong prime inheritance in Jordan systems, *Algebras Groups and Geom.,* **1**(2) (1984), 217 – 234.

[19] MONTANER, F., Local PI-theory of Jordan systems, *J. Algebra,* **216** (1999), 302 – 332.

[20] ZELMANOV, E.I., Primary Jordan triple systems III, *Sib. Math. J.,* **26** (1985), 55 – 64.