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Generalized Lie Bialgebroids and Strong Jacobi-Nijenhuis Structures

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1. Introduction

Roughly speaking, a Lie bialgebroid is a Lie algebroid A whose dual A^* is also equipped with a Lie algebroid structure which is compatible in a certain way with that on A (see [15]). An important example of a Lie bialgebroid is the associated one with a Poisson structure. More precisely, if M is a Poisson manifold with Poisson 2-vector Λ and on TM (respectively, T^*M) we consider the trivial Lie algebroid structure (resp. the cotangent Lie algebroid structure associated with Λ) then the pair (TM, T^*M) is a Lie bialgebroid. Other interesting examples of Lie bialgebroids are Lie bialgebras in the sense of Drinfeld [2] and the Lie bialgebroids associated with Poisson-Nijenhuis structures (see [11]).

On the other hand, as it is well known, a Jacobi structure on a manifold M is a 2-vector Λ and a vector field E on M such that $[\Lambda, \Lambda] = 2E \wedge \Lambda$ and $[E, \Lambda] = 0$, where $[\ ,\ ]$ is the Schouten-Nijenhuis bracket [13]. If (M, Λ, E) is a Jacobi manifold one can define a bracket of functions, the Jacobi bracket, in such a way that the space $C^\infty(M, \mathbb{R})$ endowed with the Jacobi bracket is a local Lie algebra in the sense of Kirillov [9]. Conversely, a local Lie algebra structure on $C^\infty(M, \mathbb{R})$ induces a Jacobi structure on M [5, 9]. Jacobi manifolds are natural generalizations of Poisson manifolds and of other interesting manifolds: contact and locally conformal symplectic (l.c.s.) manifolds. If M is an arbitrary manifold, the vector bundle $TM \times \mathbb{R} \to M$ possesses a natural Lie algebroid structure. Moreover, if M is a Jacobi manifold then the 1-jet bundle $T^*M \times \mathbb{R} \to M$ admits a Lie algebroid structure [8]. Since the pair

$(TM \times \mathbb{R}, T^*M \times \mathbb{R})$ is not a Lie bialgebroid, we introduced in [6] the notion of a generalized Lie bialgebroid (a natural generalization of the notion of a Lie bialgebroid) in such a way that a Jacobi manifold has associated a canonical generalized Lie bialgebroid (see Definition 2.2 and Examples 2.3). Recently, an interesting characterization of generalized Lie bialgebroids was obtained by Grabowski and Marmo [4] (see Theorem 2.4).

In [7], we studied generalized Lie bialgebras, that is, generalized Lie bialgebroids over a single point. In particular, we proved that the last ones can be considered as the infinitesimal invariants of Lie groups endowed with a certain type of Jacobi structures.

The aim of this Note is to obtain new examples of generalized Lie bialgebroids associated with strong Jacobi-Nijenhuis structures. Strong Jacobi-Nijenhuis structures may be considered as a possible Jacobi counterpart of Poisson-Nijenhuis structures (for the different definitions of Jacobi-Nijenhuis structures, we remit to [16, 19, 21]).

The paper is organized as follows. In Section 2, we recall several definitions and results about Lie algebroids and generalized Lie bialgebroids which will be used in the sequel. Using these results we describe, in Section 3, a new family of generalized Lie bialgebroids associated with strong Jacobi-Nijenhuis structures and we deduce some consequences.

2. Lie algebroids and generalized Lie bialgebroids

2.1. Lie algebroids. A *Lie algebroid* A over a manifold M is a vector bundle A over M together with a Lie algebra structure $[\![\ ,\ ]\!]$ on the space $\Gamma(A)$ of the global cross sections of $A \to M$ and a bundle map $\rho : A \to TM$, called the *anchor map*, such that, if we also denote by $\rho : \Gamma(A) \to \mathfrak{X}(M)$ the homomorphism of $C^\infty(M,\mathbb{R})$-modules induced by the anchor map, then:

(i) $\rho : (\Gamma(A), [\![\ ,\ ]\!]) \to (\mathfrak{X}(M), [\ ,\ ])$ is a Lie algebra homomorphism and

(ii) for all $f \in C^\infty(M,\mathbb{R})$ and for all $X, Y \in \Gamma(A)$, one has

$$[\![X, fY]\!] = f[\![X, Y]\!] + (\rho(X)(f))Y.$$

The triple $(A, [\![\ ,\ ]\!], \rho)$ is called a *Lie algebroid over M* (see [14]).

The *Schouten bracket of A* is defined as the unique extension $[\![\ ,\ ]\!]$ of the Lie bracket on $\Gamma(A)$ such that

$$\begin{aligned}
&[\![X, f]\!] = \rho(X)(f),\\
&[\![P, P']\!] = (-1)^{kk'}[\![P', P]\!],\\
&[\![P, P' \wedge P'']\!] = [\![P, P']\!] \wedge P'' + (-1)^{k'(k+1)}P' \wedge [\![P, P'']\!],\\
&(-1)^{kk''}[\![[\![P, P']\!], P'']\!] + (-1)^{k'k''}[\![[\![P'', P]\!], P']\!] + (-1)^{kk'}[\![[\![P', P'']\!], P]\!] = 0
\end{aligned} \tag{1}$$

for $f \in C^\infty(M, \mathbb{R})$, $X \in \Gamma(A)$, $P \in \Gamma(\wedge^k A)$, $P' \in \Gamma(\wedge^{k'} A)$ and $P'' \in \Gamma(\wedge^{k''} A)$.

Moreover, imitating the usual exterior derivative δ on the space $\Omega^*(M)$, we define the *differential of the Lie algebroid* A, $d : \Gamma(\wedge^k A^*) \to \Gamma(\wedge^{k+1} A^*)$, as follows. For $\omega \in \Gamma(\wedge^k A^*)$ and $X_0, \ldots, X_k \in \Gamma(A)$,

$$\begin{aligned}
d\omega(X_0, \ldots, X_k) = &\sum_{i=0}^{k} (-1)^i \rho(X_i)\big(\omega(X_0, \ldots, \hat{X}_i, \ldots, X_k)\big)\\
&+ \sum_{i<j} (-1)^{i+j} \omega\big([\![X_i, X_j]\!], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_k\big).
\end{aligned}$$

Furthermore, since $d^2 = 0$, we have the corresponding cohomology spaces. This cohomology is the *Lie algebroid cohomology with trivial coefficients* (see [14]). Using the above definitions, it follows that a 1-cochain $\phi_0 \in \Gamma(A^*)$ is a 1-cocycle if and only if

$$\phi_0[\![X, Y]\!] = \rho(X)(\phi_0(Y)) - \rho(Y)(\phi_0(X)) \tag{2}$$

for all $X, Y \in \Gamma(A)$.

Example 2.1. (i) A real Lie algebra of finite dimension is a Lie algebroid over a point. Another trivial example of a Lie algebroid is the triple $(TM, [\ ,\ ], Id)$, where M is a differentiable manifold and $Id : TM \to TM$ is the identity map.

(ii) If M is a differentiable manifold, then the triple $(TM \times \mathbb{R}, [\ ,\ ], \pi)$ is a Lie algebroid over M, where $\pi : TM \times \mathbb{R} \to TM$ is the canonical projection over the first factor and $[\ ,\ ]$ is the bracket given by (see [14, 17])

$$[(X, f), (Y, g)] = ([X, Y], X(g) - Y(f)) \tag{3}$$

for $(X, f), (Y, g) \in \mathfrak{X}(M) \oplus C^\infty(M, \mathbb{R}) \cong \Gamma(TM \times \mathbb{R})$. The space $\Gamma(\wedge^k(T^*M \times \mathbb{R}))$ may be identified with the direct sum $\Omega^k(M) \oplus \Omega^{k-1}(M)$ and, under this identification, the differential $\tilde{\delta}$ of $TM \times \mathbb{R}$ is

$$\tilde{\delta}(\beta, \gamma) = (\delta\beta, -\delta\gamma) \tag{4}$$

for $(\beta, \gamma) \in \Omega^k(M) \oplus \Omega^{k-1}(M)$. Note that the pair $\phi_0 = (0,1) \in \Omega^1(M) \oplus C^\infty(M, \mathbb{R})$ is a 1-cocycle of $TM \times \mathbb{R}$ (see [6]).

(iii) *The Lie algebroid* $(T^*M \times \mathbb{R}, [\![\ ,\ ]\!]_{(\Lambda,E)}, \widetilde{\#}_{(\Lambda,E)})$ *of a Jacobi manifold*: A Jacobi structure on a manifold M is a 2-vector Λ and a vector field E on M such that (see [13])

$$[\Lambda, \Lambda] = 2E \wedge \Lambda, \qquad [E, \Lambda] = 0. \tag{5}$$

Here $[\ ,\ ]$ denotes the Schouten-Nijenhuis bracket.

The manifold M endowed with a Jacobi structure is called a *Jacobi manifold*. If (M, Λ, E) is a Jacobi manifold, we can define a bracket of functions (the *Jacobi bracket*) by

$$\{f, g\} = \Lambda(\delta f, \delta g) + fE(g) - gE(f) \tag{6}$$

for all $f, g \in C^\infty(M, \mathbb{R})$. The space $C^\infty(M, \mathbb{R})$ endowed with the Jacobi bracket is a *local Lie algebra* in the sense of Kirillov (see [9]). Conversely, a structure of local Lie algebra on $C^\infty(M, \mathbb{R})$ defines a Jacobi structure on M (see [5, 9]).

If (M, Λ, E) is a Jacobi manifold, the 1-jet bundle $T^*M \times \mathbb{R} \to M$ admits a Lie algebroid structure $([\![\ ,\ ]\!]_{(\Lambda,E)}, \widetilde{\#}_{(\Lambda,E)})$, where $[\![\ ,\ ]\!]_{(\Lambda,E)}$ and $\widetilde{\#}_{(\Lambda,E)}$ are defined by

$$\begin{aligned}
&[\![(\alpha, f), (\beta, g)]\!]_{(\Lambda,E)} \\
&= \Big(\mathcal{L}_{\#_\Lambda(\alpha)}\beta - \mathcal{L}_{\#_\Lambda(\beta)}\alpha - \delta(\Lambda(\alpha, \beta)) + f\mathcal{L}_E\beta - g\mathcal{L}_E\alpha - i(E)(\alpha \wedge \beta), \\
&\qquad \Lambda(\beta, \alpha) + \#_\Lambda(\alpha)(g) - \#_\Lambda(\beta)(f) + fE(g) - gE(f)\Big), \\
&\widetilde{\#}_{(\Lambda,E)}(\alpha, f) = \#_\Lambda(\alpha) + fE
\end{aligned} \tag{7}$$

for $(\alpha, f), (\beta, g) \in \Omega^1(M) \oplus C^\infty(M, \mathbb{R})$ (see [8]). Here, $\mathcal{L}$ denotes the Lie derivative operator and $\#_\Lambda : \Omega^1(M) \to \mathfrak{X}(M)$ the homomorphism of $C^\infty(M, \mathbb{R})$-modules given by $\#_\Lambda(\alpha) = i(\alpha)\Lambda$.

In the particular case when (M, Λ) is a Poisson manifold (i.e., $E = 0$) we recover, by projection, the Lie algebroid $(T^*M, [\![\ ,\ ]\!]_\Lambda, \#_\Lambda)$, where $[\![\ ,\ ]\!]_\Lambda$ is the bracket of 1-forms defined by (see [3]):

$$[\![\alpha, \beta]\!]_\Lambda = \mathcal{L}_{\#_\Lambda(\alpha)}\beta - \mathcal{L}_{\#_\Lambda(\beta)}\alpha - \delta(\Lambda(\alpha, \beta)). \tag{8}$$

On the other hand, if $\mathcal{V}^r(M)$ is the space of r-vectors on M, then the space $\Gamma(\wedge^k(TM \times \mathbb{R}))$ may be identified with the direct sum $\mathcal{V}^k(M) \oplus \mathcal{V}^{k-1}(M)$ and, under this identification, the differential $d_{(\Lambda,E)}$ of the Lie algebroid $(T^*M \times \mathbb{R}, [\![\ ,\ ]\!]_{(\Lambda,E)}, \widetilde{\#}_{(\Lambda,E)})$ is given by

$$d_{(\Lambda,E)}(P,Q) = \big(-[\Lambda,P] + kE \wedge P + \Lambda \wedge Q,\\ [\Lambda,Q] - (k-1)E \wedge Q + [E,P]\big) \tag{9}$$

for $(P,Q) \in \mathcal{V}^k(M) \oplus \mathcal{V}^{k-1}(M)$ (see [12]). Note that the pair $X_0{=}(-E,0)$ is a 1-cocycle of $T^*M \times \mathbb{R}$.

(iv) Let $\mathcal{N} : \mathfrak{X}(M) \oplus C^\infty(M,\mathbb{R}) \to \mathfrak{X}(M) \oplus C^\infty(M,\mathbb{R})$ be a $C^\infty(M,\mathbb{R})$-linear map. We say that $\mathcal{N}$ is a *Nijenhuis operator on* $TM \times \mathbb{R}$ if it has vanishing Nijenhuis torsion $\mathcal{T}(\mathcal{N})$, where $\mathcal{T}(\mathcal{N})$ is defined by

$$\mathcal{T}(\mathcal{N})((X,f),(Y,g)) = [\mathcal{N}(X,f),\mathcal{N}(Y,g)] - \mathcal{N}[\mathcal{N}(X,f),(Y,g)]\\ - \mathcal{N}[(X,f),\mathcal{N}(Y,g)] + \mathcal{N}^2[(X,f),(Y,g)]$$

for $(X,f),(Y,g) \in \mathfrak{X}(M) \oplus C^\infty(M,\mathbb{R})$. In this case, $\mathcal{N}$ defines a deformed Lie algebroid structure $([\ ,\ ]_{\mathcal{N}}, \pi \circ \mathcal{N})$ on $TM \times \mathbb{R}$, where $\pi : TM \times \mathbb{R} \to TM$ is the canonical projection over the first factor and $[\ ,\ ]_{\mathcal{N}}$ is given by

$$[(X,f),(Y,g)]_{\mathcal{N}} = [\mathcal{N}(X,f),(Y,g)] + [(X,f),\mathcal{N}(Y,g)] - \mathcal{N}[(X,f),(Y,g)].$$

The differential $d_{\mathcal{N}}$ of the Lie algebroid $(TM \times \mathbb{R}, [\ ,\ ]_{\mathcal{N}}, \pi \circ \mathcal{N})$ is

$$d_{\mathcal{N}} = i_{\mathcal{N}} \circ \tilde{\delta} - \tilde{\delta} \circ i_{\mathcal{N}}, \tag{10}$$

$i_{\mathcal{N}} : \Omega^k(M) \oplus \Omega^{k-1}(M) \to \Omega^k(M) \oplus \Omega^{k-1}(M)$ being the contraction by $\mathcal{N}$ defined by

$$\big(i_{\mathcal{N}}(\beta,\gamma)\big)\big((X_1,f_1),\dots,(X_k,f_k)\big)\\ = \sum_{i=1}^{k} (\beta,\gamma)\big((X_1,f_1),\dots,\mathcal{N}(X_i,f_i),\dots,(X_k,f_k)\big) \tag{11}$$

for $(\beta,\gamma) \in \Omega^k(M) \oplus \Omega^{k-1}(M)$ and $(X_1,f_1),\dots,(X_k,f_k) \in \mathfrak{X}(M) \oplus C^\infty(M,\mathbb{R})$. Note that $\phi_0{=}\mathcal{N}^*(0,1) \in \Omega^1(M) \oplus C^\infty(M,\mathbb{R}) \cong \Gamma(T^*M \times \mathbb{R})$ is a 1-cocycle of the Lie algebroid $(TM \times \mathbb{R}, [\ ,\ ]_{\mathcal{N}}, \pi \circ \mathcal{N})$, where $\mathcal{N}^* : \Omega^1(M) \oplus C^\infty(M,\mathbb{R}) \to \Omega^1(M) \oplus C^\infty(M,\mathbb{R})$ is the adjoint homomorphism of $\mathcal{N} : \mathfrak{X}(M) \oplus C^\infty(M,\mathbb{R}) \to \mathfrak{X}(M) \oplus C^\infty(M,\mathbb{R})$.

2.2. Generalized Lie bialgebroids. In this Section, we will recall the definition of a generalized Lie bialgebroid (see [6]) and we will give a new characterization of this notion which has been recently obtained in [4].

First, we will exhibit some results about the differential calculus on Lie algebroids in the presence of a 1-cocycle (see [6]).

If $(A, [\![\ ,\ ]\!], \rho)$ is a Lie algebroid over M and, in addition, we have a 1-cocycle $\phi_0 \in \Gamma(A^*)$ then, using (2), we can define a representation of the Lie algebra $(\Gamma(A), [\![\ ,\ ]\!])$ on the space $C^\infty(M, \mathbb{R})$ given by $\rho_{\phi_0}(X)(f) = \rho(X)(f) + \phi_0(X)f$, for $X \in \Gamma(A)$ and $f \in C^\infty(M, \mathbb{R})$. The resultant cohomology operator d_{ϕ_0} associated with this representation is called the ϕ_0-*differential of* A and its expression, in terms of the differential d of A, is

$$d_{\phi_0}\omega = d\omega + \phi_0 \wedge \omega, \quad \text{for } \omega \in \Gamma(\wedge^k A^*). \tag{12}$$

The ϕ_0-differential of A allows us to define, in a natural way, the ϕ_0-*Lie derivative* by a section $X \in \Gamma(A)$, $(\mathcal{L}_{\phi_0})_X : \Gamma(\wedge^k A^*) \to \Gamma(\wedge^k A^*)$, as the commutator of d_{ϕ_0} and the contraction by X, that is,

$$(\mathcal{L}_{\phi_0})_X = d_{\phi_0} \circ i(X) + i(X) \circ d_{\phi_0}. \tag{13}$$

Note that if $\phi_0 = 0$ then we obtain the usual Lie derivative of A.

On the other hand, imitating the definition of the Schouten bracket of two multilinear first-order differential operators on the space of C^∞ real-valued functions on a manifold N (see [1]), we may introduce the ϕ_0-*Schouten bracket* of a k-section P and a k'-section P' as the $(k + k' - 1)$-section given by

$$[\![P, P']\!]_{\phi_0} = [\![P, P']\!] + (-1)^{k+1}(k-1)P \wedge (i(\phi_0)P') - (k'-1)(i(\phi_0)P) \wedge P', \tag{14}$$

where $[\![\ ,\ ]\!]$ is the usual Schouten bracket of A. We remark that $[\![\ ,\ ]\!]_{\phi_0}$ is skew-symmetric and that $[\![\ ,\ ]\!]_{\phi_0}$ satisfies the graded Jacobi identity. In particular, we have that

$$[\![P, f]\!]_{\phi_0} = i(d_{\phi_0} f)P \tag{15}$$

for $f \in C^\infty(M, \mathbb{R})$ and $P \in \Gamma(\wedge A)$ (for more details, see [6]) .

Now, suppose that $(A, [\![\ ,\ ]\!], \rho)$ is a Lie algebroid and that $\phi_0 \in \Gamma(A^*)$ is a 1-cocycle. Assume also that the dual bundle A^* admits a Lie algebroid structure $([\![\ ,\ ]\!]_*, \rho_*)$ and that $X_0 \in \Gamma(A)$ is a 1-cocycle. Then, DEFINITION 2.2. ([6]) The pair $((A, \phi_0), (A^*, X_0))$ is said to be a generalized Lie bialgebroid if

$$d_{*X_0}[\![X, Y]\!] = [\![X, d_{*X_0}Y]\!]_{\phi_0} - [\![Y, d_{*X_0}X]\!]_{\phi_0}, \tag{16}$$

$$(\mathcal{L}_{*X_0})_{\phi_0}P + (\mathcal{L}_{\phi_0})_{X_0}P = 0 \tag{17}$$

for $X, Y \in \Gamma(A)$ and $P \in \Gamma(\wedge^k A)$, where d_{*X_0} (respectively, $\mathcal{L}_{*X_0}$) is the X_0-differential (respectively, the X_0-Lie derivative) of A^*.

EXAMPLE 2.3. (i) In the particular case when $\phi_0 = 0$ and $X_0 = 0$, (16) and (17) are equivalent to the single condition

$$d_*[\![X, Y]\!] = [\![X, d_*Y]\!] - [\![Y, d_*X]\!].$$

Thus, the pair $((A, 0), (A^*, 0))$ is a generalized Lie bialgebroid if and only if the pair (A, A^*) is a Lie bialgebroid in the sense of Mackenzie-Xu [15].

(ii) Let (M, Λ, E) be a Jacobi manifold. In [6] we proved that the pair $((TM \times \mathbb{R}, \phi_0),(T^*M \times \mathbb{R}, X_0))$ is a generalized Lie bialgebroid, where ϕ_0 and X_0 are the 1-cocycles on $TM \times \mathbb{R}$ and $T^*M \times \mathbb{R}$ given by

$$\phi_0 = (0, 1) \in \Omega^1(M) \oplus C^\infty(M, \mathbb{R}) \cong \Gamma(T^*M \times \mathbb{R}),$$
$$X_0 = (-E, 0) \in \mathfrak{X}(M) \oplus C^\infty(M, \mathbb{R}) \cong \Gamma(TM \times \mathbb{R}).$$

In [10], it was given an alternative definition of Lie bialgebroids. Suggested by this result, Grabowski and Marmo obtained in [4] a new characterization of generalized Lie bialgebroids as follows. Consider the bracket $[\![\ ,\ ]\!]'_{\phi_0}$ of a k-section P and a k'-section P' as the $(k + k' - 1)$-section given by

$$[\![P, P']\!]'_{\phi_0} = (-1)^{k+1}[\![P, P']\!]_{\phi_0}.$$

Then, we have that

THEOREM 2.4. ([4]) *Let $(A, [\![\ ,\ ]\!], \rho)$ be a Lie algebroid and $\phi_0 \in \Gamma(A^*)$ be a 1-cocycle. Assume also that the dual bundle A^* admits a Lie algebroid structure $([\![\ ,\ ]\!]_*, \rho_*)$ and that $X_0 \in \Gamma(A)$ is a 1-cocycle. Then, $((A, \phi_0), (A^*, X_0))$ is a generalized Lie bialgebroid if and only if d_{*X_0} is a derivation with respect to $(\oplus_k\Gamma(\wedge^k A), [\![\ ,\ ]\!]'_{\phi_0})$, that is,*

$$d_{*X_0}[\![P, P']\!]'_{\phi_0} = [\![d_{*X_0}P, P']\!]'_{\phi_0} + (-1)^{k+1}[\![P, d_{*X_0}P']\!]'_{\phi_0} \tag{18}$$

for $P \in \Gamma(\wedge^k A)$ and $P' \in \Gamma(\wedge^* A)$.

3. Generalized Lie bialgebroids and strong Jacobi-Nijenhuis structures

We shall now show another class of examples of generalized Lie bialgebroids which comes from Jacobi structures and Nijenhuis operators.

Let (M, Λ, E) be a Jacobi manifold. The pair (Λ, E) may be viewed as a section of the vector bundle $\wedge^2(TM \times \mathbb{R}) \to M$ and thus it induces a homomorphism of $C^\infty(M,\mathbb{R})$-modules $\#_{(\Lambda,E)} : \Omega^1(M) \oplus C^\infty(M,\mathbb{R}) \to \mathfrak{X}(M) \oplus C^\infty(M,\mathbb{R})$ given by

$$\#_{(\Lambda,E)}(\alpha, h) = (i(\alpha)\Lambda + h\,E, -\alpha(E)) \tag{19}$$

for $(\alpha, h) \in \Omega^1(M) \oplus C^\infty(M,\mathbb{R})$. In addition, we may consider the Lie algebroid $(T^*M \times \mathbb{R}, [\![\ ,\ ]\!]_{(\Lambda,E)}, \widetilde{\#}_{(\Lambda,E)})$ and the pair $X_0 = (-E, 0)$ is a 1-cocycle of this Lie algebroid (see Example 2.1 (iii)). On the other hand, the pair $\phi_0 = (0, 1) \in \Omega^1(M) \oplus C^\infty(M,\mathbb{R})$ is a 1-cocycle of the Lie algebroid $(TM \times \mathbb{R}, [\ ,\ ], \pi)$ and the ϕ_0-differential $\tilde{\delta}_{(0,1)}$ of $TM \times \mathbb{R}$ is given by

$$\tilde{\delta}_{(0,1)}(\beta, \gamma) = (\delta\beta, \beta - \delta\gamma) \tag{20}$$

for $(\beta, \gamma) \in \Omega^k(M) \oplus \Omega^{k-1}(M)$ (see Example 2.1 (ii)).

Thus, using (7), (14), (19) and (20), it follows that the X_0-Schouten bracket $([\![\ ,\ ]\!]_{(\Lambda,E)})_{X_0}$ satisfies the following relations:

$$([\![\tilde{\delta}_{(0,1)} f, g]\!]_{(\Lambda,E)})_{X_0} = \tilde{\delta}_{(0,1)} g \cdot (\#_{(\Lambda,E)}(\tilde{\delta}_{(0,1)} f)), \tag{21}$$

$$\begin{aligned} ([\![(\alpha,h), \tilde{\delta}_{(0,1)} f]\!]_{(\Lambda,E)})_{X_0} = {} & ([\![\tilde{\delta}_{(0,1)}(\alpha,h), f]\!]_{(\Lambda,E)})_{X_0} \\ & + \tilde{\delta}_{(0,1)}([\![(\alpha,h), f]\!]_{(\Lambda,E)})_{X_0}, \end{aligned} \tag{22}$$

$$([\![\tilde{\delta}_{(0,1)}(\alpha,h), \tilde{\delta}_{(0,1)} f]\!]_{(\Lambda,E)})_{X_0} = -\tilde{\delta}_{(0,1)}([\![\tilde{\delta}_{(0,1)}(\alpha,h), f]\!]_{(\Lambda,E)})_{X_0}, \tag{23}$$

for $(\alpha, h) \in \Omega^1(M) \oplus C^\infty(M,\mathbb{R})$ and $f, g \in C^\infty(M,\mathbb{R})$.

Now, we consider a Nijenhuis operator $\mathcal{N}$ on $TM \times \mathbb{R}$ and the corresponding Lie algebroid $(TM \times \mathbb{R}, [\ ,\ ]_{\mathcal{N}}, \pi \circ \mathcal{N})$ (see Example 2.1 (iv)). As we know, $\phi_0 = \mathcal{N}^*(0,1)$ is a 1-cocycle. Furthermore, if $(d_{\mathcal{N}})_{\phi_0}$ is the ϕ_0-differential of $(TM \times \mathbb{R}, [\ ,\ ]_{\mathcal{N}}, \pi \circ \mathcal{N})$, then, using (10), (11) and (20), we have that

$$(d_{\mathcal{N}})_{\phi_0} f = \mathcal{N}^* \tilde{\delta}_{(0,1)} f, \tag{24}$$

$$(d_{\mathcal{N}})_{\phi_0}(\alpha, h) = i_{\mathcal{N}}(\tilde{\delta}_{(0,1)}(\alpha, h)) - \tilde{\delta}_{(0,1)} \mathcal{N}^*(\alpha, h), \tag{25}$$

$$(d_{\mathcal{N}})_{\phi_0}\tilde{\tilde{\delta}}_{(0,1)}f = -\tilde{\tilde{\delta}}_{(0,1)}(\mathcal{N}^*\tilde{\tilde{\delta}}_{(0,1)}f), \tag{26}$$

for $(\alpha,h)\in\Omega^1(M)\oplus C^\infty(M,\mathbb{R})$ and $f\in C^\infty(M,\mathbb{R})$, $i_{\mathcal{N}}:\Omega^k(M)\oplus\Omega^{k-1}(M)\to\Omega^k(M)\oplus\Omega^{k-1}(M)$ being the contraction by $\mathcal{N}$.

On the other hand, suppose that (Λ,E) and $\mathcal{N}$ satisfy that

$$\mathcal{N}\circ\#_{(\Lambda,E)} = \#_{(\Lambda,E)}\circ\mathcal{N}^*. \tag{27}$$

In this case, we can define the pair (Λ_1,E_1) formed by the 2-vector Λ_1 and the vector field E_1 characterized by

$$\#_{(\Lambda_1,E_1)} = \#_{(\Lambda,E)}\circ\mathcal{N}^*. \tag{28}$$

We say that the pair $((\Lambda,E),\mathcal{N})$ is a *strong Jacobi-Nijenhuis structure* if and only if (27) holds and the concomitant of (Λ,E) and $\mathcal{N}$, $C((\Lambda,E),\mathcal{N})$, identically vanishes, where $C((\Lambda,E),\mathcal{N})$ is given by

$$\begin{aligned}C((\Lambda,E),\mathcal{N})((\alpha,f),(\beta,g)) &= [\![(\alpha,f),(\beta,g)]\!]_{(\Lambda_1,E_1)} - [\![\mathcal{N}^*(\alpha,f),(\beta,g)]\!]_{(\Lambda,E)}\\ &\quad - [\![(\alpha,f),\mathcal{N}^*(\beta,g)]\!]_{(\Lambda,E)} + \mathcal{N}^*[\![(\alpha,f),(\beta,g)]\!]_{(\Lambda,E)},\end{aligned}$$

for $(\alpha,f),(\beta,g)\in\Omega^1(M)\oplus C^\infty(M,\mathbb{R})$.

Remark 3.1. In [16] is introduced the notion of a Jacobi-Nijenhuis structure imposing weaker conditions than we have adopted here. Moreover, in [19, 20] are established some local models of Jacobi-Nijenhuis manifolds and a reduction theorem is obtained. In a different direction, in [21] is given another relation between Jacobi structures and Nijenhuis operators. In addition, the author compares both approaches (see [21]).

EXAMPLE 3.2. Let M be a $2n+1$-dimensional manifold and η a 1-form on M. We say that (M,η) is a *contact manifold* if $\eta\wedge(\delta\eta)^n\neq 0$ at every point (see, for instance [13]). A contact manifold (M,η) is a Jacobi manifold whose associated Jacobi structure (Λ,E) is given by

$$\Lambda(\alpha,\beta) = \delta\eta(\flat_\eta^{-1}(\alpha),\flat_\eta^{-1}(\beta)), \qquad E=\flat_\eta^{-1}(\eta)$$

for $\alpha,\beta\in\Omega^1(M)$, $\flat_\eta:\mathfrak{X}(M)\to\Omega^1(M)$ being the isomorphism of $C^\infty(M,\mathbb{R})$-modules defined by $\flat_\eta(X)=i(X)(\delta\eta)+\eta(X)\eta$. In this particular case, we have that the homomorphism $\#_{(\Lambda,E)}$ given by (19) is an isomorphism.

Suppose that (M,η) is a contact manifold with associated Jacobi structure (Λ,E) and that (Λ_1,E_1) is a Jacobi structure on M compatible with (Λ,E), that is, $(\Lambda+\Lambda_1, E+E_1)$ is a Jacobi structure. Let us consider the $C^\infty(M,\mathbb{R})$-linear map $\mathcal{N} = \#_{(\Lambda_1,E_1)}\circ(\#_{(\Lambda,E)})^{-1}$. Then, using the results in [16] and since the homomorphism $\#_{(\Lambda,E)}$ is an isomorphism, we deduce that $((\Lambda,E),\mathcal{N})$ is a strong Jacobi-Nijenhuis structure.

An explicit example of the precedent construction is the following one.

Let M be the product manifold $\mathbb{R}\times T^*Q$, where Q is a smooth manifold of dimension m. Denote by λ_Q the Liouville 1-form of T^*Q and by Λ_Q the Poisson 2-vector associated with the canonical symplectic structure $\Omega_Q = -\delta\lambda_Q$. If $pr_1 : M \to \mathbb{R}$ and $pr_2 : M \to T^*Q$ are the canonical projections onto the first and second factor, respectively, then a direct computation proves that

$$\eta = pr_1^*(\delta t) - pr_2^*(\lambda_Q)$$

is a contact 1-form on M. In fact, if $(q^1,\ldots,q^m,p_1,\ldots,p_m)$ are fibred coordinates on T^*Q, we have that

$$\eta = \delta t - \sum_{i=1}^m p_i\,\delta q^i.$$

Thus, the Jacobi structure (Λ,E) associated with the contact 1-form η is given by

$$\Lambda = \sum_{i=1}^m \Big(\frac{\partial}{\partial q^i} + p_i\frac{\partial}{\partial t}\Big)\wedge\frac{\partial}{\partial p_i}, \quad E = \frac{\partial}{\partial t}. \tag{29}$$

On the other hand, the 2-vector Λ_Q induces, in a natural way, a Poisson structure Λ_1 on M, whose local expression is

$$\Lambda_1 = \sum_{i=1}^m \frac{\partial}{\partial q^i}\wedge\frac{\partial}{\partial p_i}. \tag{30}$$

Using (29) and (30), we deduce that the Jacobi structure (Λ,E) and the Poisson structure Λ_1 are compatible. Therefore, the pair $((\Lambda,E),\mathcal{N})$ is a strong Jacobi-Nijenhuis structure, where $\mathcal{N} : \mathfrak{X}(M)\oplus C^\infty(M,\mathbb{R}) \to \mathfrak{X}(M)\oplus C^\infty(M,\mathbb{R})$ is the map defined by

$$\mathcal{N} = \#_{(\Lambda_1,0)}\circ(\#_{(\Lambda,E)})^{-1}.$$

From (29) and (30), it follows that

$$\mathcal{N} = Id - \big(\frac{\partial}{\partial t},0\big)\otimes(\delta t,0) - (-\Delta,1)\otimes(0,1),$$

Δ being the Liouville vector field of T^*Q.

Next, we relate strong Jacobi-Nijenhuis structures and generalized Lie bialgebroids in the following result.

THEOREM 3.3. *Let (Λ, E) be a Jacobi structure on a manifold M and $\mathcal{N}$ a Nijenhuis operator on $TM \times \mathbb{R}$. Consider on $TM \times \mathbb{R}$ (resp., $T^*M \times \mathbb{R}$) the Lie algebroid structure associated with $\mathcal{N}$ (resp., (Λ, E)). Moreover, consider the 1-cocycle $\phi_0 = \mathcal{N}^*(0,1)$ (resp., $X_0 = (-E, 0)$) on $TM \times \mathbb{R}$ (resp., $T^*M \times \mathbb{R}$). Then, $((\Lambda, E), \mathcal{N})$ is a strong Jacobi-Nijenhuis structure if and only if $((TM \times \mathbb{R}, \phi_0), (T^*M \times \mathbb{R}, X_0))$ is a generalized Lie bialgebroid.*

Proof. Let us set

$$\begin{aligned} A((\beta,\gamma),(\beta',\gamma')) = {} & (d_{\mathcal{N}})_{\phi_0}([\![(\beta,\gamma),(\beta',\gamma')]\!]_{(\Lambda,E)})_{X_0} \\ & + ([\![(d_{\mathcal{N}})_{\phi_0}(\beta,\gamma),(\beta',\gamma')]\!]_{(\Lambda,E)})_{X_0} \\ & + (-1)^k([\![(\beta,\gamma),(d_{\mathcal{N}})_{\phi_0}(\beta',\gamma')]\!]_{(\Lambda,E)})_{X_0} \end{aligned} \tag{31}$$

for $(\beta, \gamma) \in \Omega^k(M) \oplus \Omega^{k-1}(M)$ and $(\beta', \gamma') \in \Omega^*(M) \oplus \Omega^{*-1}(M)$.

Using (1), (14) and the properties of the differential $d_{\mathcal{N}}$, we deduce that A=0 if and only if

$$A(f,g) = 0, \quad A(\tilde{\delta}_{(0,1)}f, g) = 0, \quad A(\tilde{\delta}_{(0,1)}f, \tilde{\delta}_{(0,1)}g) = 0 \tag{32}$$

for $f, g \in C^\infty(M, \mathbb{R})$. Note that if $(\beta, \gamma) \in \Omega^k(M) \oplus \Omega^{k-1}(M)$ then for every point x of M there exists an open subset U of M, $x \in U$, such that on U

$$(\beta,\gamma) = \sum_{i=1,\ldots,r} f_1^i\, \tilde{\delta}_{(0,1)} f_2^i \wedge \ldots \wedge \tilde{\delta}_{(0,1)} f_k^i,$$

with $f_j^i \in C^\infty(U, \mathbb{R})$, for all i and j.

Now suppose that f and g are real C^∞-functions on M. Then, using (21), (24) and (31), we get that

$$A(f,g) = \tilde{\delta}_{(0,1)}g \cdot ((\#_{(\Lambda,E)} \circ \mathcal{N}^* - \mathcal{N} \circ \#_{(\Lambda,E)})\tilde{\delta}_{(0,1)}f). \tag{33}$$

On the other hand, from (21), (22), (24), (26) and (31), we have that

$$A(\tilde{\delta}_{(0,1)}f, g) = C((\Lambda,E),\mathcal{N})(\tilde{\delta}_{(0,1)}f, \tilde{\delta}_{(0,1)}g) + \tilde{\delta}_{(0,1)}A(f,g). \tag{34}$$

Finally, using (22), (23), (25), (26) and (31), we obtain that

$$A(\tilde{\delta}_{(0,1)}f, \tilde{\delta}_{(0,1)}g) = -\tilde{\delta}_{(0,1)}(C((\Lambda,E),\mathcal{N})(\tilde{\delta}_{(0,1)}f, \tilde{\delta}_{(0,1)}g)). \tag{35}$$

Therefore, from (33), (34), (35) and Theorem 2.4, we conclude the result. ∎

Remark 3.4. After finishing this paper, Nunes da Costa sent to us a recent preprint [18] where the above theorem was proved by using other techniques.

As a consequence of Theorem 3.3, we recover a result obtained, with weaker hypotheses, in [16, 19].

Corollary 3.5. *Let $((\Lambda, E), \mathcal{N})$ be a strong Jacobi-Nijenhuis structure on a manifold M. Then the 2-vector Λ_1 and the vector field E_1 characterized by (28) define a Jacobi structure on M.*

Proof. Since $((TM \times \mathbb{R}, \phi_0), (T^*M \times \mathbb{R}, X_0))$ is a generalized Lie bialgebroid, we can define a Jacobi bracket $\{\ ,\ \}_1$ on M given by

$$\{f, g\}_1 = (d_{\mathcal{N}})_{\phi_0} f \cdot d_{*X_0} g$$

for $f, g \in C^\infty(M, \mathbb{R})$ (see Theorem 3.7 in [6]). Using (21) and (24), we deduce that

$$\begin{aligned}\{f, g\}_1 &= ([\![\mathcal{N}^* \tilde{\delta}_{(0,1)} f, g]\!]_{(\Lambda,E)})_{X_0} = \tilde{\delta}_{(0,1)} g \cdot ((\#_{(\Lambda,E)} \circ \mathcal{N}^*)(\tilde{\delta}_{(0,1)} f)) \\ &= \Lambda_1(\delta f, \delta g) + f E_1(g) - g E_1(f).\end{aligned}$$

Therefore, we conclude our result. ∎

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