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The Set of First-Order Differential Equations with Periodic or Bounded Solutions

Jose L. Bravo[1], Manuel Fernández[1,*], Antonio Tineo[2,**]

[1] *Departamento de Matemáticas, Universidad de Extremadura,* 06071-*Badajoz, Spain*

[2] *Departamento de Matemáticas, Facultad de Ciencias, Universidad de los Andes,* 5101-*Mérida, Venezuela*

e-mail: joseluisb@wanadoo.es, ghierro@unex.es, atineo@ciens.ula.ve

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The objective of this note is the announcement of two results of Ambrosetti-Prodi type concerning the existence of periodic (respectively bounded) solutions of the first order differential equation $x' = f(t, x)$

1. Periodic solutions

Let us fix a real number $T > 0$ and define $\mathcal{C}$ as the set of all continuous functions $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that:

A_1) $f(t, x)$ is T-periodic in t.

A_2) $f(t, x)$ is locally Lipschitz continuous in x.

A_3) $f(t, x)$ is concave in x and there exists $t_0 = t_0(f) \in \mathbb{R}$ such that $f(t_0, x)$ is strictly concave in x.

A_4) $\lim_{|x|\to\infty} f(t, x) = -\infty$ uniformly on $t \in \mathbb{R}$.

In $\mathcal{C}$ we shall consider the topology of uniform convergence on compact sets. We also define $\mathcal{C}_0$ as the subset of $\mathcal{C}$ consisting of all points f such that the equation

$$x' = f(t, x) \tag{1}$$

has a unique T-periodic solution.

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THEOREM 1. *The map* $H : \mathcal{C}_0 \times \mathbb{R} \to \mathcal{C}$, $H(g, a) = g + a$, *is a homeomorphism onto* $\mathcal{C}$. *Moreover, if* $f \in H(\mathcal{C}_0 \times (-\infty, 0))$ (*resp.* $f \in H(\mathcal{C}_0 \times (0, \infty))$) *then, Eq.* (1) *has exactly two (resp. zero)* T*-periodic solutions.*

To prove Theorem 1 we first use the arguments in [1] to obtain for every $f \in \mathcal{C}$ the existence of a (unique) real number $\lambda_0 = \lambda_0(f)$ such that equation

$$x' = f(t, x) + \lambda \tag{2}$$

has exactly zero, one or two T-periodic solutions according to $\lambda < \lambda_0$, $\lambda = \lambda_0$ or $\lambda > \lambda_0$. Thus we prove that $\lambda_0(f)$ depends continuously on f, with respect to the topology of the uniform convergence on compact sets.

THEOREM 2. *Let* $X \subset \mathcal{C}$ *be an affine manifold such that* $X + \mathbb{R} = X$. *Then* $X_0 := X \cap \mathcal{C}_0$ *is the graph of a continuous function* $\mu : F \to \mathbb{R}$, *defined on a closed hyperplane* F *of* X.

2. Bounded separated solutions

Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function. We say that f is *s-concave* in x if given $R, \epsilon > 0$, there exists a continuous function $b : \mathbb{R} \to [0, \infty)$ such that $A_L(b) > 0$ and

$$f(t, (1 - \lambda)x + \lambda y) \geq (1 - \lambda)f(t, x) + \lambda f(t, y) + \lambda(1 - \lambda)b(t) \tag{3}$$

if $|x - y| \geq \epsilon$, $|x|, |y| \leq R$, $\lambda \in [0, 1]$, and $t \in \mathbb{R}$.

Here $A_L(b)$ denotes the lower average of b in the sense of [2]. That is,

$$A_L(b) = \lim_{r \to +\infty} \inf \left\{ \frac{1}{t - s} \int_s^t b(\tau) d\tau : t - s \geq r \right\}. \tag{4}$$

We say that f is *locally equicontinuous in* x if for each compact set K of $\mathbb{R}$ and each $\epsilon > 0$ there exists $\delta > 0$ such that

$$|f(t, x) - f(t, y)| \leq \epsilon \qquad \text{if } \; t \in \mathbb{R}, \; x, y \in K, \; |x - y| \leq \delta.$$

We define $\mathcal{D}$ as the subset of all continuous functions $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that:

- H_1) f is locally equicontinuous in x and bounded on $\mathbb{R} \times K$ for any compact subset K of $\mathbb{R}$.
- H_2) $f(t, x)$ is locally Lipschitz continuous in x.

H_3) $f(t,x)$ is s-concave in x.

H_4) $\lim_{|x|\to\infty} f(t,x) = -\infty$ uniformly on $t \in \mathbb{R}$.

We define $\mathcal{D}_+$ (resp. $\mathcal{D}_-$) as the subset of $\mathcal{D}$ consisting of all points f such that the equation

$$x' = f(t,x) \tag{5}$$

has two (resp. zero) bounded solutions $u_0 < u_1$ and $\inf(u_1 - u_0) > 0$. We also define $\mathcal{D}_0$ as the subset of $\mathcal{D}$ consisting of all points f such that Eq. (5) has a bounded solution and $\inf(|u - v|) = 0$ if u, v are bounded solutions of this equation.

THEOREM 3. *Theorems* 1 *and* 2 *remain true if we replace* $\mathcal{C}$ *by* $\mathcal{D}$.

The proof uses theorem 3.7 of [3] that with this notation can be stated as follows:

Let $f \in \mathcal{D}$. Then there exists $\lambda_0 = \lambda_0(f)$ such that $f + \lambda \in \mathcal{D}_+$ for all $\lambda > \lambda_0$, $f + \lambda_0 \in \mathcal{D}_0$ and $f + \lambda \in \mathcal{D}_-$ for all $\lambda < \lambda_0$.

REFERENCES

[1] MAWHIN, J. First order ordinary differential equations with several periodic solutions, *J. Appl. Math. Phys.,* **38** (1987), 257–265.

[2] TINEO, A., An iterative scheme for the N-competing species problem, *J. Differential Equations,* **116** (1995), 1–15.

[3] TINEO, A. First order ordinary differential equations with several bounded separate solutions, *J. Math. Anal. Appl.,* **225** , (1998), 359–372.