Fix-Finite Approximation Property
in Normed Vector Spaces

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(Research paper presented by J.P. Moreno)

AMS Subject Class. (2000): 46A55, 52A07, 54H25

Received November 22, 2000

1. Introduction

Let $D$ and $A$ be two nonempty subsets in a metric space. We say that the pair $(D, A)$ satisfies the fix-finite approximation property (in short F.F.A.P.) for a family $F$ of maps (or multifunctions) from $D$ to $A$, if for every $f \in F$ and all $\varepsilon > 0$ there exists $g \in F$ which is $\varepsilon$-near to $f$ and has only a finite number of fixed points. In the particular case where $D = A$, we say that $A$ satisfies the F.F.A.P. for $F$.


In this paper we study the fix-finite approximation property in normed vector spaces. We work with the pair $(D, A)$ such that $A$ satisfies the Schauder condition.

If $x$ is a point of a normed space $X$ and $r > 0$, then we denote by $B(x, r)$ the open ball of radius $r$ and center $x$. A subset $K$ of $X$ is said to be relatively compact if its closure $\overline{K}$ is compact. The convex hull of a subset $\{x_1, \ldots, x_n\}$ of $X$ is defined by

$$\text{conv} \{x_1, \ldots, x_n\} = \left\{ \sum_{i=1}^{n} \alpha_i x_i : \alpha_i \in [0, 1] \text{ for } i = 1, \ldots, n \text{ and } \sum_{i=1}^{n} \alpha_i = 1 \right\}.$$
A subset $A$ of a normed space $X$ is said to enjoy the Schauder condition if for any nonempty relatively compact subset $K$ of $A$ and every $\varepsilon > 0$ there exists a finite cover \( \{ B(x_i, \eta_{x_i}) : x_i \in A, 0 < \eta_{x_i} < \varepsilon, i = 1, \ldots, n \} \) of $K$ such that for any subset \( \{ x_{i_1}, \ldots, x_{i_k} \} \) of \( \{ x_1, \ldots, x_n \} \) with \[
\bigcap_{j=1}^{k} B(x_{i_j}, \eta_{x_{i_j}}) \cap K \neq \emptyset
\] the convex hull of \( \{ x_{i_1}, \ldots, x_{i_k} \} \) is contained in $A$.

For example, any nonempty convex subset of a normed space $X$ and any open subset of $X$ satisfies the Schauder condition (see [6]). Also, all finite-union of closed convex subsets of a Banach space satisfies the Schauder condition (see [1]).

In the present work we first establish the following result (Theorem 3.1): if $A$ is a nonempty subset of a normed space $X$ satisfying the Schauder condition and $D$ is a compact subset of $X$ containing $A$, then the pair $(D, A)$ satisfies the F.F.A.P. for any $n$-function.

Secondly we prove (Theorem 3.2): if $A$ is a nonempty subset of a normed space $X$ satisfying the Schauder condition and $D$ is a path and simply connected compact subset of $X$ containing $A$, then the pair $(D, A)$ satisfies the F.F.A.P. for any $n$-valued continuous multifunction. As consequence we obtain a generalization of the Schrimer’s result [5, Theorem 4.6].

2. Preliminaries

In this section we recall some definitions for subsequent use.

Let $X$ and $Y$ be two Hausdorff topological spaces. A multifunction $F : X \to Y$ is a map from $X$ into the set $2^Y$ of nonempty subsets of $Y$. The range of $F$ is $F(X) = \cup_{x \in X} F(x)$.

The multifunction $F : X \to Y$ is said to be upper semi-continuous (usc) if for each open subset $V$ of $Y$ with $F(x) \subset V$ there exists an open subset $U$ of $X$ with $x \in U$ and $F(U) \subset V$.

The multifunction $F : X \to Y$ is called lower semi-continuous (lsc) if for every $x \in X$ and open subset $V$ of $Y$ with $F(x) \cap V \neq \emptyset$ there exists an open subset $U$ of $X$ with $x \in U$ and $F(x') \cap V \neq \emptyset$ for all $x' \in U$.

The multifunction $F : X \to Y$ is continuous if it is both upper semi-continuous and lower semi-continuous.

The multifunction $F$ is compact if it is continuous and the closure of its range $\overline{F(X)}$ is a compact subset of $Y$. 
A point \( x \) of \( X \) is said to be a fixed point of a multifunction \( F : X \to Y \) if \( x \in F(x) \). We denote by \( \text{Fix}(F) \) the set of all fixed points of \( F \).

Let \( X \) and \( Y \) be two normed spaces. We denote by \( C(X) \) the set of nonempty compact subsets of \( X \). Let \( A \) and \( B \) be two elements of \( C(X) \). The Hausdorff distance between \( A \) and \( B \), \( d_H(A, B) \), is defined by setting:

\[
d_H(A, B) = \max \{ \rho(A, B), \rho(B, A) \}
\]

where

\[
\rho(A, B) = \sup \{ d(x, B) : x \in A \},
\]

\[
\rho(B, A) = \sup \{ d(y, A) : y \in B \}
\]

and

\[
d(x, B) = \inf \{ \| y - x \| : y \in B \}.
\]

Let \( F \) and \( G \) be two compact multifunctions from \( X \) to \( Y \). We define the Hausdorff distance between \( F \) and \( G \) by setting:

\[
d_H(F, G) = \sup \{ d_H(F(x), G(x)) : x \in X \}.
\]

Let \( \varepsilon > 0 \) and \( F \) and \( G \) be two compact multifunctions from \( X \) to \( Y \). We say that \( F \) and \( G \) are \( \varepsilon \)-near if \( d_H(F, G) < \varepsilon \).

3. Fix-finite approximation property

3.1. Fix-finite approximation property for \( n \)-functions. In this subsection we study the fix-finite approximation property for \( n \)-functions. First, we recall the definition of an \( n \)-function.

**Definition 3.1.** Let \( X \) and \( Y \) be two Hausdorff topological spaces. A multifunction \( F : X \to Y \) is said to be an \( n \)-function if there exist \( n \) continuous maps \( f_i : X \to Y \), where \( i = 1, \ldots, n \), such that \( F(x) = \{ f_1(x), \ldots, f_n(x) \} \) for all \( x \in X \) and \( f_i(x) \neq f_j(x) \) for all \( x \in X \) and \( i, j = 1, \ldots, n \) with \( i \neq j \).

In this subsection we shall prove the following:

**Theorem 3.1.** Let \( A \) be a nonempty subset of a normed space \( X \) satisfying the Schauder condition. If \( D \) is a compact subset of \( X \) containing \( A \), then the pair \( (D, A) \) satisfies the F.F.A.P. for any \( n \)-function \( F : D \to A \).

In order to prove Theorem 3.1, we shall need the following lemmas.
Lemma 3.1. If a nonempty subset $A$ of a normed space $X$ satisfies the Schauder condition, then for any relatively compact subset $K$ of $A$ and every $\varepsilon > 0$ there exist a finite polyhedron $P$ contained in $A$ and a continuous map $\pi : K \to P$ such that $\|\pi(x) - x\| < \varepsilon$ for all $x \in K$.

Proof. Let $\varepsilon > 0$ and $K$ be a nonempty relatively compact subset of $A$. Since $A$ satisfies the Schauder condition, then there exists a finite cover $\{B(x_i, \eta_{x_i}) : x_i \in A, 0 < \eta_{x_i} < \varepsilon, i = 1, \ldots, n\}$ of $K$ such that for all subset $\{x_{i_1}, \ldots, x_{i_k}\}$ of $\{x_1, \ldots, x_n\}$ with $\bigcap_{j=1}^k B(x_{i_j}, \eta_{x_{i_j}}) \cap K \neq \emptyset$ the convex hull of $\{x_{i_1}, \ldots, x_{i_k}\}$ is contained in $A$.

For all $i = 1, \ldots, n$, let $\mu_i$ be the continuous function defined by $\mu_i(x) = \max(0, \eta_{x_i} - \|x - x_i\|)$, for all $x \in K$. Since for all $x \in K$ there exists $i \in \{1, \ldots, n\}$ such that $\|x - x_i\| < \eta_{x_i}$, then $\sum_{i=1}^n \mu_i(x) > 0$. Now we can define a continuous function $\alpha_i$ on $K$ by setting:

$$\alpha_i(x) = \frac{\mu_i(x)}{\sum_{i=1}^n \mu_i(x)}, \quad i = 1, \ldots, n,$$

for all $x \in K$.

Let

$$Q = \left\{\{x_{i_1}, \ldots, x_{i_k}\} \subset \{x_1, \ldots, x_n\} : \bigcap_{j=1}^k B(x_{i_j}, \eta_{x_{i_j}}) \cap K \neq \emptyset\right\}$$

and

$$P = \bigcup_{\{x_{i_1}, \ldots, x_{i_k}\} \in Q} \text{conv}\{x_{i_1}, \ldots, x_{i_k}\}.$$

Let $\pi$ be the map from $K$ to $P$ defined by $\pi(x) = \sum_{i=1}^n \alpha_i(x)x_i$, for all $x \in K$. Then, the map $\pi$ is continuous and satisfies the property $\|\pi(x) - x\| < \varepsilon$ for all $x \in K$. \[\square\]

In [6] we introduced the notion of Hopf spaces. These are metric spaces satisfying the F.F.A.P. for any compact self-map. By using [6, Theorem 1.3] and the Schauder condition we obtain the following lemma.

Lemma 3.2. Let $A$ be a nonempty subset of a normed space $X$ satisfying the Schauder condition. If $D$ is a compact subset of $X$ containing $A$, then for all continuous map $f : D \to A$ and for every $\varepsilon > 0$, there exist a finite polyhedron $P$ contained in $A$ and a continuous map $g : D \to P$ which is $\varepsilon$-near to $f$ and has only a finite number of fixed points. In particular every nonempty compact subset of a normed space satisfying the Schauder condition is a Hopf space.
Proof. Since \( f(D) \) is a relatively compact subset of \( A \), then by Lemma 3.1 for a given \( \varepsilon > 0 \), there exist a finite polyhedron \( P \) contained in \( A \) and a continuous map \( \pi_\varepsilon : f(D) \to P \) such that \( \|\pi_\varepsilon(y) - y\| < \frac{1}{2}\varepsilon \), for all \( y \in f(D) \). Set \( f_\varepsilon = \pi_\varepsilon \circ f \), then the map \( f_\varepsilon : D \to P \) is continuous and satisfies \( \|f_\varepsilon(x) - f(x)\| < \frac{1}{2}\varepsilon \), for all \( x \in D \).

By [6, Theorem 1.3] there exists a continuous map \( g : D \to P \) which is \( \frac{1}{2}\varepsilon \)-near to \( f_\varepsilon \) and has only a finite number of fixed points. Then, the map \( g \) is \( \varepsilon \)-near to \( f \) because for all \( x \in D \), we have:

\[
\|f(x) - g(x)\| \leq \|f(x) - f_\varepsilon(x)\| + \|f_\varepsilon(x) - g(x)\| < \varepsilon.
\]

Proof of Theorem 3.1. Let \( \varepsilon > 0 \) and \( F : D \to A \) be an \( n \)-function. Then, there exist \( n \) continuous maps \( f_i : D \to A \) such that \( F(x) = \{f_1(x), \ldots, f_n(x)\} \) for all \( x \in D \) and \( f_i(x) \neq f_j(x) \) for all \( x \in D \) and \( i, j = 1, \ldots, n \) with \( i \neq j \).

For all \( i, j = 1, \ldots, n \) with \( i \neq j \), we define \( \delta_{i,j}(F) = \min\{\|f_i(x) - f_j(x)\| : x \in D\} \). As each \( f_i \) is continuous for all \( i = 1, \ldots, n \) and \( D \) is compact, then for each \( i, j = 1, \ldots, n \) with \( i \neq j \), we have \( \delta_{i,j}(F) > 0 \). Therefore,

\[
\delta(F) = \min\{\delta_{i,j}(F) : i, j = 1, \ldots, n, \ i \neq j\} > 0.
\]

For a given \( \varepsilon > 0 \), we set \( \lambda = \min\{\frac{1}{2}\delta(F), \frac{1}{2}\varepsilon\} \). By Lemma 3.2, for each \( i = 1, \ldots, n \), there exists a map \( g_i : D \to A \) which is \( \lambda \)-near to \( f_i \) and has only a finite number of fixed points. Let \( G : D \to A \) be the multifunction defined by \( G(x) = \{g_1(x), \ldots, g_n(x)\} \), for all \( x \in D \).

Claim 1. The multifunction \( G \) is an \( n \)-function. Indeed, if there exists \( x_0 \in D \) and \( i, j = 1, \ldots, n \) with \( i \neq j \), such that \( g_i(x_0) = g_j(x_0) \), then,

\[
\|f_i(x_0) - f_j(x_0)\| \leq \|f_i(x_0) - g_i(x_0)\| + \|f_j(x_0) - g_j(x_0)\| < 2\lambda.
\]

Therefore, \( \delta_{i,j}(F) < \delta(F) \). This is a contradiction and our claim is proved.

Claim 2. The multifunction \( G \) is \( \varepsilon \)-near to \( F \). Indeed, for all \( i = 1, \ldots, n \) and for every \( x \in D \), we have, \( \|f_i(x) - g_i(x)\| < \frac{1}{2}\varepsilon \). Then, \( d_H(F, G) < \varepsilon \).

Claim 3. The multifunction \( G \) has only a finite number of fixed points. Indeed, \( \text{Fix}(G) = \bigcup_{i=1}^m \text{Fix}(g_i) \) and for all \( i = 1, \ldots, n \) the maps \( g_i \) has only a finite number of fixed points.

Corollary 3.1. Let \( C_i \), for \( i = 1, \ldots, m \), be a finite family of nonempty convex compact subsets of a normed space, then \( \bigcup_{i=1}^m C_i \) satisfies the F.F.A.P. for any \( n \)-function \( F : \bigcup_{i=1}^m C_i \to \bigcup_{i=1}^m C_i \).
3.2. Fix-finite approximation property for $n$-valued continuous multifunctions. To start this subsection, we give the definition of an $n$-valued multifunction.

**Definition 3.2.** Let $X$ and $Y$ be two Hausdorff topological spaces. A multifunction $F : X \rightarrow Y$ is said to be $n$-valued if for all $x \in X$, the subset $F(x)$ of $Y$ consists of $n$ points.

Now we recall the definition of the gap of a $n$-valued multifunction. Let $X$ and $Y$ be two Hausdorff topological spaces and let $F : X \rightarrow Y$ be a $n$-valued continuous multifunction. Then, we can write $F(x) = \{y_1, \ldots, y_n\}$ for all $x \in X$. We define a real function $\gamma$ on $X$ by

$$\gamma(x) = \inf\{\|y_i - y_j\| : y_i, y_j \in F(x), i, j = 1, \ldots, n, i \neq j\},$$

for all $x \in X$, and the gap of $F$ by

$$\gamma(F) = \inf\{\gamma(x) : x \in X\}.$$ 

Since the multifunction $F$ is continuous then the function $\gamma$ is also continuous [5, p.76]. If $X$ is compact, then $\gamma(F) > 0$.

In this subsection we show the following:

**Theorem 3.2.** Let $A$ be a nonempty subset of a normed space $X$ satisfying the Schauder condition. If $D$ is a path and simply connected compact subset of $X$ containing $A$, then the pair $(D, A)$ satisfies the F.F.A.P. for any $n$-valued continuous multifunction $F : D \rightarrow A$.

We recall the following Lemma due to H. Schrimer [5] which is useful for the proof of our result.

**Lemma 3.3.** Let $X$ and $Y$ be two compact Hausdorff topological spaces. If $X$ is path and simply connected and $F : X \rightarrow Y$ is a $n$-valued continuous multifunction, then $F$ is an $n$-function.

**Proof of Theorem 3.2.** Let $\varepsilon > 0$ and $F : D \rightarrow A$ be a $n$-valued continuous multifunction. Then, $\gamma(F) > 0$ and $\lambda = \min\{\frac{1}{4}\varepsilon : \frac{1}{2}\gamma(F)\} > 0$. By Lemma 3.1 there exist a finite polyhedron $P$ contained in $A$ and a continuous map $\pi : F(D) \rightarrow P$ such that $\|\pi(y) - y\| < \lambda$ for all $y \in F(D)$. Now we define a continuous multifunction $G : D \rightarrow P$ by $G(x) = (\pi \circ F)(x)$, for all $x \in D$.

Claim 1. The multifunction $G$ is $n$-valued and $\frac{1}{2}\varepsilon$-near to $F$. Indeed, if $x \in D$ such that $F(x) = \{y_1, \ldots, y_n\}$, then $G(x) = \{\pi(y_1), \ldots, \pi(y_n)\}$ with $\|y_i - \pi(y_i)\| < \frac{1}{4}\varepsilon$ for all $i = 1, \ldots, n$. 
Claim 2. There exists an $n$-function $L : D \rightarrow A$ which is $\varepsilon$-near to $F$ and has only a finite number of fixed points. Indeed, from Lemma 3.3 the multifunction $G : D \rightarrow P$ is an $n$-function and by Theorem 3.1 there exists an $n$-function $L : D \rightarrow P$ which is $\frac{1}{2}\varepsilon$-near to $G$ and has only a finite number of fixed points. Then, the multifunction $L : D \rightarrow P$ is $\varepsilon$-near to $F$ and has only a finite number of fixed points.

As a consequence of Theorem 3.1 and Theorem 3.2 we obtain the following:

**Corollary 3.2.** Let $C_i$, for $i = 1, \ldots, m$, be a finite family of nonempty convex compact subsets of a normed space such that $\cap_{i=1}^{m} C_i \neq \emptyset$ or $C_i \cap C_j = \emptyset$ for $i \neq j$, then $\cup_{i=1}^{m} C_i$ satisfies the F.F.A.P. for any $n$-valued continuous multifunction $F : \cup_{i=1}^{m} C_i \rightarrow \cup_{i=1}^{m} C_i$.

**Proof.** Let $\varepsilon > 0$ and $F : \cup_{i=1}^{m} C_i \rightarrow \cup_{i=1}^{m} C_i$ be a $n$-valued continuous multifunction. For the proof we distinguish the following two cases.

First Case. $C_i \cap C_j = \emptyset$ for $i, j = 1, \ldots, m$ and $i \neq j$. We have, $F|_{C_i} : C_i \rightarrow \cup_{i=1}^{m} C_i$ is an $n$-function for $i = 1, \ldots, m$. From Lemma 3.3, the multifunction $F|_{C_i}$ is a $n$-function for $i = 1, \ldots, m$. Therefore, for each $i \in \{1, \ldots, m\}$, there exist $n$ continuous maps $f_{ij} : C_i \rightarrow \cup_{i=1}^{m} C_i$ such that $F(x) = \{f_{i1}(x), \ldots, f_{in}(x)\}$ for all $x \in C_i$. Now for each $j \in \{1, \ldots, n\}$ we can define a continuous map $h_j : \cup_{i=1}^{m} C_i \rightarrow \cup_{i=1}^{m} C_i$ by $h_j(x) = f_{ij}(x)$ if $x \in C_i$. It follows that for all $x \in \cup_{i=1}^{m} C_i$, we have $F(x) = \{h_1(x), \ldots, h_n(x)\}$. Thus, the multifunction $F$ is an $n$-function. By Corollary 3.1 there exists a $n$-multifunction $G : \cup_{i=1}^{m} C_i \rightarrow \cup_{i=1}^{m} C_i$ which is $\varepsilon$-near to $F$ and has only a finite number of fixed points.

Second Case. $\cap_{i=1}^{m} C_i \neq \emptyset$. It follows from Theorem 3.2 that $\cup_{i=1}^{m} C_i$ satisfies the F.F.A.P. for any $n$-valued continuous multifunction.

As a particular case of Corollary 3.2 we obtain a generalization of the Schirmer’s result [5, Theorem 4.6].

**Corollary 3.3.** If $C_1$ and $C_2$ are two nonempty convex compact subsets of a normed space, then $C_1 \cup C_2$ satisfies the F.F.A.P. for any $n$-valued continuous multifunction $F : C_1 \cup C_2 \rightarrow C_1 \cup C_2$.

Acknowledgements

The author is thankful to the learned referee for extremely attentive reading and useful critical remarks to improve the presentation of the paper.
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