The Set of First-Order Differential Equations
with Periodic or Bounded Solutions

JOSE L. BRAVO¹, MANUEL FERNÁNDEZ¹,* , ANTONIO TINEO²,**

¹Departamento de Matemáticas, Universidad de Extremadura, 06071-Badajoz, Spain
²Departamento de Matemáticas, Facultad de Ciencias, Universidad de los Andes,
5101-Mérida, Venezuela
e-mail: joseluisb@wanadoo.es, ghierro@unex.es, atineo@ciens.ula.ve

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The objective of this note is the announcement of two results of Ambrosetti-Prodi type concerning the existence of periodic (respectively bounded) solutions of the first order differential equation \( x' = f(t, x) \)

1. Periodic solutions

Let us fix a real number \( T > 0 \) and define \( \mathcal{C} \) as the set of all continuous functions \( f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) such that:

- \( A_1 \) \( f(t, x) \) is \( T \)-periodic in \( t \).
- \( A_2 \) \( f(t, x) \) is locally Lipschitz continuous in \( x \).
- \( A_3 \) \( f(t, x) \) is concave in \( x \) and there exists \( t_0 = t_0(f) \in \mathbb{R} \) such that \( f(t_0, x) \) is strictly concave in \( x \).
- \( A_4 \) \( \lim_{|x| \to \infty} f(t, x) = -\infty \) uniformly on \( t \in \mathbb{R} \).

In \( \mathcal{C} \) we shall consider the topology of uniform convergence on compact sets. We also define \( \mathcal{C}_0 \) as the subset of \( \mathcal{C} \) consisting of all points \( f \) such that the equation

\[
x' = f(t, x)
\]

has a unique \( T \)-periodic solution.

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Theorem 1. The map \( H : C_0 \times \mathbb{R} \to C, \) \( H(g, a) = g + a, \) is a homeomorphism onto \( C. \) Moreover, if \( f \in H(C_0 \times (-\infty, 0)) \) (resp. \( f \in H(C_0 \times (0, \infty)) \)) then, Eq. (1) has exactly two (resp. zero) \( T \)-periodic solutions.

To prove Theorem 1 we first use the arguments in [1] to obtain for every \( f \in C \) the existence of a (unique) real number \( \lambda_0(f) \) such that equation
\[
x' = f(t, x) + \lambda
\]
has exactly zero, one or two \( T \)-periodic solutions according to \( \lambda < \lambda_0, \lambda = \lambda_0 \) or \( \lambda > \lambda_0. \) Thus we prove that \( \lambda_0(f) \) depends continuously on \( f, \) with respect to the topology of the uniform convergence on compact sets.

Theorem 2. Let \( X \subset C \) be an affine manifold such that \( X + \mathbb{R} = X. \) Then \( X_0 := X \cap C_0 \) is the graph of a continuous function \( \mu : F \to \mathbb{R}, \) defined on a closed hyperplane \( F \) of \( X. \)

2. Bounded separated solutions

Let \( f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) be a continuous function. We say that \( f \) is \( s \)-concave in \( x \) if given \( R, \epsilon > 0, \) there exists a continuous function \( b : \mathbb{R} \to [0, \infty) \) such that \( A_L(b) > 0 \) and
\[
f(t, (1 - \lambda)x + \lambda y) \geq (1 - \lambda)f(t, x) + \lambda f(t, y) + \lambda(1 - \lambda)b(t)
\]
if \( |x - y| \geq \epsilon, \) \( |x|, |y| \leq R, \) \( \lambda \in [0, 1], \) and \( t \in \mathbb{R}. \)

Here \( A_L(b) \) denotes the lower average of \( b \) in the sense of [2]. That is,
\[
A_L(b) = \lim_{r \to +\infty} \inf \left\{ \frac{1}{t-s} \int_s^t b(\tau) \, d\tau : t - s \geq r \right\}.
\]

We say that \( f \) is locally equicontinuous in \( x \) if for each compact set \( K \) of \( \mathbb{R} \) and each \( \epsilon > 0 \) there exists \( \delta > 0 \) such that
\[
|f(t, x) - f(t, y)| \leq \epsilon \quad \text{if} \quad t \in \mathbb{R}, \ x, y \in K, \ |x - y| \leq \delta.
\]

We define \( D \) as the subset of all continuous functions \( f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) such that:

\( H_1 \) \( f \) is locally equicontinuous in \( x \) and bounded on \( \mathbb{R} \times K \) for any compact subset \( K \) of \( \mathbb{R}. \)

\( H_2 \) \( f(t, x) \) is locally Lipschitz continuous in \( x. \)
$H_3$) $f(t, x)$ is $s$-concave in $x$.

$H_4$) $\lim_{|x| \to \infty} f(t, x) = -\infty$ uniformly on $t \in \mathbb{R}$.

We define $\mathcal{D}_+$ (resp. $\mathcal{D}_-$) as the subset of $\mathcal{D}$ consisting of all points $f$ such that the equation

$$x' = f(t, x) \quad (5)$$

has two (resp. zero) bounded solutions $u_0 < u_1$ and $\inf(u_1 - u_0) > 0$. We also define $\mathcal{D}_0$ as the subset of $\mathcal{D}$ consisting of all points $f$ such that Eq. (5) has a bounded solution and $\inf(|u - v|) = 0$ if $u, v$ are bounded solutions of this equation.

**Theorem 3.** Theorems 1 and 2 remain true if we replace $\mathcal{C}$ by $\mathcal{D}$.

The proof uses theorem 3.7 of [3] that with this notation can be stated as follows:

Let $f \in \mathcal{D}$. Then there exists $\lambda_0 = \lambda_0(f)$ such that $f + \lambda \in \mathcal{D}_+$ for all $\lambda > \lambda_0$, $f + \lambda_0 \in \mathcal{D}_0$ and $f + \lambda \in \mathcal{D}_-$ for all $\lambda < \lambda_0$.

**References**

