EXTRACTA MATHEMATICAE Vol. **17**, Núm. 3, 303 – 329 (2002)
IV *Curso Espacios de Banach y Operadores.*
Laredo, Agosto de 2001.

An Introduction to Polynomials on Banach Spaces †

Richard Aron

Department of Mathematical Sciences, Kent State University,
Kent, Ohio 44242, *USA*

e-mail: aron@math.kent.edu

AMS *Subject Class.* (2000): 46B28, 46B04

1. Introduction.

Probably the principal reason for interest in polynomials is that they are much easier to work with than the functions which they approximate. Another excellent reason to study polynomials is that they play a crucial 'intermediate' role between linear mappings and arbitrary continuous or differentiable functions.

In this series of lectures, we give a personal introduction to the study of polynomials, which we hope will clarify the statements made in the above paragraph. In §2, we define what we mean by polynomials on a Banach space and we give some fundamental results on polynomials and their norms. We also describe several approximation results and questions, including the one (concerning a generalization of the Stone-Weierstrass theorem) which originally piqued our interest in this area over 25 years ago and which remains unsolved. In §3, we study norms of polynomials, examining the relation between the norm of a polynomial in several variables and its coefficients. Our principal interest will be in finding *lower bounds* for the norm of a polynomial. In §4, we study the problem of finding 'large' subspaces on which a polynomial is constant. This topic may at first seem peculiar, since it is by no means clear (nor true!) that if $P(x) = P(y) = c$, then $P(x + y) = c$. As we will see, there are somewhat surprising results which can be obtained in both the case of real and complex polynomials. Finally, in §5, we study problems involving the search for Hahn-Banach type theorems concerning extensions of polynomials from a subspace of a Banach space to the entire space.

†Supported in part by a grant of the Ministerio de Educación y Cultura of Spain (SAB1999-0214).

The primary purpose of these notes is to introduce (hopefully) interested people to this nice area. As a consequence, we have omitted mention of a number of beautiful, sometimes more technical, current results of colleagues, to whom we unreservedly apologize. A general work on polynomials in infinite dimensional spaces, containing this material and much more, is the recent book by Sean Dineen [13], which also contains an invaluable list of references.

Finally, we thank Manuel González for giving us this opportunity to 'advertise' this material.

2. Fundamental notions

Throughout, E and F will be complex Banach spaces over $\mathbb{R}$ or $\mathbb{C}$. To define a polynomial on E, we first need to recall the notion of multilinear form.

Definition 2.1. (a) For $n \in \mathbb{N}$, $\mathcal{L}({}^nE, F)$ denotes the space of continuous n-linear mappings $A : E \times \cdots \times E \to F$. It is easy that $\mathcal{L}({}^nE, F)$ is a Banach space with norm given by

$$A \to ||A|| \equiv \sup_{x_j \in E,\, ||x_j|| \leq 1,\, 1 \leq j \leq n} ||A(x_1, \dots, x_n)||.$$

(b) For $n \in \mathbb{N}$, let $\mathcal{P}({}^nE, F) = \{P : E \to F\colon$ for some $A \in \mathcal{L}({}^nE, F)$, we have $P(x) = A(x, \dots, x)$ for every $x \in E\}$. This space, of n-homogeneous polynomials, is a Banach space under the norm

$$P \to ||P|| \equiv \sup_{x \in E,\, ||x|| \leq 1} ||P(x)||.$$

We say that P is a polynomial on E, writing $P \in \mathcal{P}(E)$, if P is a finite sum of homogeneous polynomials.

We write $\mathcal{L}({}^nE)$ for $\mathcal{L}({}^nE, \mathbb{K})$. By definition, $\mathcal{L}({}^0E, F) = F$; also, $\mathcal{L}({}^1E, F)$ is just the continuous linear mappings from E to F. Similar remarks apply to $\mathcal{P}({}^nE)$.

Given an n-homogeneous polynomial P, it is straightforward that we may take the associated n-linear mapping A to be *symmetric;* in fact it is not hard to see that $\mathcal{P}({}^nE; F)$ and the space of symmetric n-linear mappings $\mathcal{L}_s({}^nE, F)$ are isomorphic as Banach spaces. Note in particular that if P is a scalar valued 2-homogeneous polynomial on $\mathbb{K}^n$, then P arises from a symmetric bilinear form $A : \mathbb{K}^n \times \mathbb{K}^n \to \mathbb{K}$, which we may in turn associate with a symmetric matrix A via $A(x, y) = x^tAy$.

If we start with a symmetric n-linear mapping, it is trivial to find the associated n-homogeneous polynomial. The converse is the content of the following, which we state in probabilistic form:

THEOREM 2.2. (Polarization Formula) *Let $r_1, \dots, r_n$ be n independent normalized random variables on a probability space Ω. Let $x_1, \dots, x_n \in E$, and let $P \in \mathcal{P}(^nE, F)$. Then the associated symmetric n-linear mapping A is given by the formula*

$$A(x_1, \dots, x_n) = \frac{1}{n!} \int_\Omega \left[\overline{r_1(t)} \cdots \overline{r_n(t)} P\big(\sum_{j=1}^n r_j(t) x_j \big) \right].$$

Proof. Recall that our hypotheses on the random variables simply mean that $\int_\Omega |r|^2 = 1, \int_\Omega r = 0$, and that $\int_\Omega r_j^s r_k^t = \int_\Omega r_j^s \int_\Omega r_k^t$ for $j \neq k$. The right hand side of the equality is $\frac{1}{n!}$ times $\int_\Omega \overline{r_1} \cdots \overline{r_n} P(\sum_{j=1}^n r_j x_j) = \int_\Omega \overline{r_1} \cdots \overline{r_n} A(\sum r_j x_j, \dots, \sum r_j x_j)$, which equals

$$\sum_{0 \le j_1, \dots, j_n,\ \sum_{t=1}^n j_t = n} \binom{n}{j_1, \dots, j_n} \int_\Omega \overline{r_1} r_1^{j_1} \cdots \overline{r_n} r_n^{j_n} A(x_1^{j_1}, \dots, x_n^{j_n})$$
$$= \sum_{\text{same indices}} \binom{n}{j_1, \dots, j_n} A(x_1^{j_1}, \dots, x_n^{j_n}) \int_\Omega \overline{r_1} r_1^{j_1} \cdots \overline{r_n} r_n^{j_n}.$$

If some $j_t > 1$, then some other $j_s = 0$, and so the corresponding integral is 0. Consequently, the only non-zero term occurs when all $j_t = 1$, in which case the above expression is simply $n! A(x_1, \dots, x_n)$. ∎

One case of the polarization formula merits special mention.

COROLLARY 2.3. *For $x_1, \dots, x_n, P$, and A as above,*

$$A(x_1, \dots, x_n) = \frac{1}{n! 2^n} \sum_{\epsilon_j = \pm 1, j = 1, \dots, n} \epsilon_1 \cdots \epsilon_n P\big(\sum_{j=1}^n \epsilon_j x_j \big).$$

The proof of this corollary consists merely of taking n (Bernoulli) random variables ϵ_j, each taking values ± 1 independently.

Applying Corollary 2.3 to arbitrary unit vectors $x_1, \dots, x_n$, we see that

$$||A(x_1, \dots, x_n)|| \le \frac{1}{n!} \sup_{\epsilon_j = \pm 1} ||P(\sum_{j=1}^n \epsilon_j x_j)|| \le \frac{1}{n!} ||P|| n^n,$$

since each $|| \sum_{j=1}^n \epsilon_j x_j ||$ has norm at most n. We have proved the following:

COROLLARY 2.4. *Let P be an n-homogeneous polynomial with associated symmetric n-linear mapping A. Then $||A|| \leq \frac{n^n}{n!}||P||$.*

A lot of work has been done on finding the best constant C, as a function of the Banach space E, and extremal polynomials P such that $||A|| \leq C||P||$. For instance, if E is Hilbert space, then $||A|| = ||P||$ for every n and every $P \in \mathcal{P}(^nE)$; on the other hand, the polynomial $P : \ell_1 \to \mathbb{K}$, $P(x) = x_1 \cdots x_n$, is such that $||A|| = \frac{n^n}{n!}||P||$.

Thus far, we have not given any example of a polynomial on an arbitrary Banach space. We'll now correct this omission, restricting to the scalar valued case.

EXAMPLE 2.5. For all $\phi \in E^*$ and all n, $\phi^n \in \mathcal{P}(^nE)$. For certain Banach spaces, for example $E = c_0$, every $P \in \mathcal{P}(^nE)$ is a limit of so-called *finite type polynomials,* of the form $\sum_{j=1}^k c_j \phi_j^n$. The same holds for k-homogeneous polynomials $P : \ell_p \to \mathbb{K}$, provided $k < p$. However, this is by no means true in general. For instance, the polynomial $\sum_{j=1}^\infty x_j^2$ on ℓ_2 cannot be approximated by such finite-type polynomials.

Our interest in the area of polynomials originated in a question which was studied by Nemirovskii and Semenov [19] in the late 1960's and early 1970's.

QUESTION 2.6. Let E be a Banach space. Let the space of polynomials on E, $\mathcal{P}(E)$, be given the sup-norm on the open unit ball B_E of E. What is the completion of $\mathcal{P}(E)$?

We will first study Question 2.6 when the underlying field is $\mathbb{R}$. Note that if $\dim E < \infty$, then the answer is given precisely by the Stone-Weierstrass theorem. Also, for special infinite dimensional E such as c_0, the problem has an easy solution, which we will describe in Remark 2.10. However, the problem remains open for $E = \ell_2$. Note that any limit of a sequence of polynomials must be uniformly continuous on bounded subsets of E. However, we show here that there are uniformly continuous functions on ℓ_2 which are not approximable by polynomials. In fact, the following is true.

EXAMPLE 2.7. (A.S. Nemirovski and S. M. Semenov) There is a $\mathcal{C}^\infty$ function g on ℓ_2, which is uniformly continuous and which has uniformly continuous derivatives, which cannot be approximated uniformly on the unit ball B of ℓ_2 by any polynomial.

Proof. For convenience, we prove the result with B being the ball in ℓ_2 of radius 4. Denote by $\mathcal{P}_{\leq s}(\mathbb{R}^n)$ the vector space of all polynomials $P : \mathbb{R}^n \to \mathbb{R}$ of degree at most s. For any $t \leq s$, the set of monomials $\{x_{j_1} \cdots x_{j_t}\}$ forms a basis for the t-homogeneous polynomials on $\mathbb{R}^n$, and there are at most $n^t \leq n^s$ such monomials. Consequently, $\dim \mathcal{P}_{\leq s}(\mathbb{R}^n)$ is at most $1 + sn^s$. Call Γ_n any collection of points in $B_n = \{x \in \mathbb{R}^n : ||x|| \leq 1\}$ which are maximal with respect to the following property: For any two different points x and y in Γ_n, $||x - y|| \geq 1/2$. The first thing to notice is that there are at least 2^n points in Γ_n. Indeed, by maximality of Γ_n, any point in B_n is within 1/2 of at least one point of Γ_n. In other words, B_n is a subset of the union $\cup_{x\in\Gamma_n} B(x, \frac{1}{2})$ of the balls in $\mathbb{R}^n$ of radius 1/2. If μ denotes Lebesgue measure, it follows that $\mu(B_n) \leq \sum_{x\in\Gamma_n} \mu(B(x, \frac{1}{2})) = |\Gamma_n|\mu(B(0, \frac{1}{2}))$, since all the balls have the same measure, $= |\Gamma_n|\frac{1}{2^n}\mu(B_n)$ by homogeneity of μ.

For each $s = 1, 2, \ldots$, let $n = n(s)$ be any integer such that $2^n > sn^s + 1$. Thus, if $\mathcal{C}(\Gamma_n) = \{f : \Gamma_n \to \mathbb{R}\}$ with the max-norm, then $\dim \mathcal{C}(\Gamma_n) \geq 2^n > \dim \mathcal{P}_{\leq s}(\mathbb{R}^n)$. What this means is that the restriction mapping $r : \mathcal{P}_{\leq s}(\mathbb{R}^n) \to \mathcal{C}(\Gamma_n)$, $r(P) \equiv P|_{\Gamma_n}$, maps onto a *proper* subspace of $\mathcal{C}(\Gamma_n)$. Thus, there must be a function $f_s \in \mathcal{C}(\Gamma_n)$, $||f_s|| = 1$, such that $||f_s - r(P)|| \geq 1$ for every $P \in \mathcal{P}_{\leq s}(\mathbb{R}^n)$. In other words, there is a function defined on a finite subset of B_n which cannot be approximated on B_n by any polynomial of degree at most s. For each s, we place $B_{n(s)} \subset B$ in such a way that the distance between any two such balls is at least 1/2. Call $\Gamma = \cup_s \Gamma_{n(s)}$, and rename the points of Γ as $\{x_j : j \in \mathbb{N}\}$. Define $f : \Gamma \to \mathbb{R}$ by $f(x_j) \equiv f_s(x_j)$ provided $x_j \in \Gamma_{n(s)}$. Then there can be no polynomial P which can approximate f within 1/2 on B. Otherwise, if P were such a polynomial of degree s, say, then f_s would have a polynomial approximation to within 1/2, which is false.

Finally, let $\theta : \mathbb{R} \to [-1, 1]$ be a $\mathcal{C}^\infty$ function which is such that $\theta(0) = 1$ and $\theta(t) = 0$ if $|t| > 1/8$. The function $g : \ell_2 \to \mathbb{R}$, $g(x) = \sum_j f(x_j)\theta(||x - x_j||^2)$, is the required function. ∎

The answer to Question 2.6 in the complex case is considerably easier. First, we need a definition.

Definition 2.8. Let $U \subset E$ be an open subset of the complex Banach space E. A function $f : U \to \mathbb{C}$ is said to be analytic or holomorphic if one of the following equivalent conditions holds:

(i) The function f has a Fréchet derivative at every point of U.

(ii) For every $b \in U$, there is a Taylor expansion about b which converges to f near b. Specifically, for all x in some neighborhood of b, we can write

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(b)}{n!}(x-b),$$

where each $f^{(n)}(b)\in\mathcal{P}(^nE)$.

We denote the space of holomorphic functions on U by $\mathcal{H}(U)$. For us, the only cases of interest will be $U=B_E$, the open unit ball of E, and $U=E$. It is important to note that if $\dim E=\infty$, then there are plenty of $f\in\mathcal{H}(E)$ such that $\sup_{x\in B_E}|f(x)|=\infty$. For instance, if $E=c_0$ or ℓ_p, $1\le p<\infty$, the function $f(x)=\sum_{n=1}^{\infty}x_n^n$ is an entire function which is unbounded on the unit ball. Let $\mathcal{H}_b(E)$ denote those functions $f\in\mathcal{H}(E)$ such that $\sup_{||x||\le k}|f(x)|<\infty$ for every $k>0$, endowed with the natural metric induced by the family $f\rightsquigarrow\sup_{||x||\le k}|f(x)|$.

Proposition 2.9. (See, e.g., [13]) *If E is a complex Banach space, then the completion of the space of complex polynomials $\mathcal{P}(E)$ is $\mathcal{H}_b(E)$.*

Proof. This is an immediate consequence of the following easily proved facts: (i) $\mathcal{H}_b(E)$ is complete, (ii) For any $f\in\mathcal{H}_b(E)$, the Taylor series $x\to\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}(x)$ converges uniformly to f on any ball in E. ∎

To conclude this section, we mention some positive results related to Question 2.6, as well as another open problem.

Suppose that we restrict our attention to the completion of the finite type polynomials on E. It will be a bit more convenient to consider the space obtained by taking uniform limits with respect to *every* ball in E, rather than just the unit ball B_E of E. We have a reasonably satisfactory answer in the case of both real and complex Banach spaces.

Proposition 2.10. [8] ($\mathbb{R}$-*case*) *If E is a real Banach space, then a function $f:E\to\mathbb{R}$ is a uniform limit of finite type polynomials on balls of E if and only if the restriction of f to each bounded subset of E is weakly uniformly continuous.*

($\mathbb{C}$-*case*) *If E is a complex Banach space whose dual has the approximation property, then a holomorphic function $f:E\to\mathbb{C}$ is a uniform limit of finite type polynomials on the ball of E if and only if the restriction of f to each bounded subset of E is weakly uniformly continuous.*

Proof. The $\Rightarrow$ direction is straightforward in both the real and complex cases. For the 'real' converse, fix a bounded set $B \subset E$ and $\epsilon > 0$. Our assumption on f means that there are $\delta > 0$ and functionals $\phi_1, \dots, \phi_k \in E^*$ such that if $x, y \in B$ satisfy $|\phi_j(x-y)| < 2\delta$ $(j = 1, \dots, k)$, then $|f(x)-f(y)| < \epsilon$. Call $\Phi : E \to \mathbb{R}^k$ the function taking $x \in E$ to $(\phi_1(x), \dots, \phi_k(x))$. Since $\overline{\Phi(B)}$ is compact, there are $x_1, \dots, x_m$ in B such that for all $x \in B$, $||\Phi(x) - \Phi(x_j)||_\infty < \delta$ for some $j = 1, \dots, m$. For each such j, let $h_j : \mathbb{R}^k \to [0, 1]$ be a continuous function such that supp $h_j \subset B_\infty(\Phi(x_j), 2\delta)$ and such that $\sum_{j=1}^m h_j(y) = 1$ for every $y \in \cup_{j=1}^m B_\infty(\Phi(x_j), \delta)$. (Such a collection is called a *partition of unity*.) Thus, for every $x \in B$,

$$\Big|\sum_{j=1}^m h_j \circ \Phi(x) f(x_j) - f(x)\Big| = \Big|\sum_{j=1}^m h_j \circ \Phi(x)(f(x_j) - f(x))\Big| < \epsilon.$$

Indeed, for each such x, $||\Phi(x) - \Phi(x_i)||_\infty < \epsilon$ for some i and so $\sum_{j=1}^m h_j \circ \Phi(x) = 1$. Also, if $||\Phi(x) - \Phi(x_i)||_\infty < 2\delta$, then $|f(x_i) - f(x)| < \epsilon$, while if $||\Phi(x) - \Phi(x_i)||_\infty \geq 2\delta$, then $h_i \circ \Phi(x) = 0$.

To complete this part of the argument, we approximate each h_j by a polynomial $P_j : \mathbb{R}^k \to \mathbb{R}$ uniformly within $\epsilon \cdot (m \max_j\{|f(x_j)|\})^{-1}$ on $\overline{\Phi(B)}$. We will therefore have the finite type polynomial $\sum_{j=1}^m f(x_j)(P_j \circ \Phi)$ within 2ϵ of f on B.

For the 'complex' converse, which we sketch, we will need the following notions: Each n-linear mapping $A : E \times \cdots \times E \to \mathbb{K}$ can be associated to a linear mapping $C : E \to \mathcal{L}(^{n-1}E)$, given by $C(x)(x_1, \dots, x_{n-1}) = A(x, x_1, \dots, x_{n-1})$. Moreover, let $\mathcal{L}_{wu}(^mE)$ denote the space of all $A \in \mathcal{L}(^mE)$ such that the restriction of A to $B \times \cdots \times B$ is weakly uniformly continuous on the unit ball B of E. It is an exercise that the above association will take an $A \in \mathcal{L}_{wu}(^2E)$ to a compact linear $C : E \to E^*$.

Let f be a holomorphic function on E which is weakly uniformly continuous on bounded subsets of E, and let a bounded set $B \subset E$ and $\epsilon > 0$ be given. By straightforward arguments involving the Cauchy integral formula and Cauchy's inequalities, if $f = \sum_{n=0}^\infty P_n$ is the Taylor expansion of f into n-homogeneous polynomials, then each P_n is itself weakly uniformly continuous on bounded subsets of E. Also, since f is weakly uniformly continuous on bounded sets, it is bounded on any ball in E. Therefore, its Taylor series converges uniformly to f on B. Thus, it suffices to show that each P_n can be uniformly approximated by a finite type n-homogeneous polynomial on B.

We now use a typical argument which links n-homogeneous polynomials to certain linear mappings. Namely, to each n-homogeneous polynomial $P : E \to \mathbb{K}$ we first associate the symmetric n-linear A, and this in turn is associated to the linear mapping $C : E \to \mathcal{L}({}^{n-1}E)$ as indicated above. It isn't hard to show that our original polynomial P is weakly uniformly continuous on bounded subsets of E if and only if A is weakly uniformly continuous on bounded subsets of $E \times \cdots \times E$, which in turn is equivalent to the mapping C being compact with range contained in $\mathcal{L}_{wu}({}^{n-1}E)$. We now use our hypothesis of E^* having the approximation property to see that $C(\cdot)$ can be uniformly approximated on B by a finite sum of the form $\sum \phi_j(\cdot)A_j$, where $\phi_j \in E^*$ and $A_j \in \mathcal{L}_{wu}({}^{n-1}E)$. An induction argument is all that is needed to complete the proof. ∎

Remark 2.11. For the Banach space c_0, every polynomial is a uniform limit of finite type polynomials (see, e.g., [20]). Consequently, Proposition 2.10 provides a complete solution to the 'Stone-Weierstrass' problem for c_0. In connection with Proposition 2.10, one can show ([2]) that a holomorphic function $f : E \to \mathbb{C}$ is weakly uniformly continuous on balls of E if and only if its Fréchet derivative $df : E \to E^*$ takes bounded subsets of E to relatively compact subsets of E^*.

Finally, we mention that a 'neat' analogous characterization is unknown for the completion of the algebras generated by all polynomials of degree at most k, if $k > 1$.

3. Norms of polynomials

Let's fix the max norm on $\mathbb{K}^n$, and consider a polynomial $P : \mathbb{K}^n \to \mathbb{K}$ of degree k,

$$P(x) = \sum_{|\alpha| \leq k} b_\alpha x^\alpha.$$

In estimating the norm of P *from above*, there is no better estimate possible than $||P|| \leq \sum_\alpha |b_\alpha|$ (just take a polynomial with all $b_\alpha > 0$). However, we can get some interesting and useful estimates on $||P||$ *from below* in many situations. This is the theme of this short section.

Although our results will depend heavily on whether $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, the central theme will be the same: It is possible to obtain estimates of $||P||$ from below which are independent of n, the number of variables. In fact, in the complex case, our estimate is even independent of the degree k of the polynomial.

Let's begin with the complex case. A typical polynomial $P : \mathbb{C}^n \to \mathbb{C}$ of degree k can be written as follows:

$$\begin{aligned} P(z) = a_0 + [b_1 z_1 + \ldots + b_n z_n] + \cdots + [(a_1 z_1^m + \cdots + a_n z_n^m) \\ + \text{other homogeneous terms of degree } m] \\ + \cdots + [\text{terms of degree } k]. \end{aligned}$$

THEOREM 3.1. [5] *Let $P : \mathbb{C}^n \to \mathbb{C}$ be a polynomial of degree k and let m be an integer, $k/2 < m \leq k$. Then, with the above notation,* $|a_0| + |a_1| + \cdots + |a_n| \leq ||P||$.

Sketch of Proof. Recall that we are taking $||P|| = \max_{|z_1|,\ldots,|z_n| \leq 1} |P(z)|$. The basic idea of the proof is already apparent in the case $k = 4$ and $m = 3$, and we will work with this notationally easier situation. Without loss of generality, $a_0 \geq 0$. Let $s_1, s_2, \ldots, s_n$ be defined on $[0,1]$ in the following way.

$$s_1(t) = \begin{cases} 1 & \text{if } 0 \leq t \leq 1/3 \\ \alpha & \text{if } 1/3 < t \leq 2/3 \\ \beta & \text{if } 2/3 < t \leq 1. \end{cases}$$

Here, $1, \alpha$ and β are the cube roots of unity. Assuming that s_{j-1} has been defined, we define s_j as follows: Divide each subinterval of $[0,1]$ on which s_{j-1} is constant into three equal subintervals, and let s_j be equal to 1 on the first of these, α on the second, and β on the third. Like the analogous Rademacher functions, the functions (s_j) have the following properties:

(i) $|s_j(t)| = 1$ and $\int_0^1 s_j^3(t)dt = 1$ (all t and all j).

(ii) $\int_0^1 s_{j_1}(t)s_{j_2}(t)\cdots s_{j_l}(t)dt = 0$, provided l is not a multiple of 3,

(iii) $\int_0^1 s_{j_1}(t)s_{j_2}(t)s_{j_3}(t)dt = 0$, provided the 3 indices j_1, j_2, j_3 are not all equal.

Choose constants $d_1, \ldots, d_k$ of modulus one so that $a_j d_j^3 \geq 0$. Then,

$$\begin{aligned}||P|| &\geq \Big|\int_0^1 P(d_1 s_1(t),\ldots,d_n s_n(t))dt\Big| \\ &= \Big|\int_0^1 a_0 + [\sum_{j=1}^n b_j d_j \int_0^1 s_j(t)dt] + \cdots + [\sum_{j=1}^n a_j d_j^3 \int_0^1 s_j^3(t)dt \\ &\qquad + \int_0^1 \text{other homogeneous terms of degree 3}] \\ &\qquad\quad + \int_0^1 \text{homogeneous terms of degree 4}\Big|.\end{aligned}$$

The properties of our s_j's and our choice of d_j's show that the above integral is merely $a_0 + |a_1| + \cdots + |a_n|$. ∎

We remark that we don't have an estimate in the situation $1 \leq m \leq k/2$.

The real case is substantially more complicated. For one thing, the 'neat' estimate of the complex case fails.

EXAMPLE 3.2. Let $Q_1 : \mathbb{R}^2 \to \mathbb{R}$ be given by $Q_1(x_1,x_2) = x_1^2 - x_2^2$. Then, using the max-norm on $\mathbb{R}^2$, $||Q_1|| = 1$ although $|a_1| + |a_2| = |1| + |-1| = 2$. Iterating, if $Q_{m+1} : \mathbb{R}^{2^{m+1}} \to \mathbb{R}$ is the $2^{m+1}-$homogeneous polynomial given by

$$\begin{aligned}Q_{m+1}(x_1,\ldots,x_{2^m},x_{2^m+1},\ldots,x_{2^{m+1}}) &= Q_m(x_1,\ldots,x_{2^m})^2 \\ &\quad - Q_m(x_{2^m+1},\ldots,x_{2^{m+1}})^2,\end{aligned}$$

then each $Q_m \in \mathcal{P}({}^{2^m}\mathbb{R}^{2^m})$. Furthermore, adding the absolute values of the 2^m coefficients a_i of $x_i^{2^m}$, we get $2^m = \sum_{i=1}^{2^m} |a_i|$ although $||Q_m|| = 1$.

In fact, a similar iterative example, based on $Q_0 : \mathbb{R}^3 \to \mathbb{R}$, $Q_0(x_1,x_2,x_3) = x_1^2+x_2^2+x_3^2-x_1x_2-x_2x_3-x_3x_1$, gives a better estimate. One can show that in this situation, $\max_{0\leq x_i\leq 1, i=1,2,3} |Q_0(x_1,x_2,x_3)| = 1$. As a result, $Q_1 \in \mathcal{P}({}^4\mathbb{R}^6)$ given by $Q_1(x_1,\ldots,x_6) = Q_0(x_1,x_2,x_3)^2 - Q_0(x_4,x_5,x_6)^2$ is such that the sum of the absolute values of the 'pure' coefficients of the x_i^4 gives 6, with $||Q_1||$ still equal to 1.

Nevertheless, there are estimates which one can obtain which are independent of the number of variables, such as the following:

THEOREM 3.3. [5] *Let $P : \mathbb{R}^n \to \mathbb{R}$ be a k-homogeneous polynomial,*

$$P(x) = a_1 x_1^k + \cdots + a_n x_n^k + \text{ other terms.}$$

Then

$$|a_1| + \cdots + |a_n| \le 4k^2 \max_{0 \le x_i \le 1, i=1,\ldots,n} |P(x)|.$$

Proof. The proof begins with several reductions. First, there is a subset L of $\{1, 2, \ldots, n\}$ consisting of coefficients a_i having the same sign, such that $|\sum_{i \in L} a_i| \ge 1/2 \sum_{i=1}^n |a_i|$. Without loss of generality $L = \{1, 2, \ldots, m\}$ for some m and for each $i \in L$, $a_i \ge 0$. Let's normalize so that $\sum_{i=1}^m a_i = 1$. We'll fix $x_i = 0$ for $i > m$.

The object of the proof is to show that one can choose values 0 or 1 for the m variables $x_1, \ldots, x_m$ in such a way that the required inequality holds. To do this, we allow each of these m variables to assume values 0 and 1 independently, with respective probabilities $1 - t$ and t. The proof consists in applying Markov's inequality to show that for some choice of $t \in [0, 1]$, the expected value of $P(x_1, \ldots, x_m, 0, \ldots, 0)$ will be at least $\frac{1}{2k^2}$. It will be easy to proceed from here to the required inequality.

Note that the expected value of $x_i^{j_i}$ is t, regardless of which positive integer j_i we choose. Analogously, the expected value of $x_i^{j_i} x_l^{j_l}$ is t^2, etc. Let's write P in the following form:

$$\begin{aligned} P(x) = &\sum_{i=1}^m a_i x_i^k + \sum \{a_\alpha x^\alpha \colon |\alpha| = k, \ 2 \ \text{ indices of } \alpha \text{ are } \ne 0\} \\ &+ \sum \{a_\alpha x^\alpha \colon |\alpha| = k, \ 3 \text{ indices of } \alpha \text{ are } \ne 0\} + \cdots \\ &+ \sum \{a_\alpha x^\alpha \colon |\alpha| = k, \ k \text{ indices of } \alpha \text{ are } \ne 0\}. \end{aligned}$$

Thus, since expectation is a linear function, the expectation of $P(x_1, \ldots, x_m, 0, \ldots, 0)$ is

$$S(t) \equiv \sum_{i=1}^m a_i t + A_2 t^2 + \cdots + A_k t^k = t + A_2 t^2 + \cdots + A_k t^k,$$

where each

$$A_j = \sum \{a_\alpha \colon \ \text{exactly } j \text{ indices of } \alpha \text{ are } \ne 0\}.$$

Let $f(y) \equiv S(\frac{y+1}{2})$, where $y \in [-1,1]$. Then $\max_{t\in[0,1]} |S(t)| = \max_{y\in[-1,1]} |f(y)|$, which by Markov's inequality is $\geq \frac{1}{k^2} \max_{[-1,1]} |f'(y)| = \frac{1}{2k^2} \max_{t\in[0,1]} |S'(t)| \geq \frac{1}{2k^2} S'(0) = \frac{1}{2k^2}$. Thus, since the maximum value of S is at least $\frac{1}{2k^2}$, there must be some t for which $|S(t)| \geq \frac{1}{2k^2}$. In other words, for some choice of $x_1 = 0$ or $1, \dots, x_m = 0$ or $1, x_{m+1} = \cdots = x_n = 0$, $|P(x_1, \dots, x_m, x_{m+1}, \dots, x_n)| \geq \frac{1}{2k^2} \geq \frac{1}{4k^2} \sum_{i=1}^n |a_i|$. ∎

Remarks 3.4. With somewhat more care, the constant 4 in Theorem 3.3 can be reduced. More importantly, in a number of cases, the exponent k^2 can be reduced; however, we don't know whether k^2 can be replaced by something smaller in general. Let us denote by $c(k)$ the smallest constant which satisfies the estimate of Theorem 3.3: $|a_1| + \cdots + |a_n| \leq 4k^2 \max_{0\leq x_i \leq 1, i=1,\dots,n} |P(x)|$. We don't know the value of $c(k)$ for $k \geq 3$; in fact, we don't even know whether $(c(k))$ is an increasing sequence.

Note that the maximum we took in Theorem 3.3 was over the first 'quadrant' only, in n-space; that is, our x_i varied only in $[0,1]$ rather than in $[-1,1]$. A modification of the argument yields the following:

THEOREM 3.5. [5] *Let $P : \mathbb{R}^n \to \mathbb{R}$ be a polynomial of degree k, $P(x) = \sum_{i=1}^n a_i x_i^k$ + other terms. Then*

$$\sum_{i=1}^n |a_i| \leq 2^{k+2} \max_{x_1,\dots,x_n \in [-1,1]} |P(x_1, \dots, x_n)|.$$

We conclude this section with a brief remark on a result in a very similar spirit. Namely, the following is true concerning norms of products of polynomials:

THEOREM 3.6. [10] *Let P_1 and P_2 be homogeneous polynomials on a complex Banach space E of degree m_1 and m_2, respectively. Then*

$$||P_1|| ||P_2|| \leq \frac{(m_1 + m_2)^{(m_1+m_2)}}{m_1^{m_1} m_2^{m_2}} ||P_1 P_2||.$$

Proof. We prove the result in the simplest non-trivial case, namely when P_1 and P_2 are elements of E^*. The argument in the general case is similar.

Fix x and y in B, the closed unit ball of E. We have the following easily verified equality:

$$\phi_1(x)\overline{\phi_2(y)} = \frac{1}{2\pi}\int_0^{2\pi} e^{i\theta}\phi_1(x+e^{i\theta}y)\overline{\phi_2(x+e^{i\theta}y)}d\theta.$$

Thus,

$$\begin{aligned}||\phi_1||||\phi_2|| = \sup_{x,y\in B}|\phi_1(x)\overline{\phi_2(y)}| &\leq \sup_{x,y,\ \theta\in[0,2\pi]}|\phi_1(x+e^{i\theta}y)\overline{\phi_2(x+e^{i\theta}y)}| \\ &\leq ||\phi_1\phi_2|| \sup_{x,y,\theta}||x+e^{i\theta}y||^2 = 4||\phi_1\phi_2||.\end{aligned}$$

∎

4. Zeros of polynomials

We next turn our attention to zeros of polynomials, dealing first with the complex and then the real case. Our interest will be in finding 'large' subspaces of E on which a polynomial is constant, in a way we now make precise.

It is trivial that the nullspace of a linear form on a vector space is a hyperplane. In our terminology, to every 1-homogeneous scalar valued polynomial, there is a vector space of codimension 1 on which the polynomial vanishes. A direct generalization of this result to polynomials having higher homogeneity cannot hold: Take $P : \mathbb{C}^4 \to \mathbb{C}$ given by $P(z_1, z_2, z_3, z_4) = z_1^2 + \cdots + z_4^2$. Although $(1, i, 0, 0)$ and $(0, 1, i, 0)$ are both in $P^{-1}(0)$, $P(1, 1+i, i, 0) \neq 0$. Nevertheless, the 2-dimensional subspace $\{(a, ia, b, ib)\colon a, b \in \mathbb{C}\}$ is contained in $P^{-1}(0)$, and it is the generalization of this fact to polynomials which we study here. Specifically, we study $P^{-1}(0)$ for polynomials $P : E \to \mathbb{K}$, showing that these polynomials are constant on large subspaces. Moreover, the size of these subspaces depends only on the degree of P and not on P itself. These results appear to be known to algebraic geometers, but not to very many analysts. The technique of proof we present here is purely analytical, and uses only one 'ingredient:' Given a non-constant entire function $f : \mathbb{C}^k \to \mathbb{C}$, where $k \geq 2$, either $f^{-1}(0) = \emptyset$ or $f^{-1}(0)$ is non-compact (see, e.g., [16], p.21).

Theorem 4.1. [21],[4] *There is a function $\Theta : \mathbb{N}\times\mathbb{N}\to\mathbb{N}$, $\Theta(m, d) = n$, with the following property: For every polynomial $P : \mathbb{C}^n \to \mathbb{C}$, deg $P \leq d$, there is an m-dimensional subspace $X \subset \mathbb{C}^n$ such that $P|_X \equiv P(0)$.*

Idea of Proof. There is no loss if we assume that $P(0) = 0$. Also, without loss of generality, we will only work with homogeneous polynomials, obtaining for each d a function Θ_d which is Θ restricted to the d-homogeneous polynomials. Indeed, suppose that for each d, we have found a function $\Theta_d : \mathbb{N} \to \mathbb{N}$ such that for each $m \in \mathbb{N}$, every d-homogeneous polynomial in $\Theta_d(m)$ variables vanishes on an m-dimensional subspace. Then a possible value for $\Theta(m, 2)$ is $\Theta_2(\Theta_1(m))$. To see this let $P = P_1 + P_2 : \mathbb{C}^n \to \mathbb{C}$ be a polynomial of degree 2 with linear part P_1 and 2-homogeneous part P_2, where $n = \Theta_2(\Theta_1(m))$. Then $P_2 \equiv 0$ on a subspace X of $\mathbb{C}^n$ of dimension $\Theta_1(m)$, and $P_1 \equiv 0$ on a subspace of X of dimension m. Therefore, $P|_X \equiv 0$ on an m-dimensional subspace of $\mathbb{C}^n$. In general one possible value for $\Theta(m, d)$ is $\Theta(m, d) = \Theta_d(\Theta_{d-1}(\ldots(\Theta_1(m))\ldots))$.

As we remarked, $\Theta_1(m) = m + 1$. To find Θ_2, let $P : \mathbb{C}^n \to \mathbb{C}$ be a 2-homogeneous polynomial (where n is yet to be determined). If $n \geq 2$, then the 'ingredient' described above applied to the entire function P yields $z_1 \in \mathbb{C}^n, z_1 \neq 0$, such that $P(z_1) = 0$. Let $S_1 = \{z \in \mathbb{C}^n : A(z, z_1) = 0\}$, where A is as usual the symmetric bilinear form associated to P. Note that S_1 can be written as the direct sum of $[z_1] \bigoplus Y_{n-2}$, where we associate Y_{n-2} with $\mathbb{C}^{n-2}$. Consider $P|_{\mathbb{C}^{n-2}} : \mathbb{C}^{n-2} \to \mathbb{C}$. If $n - 2 \geq 2$, then there is $z_2 \in \mathbb{C}^{n-2}, z_2 \neq 0$, such that $P(z_2) = 0$. Thus for any scalars a_1 and a_2, $P(a_1 z_1 + a_2 z_2) = a_1^2 P(z_1) + 2a_1 a_2 A(z_1, z_2) + a_2^2 P(z_2)$. By our choices of z_1 and $z_2 \in S_1$, we see that $P(a_1 z_1 + a_2 z_2) = 0$. Proceeding in this way, we see that $\Theta_2(m) \leq 2m$ for every m; that is, every 2-homogeneous polynomial in $2m$ variables vanishes on an m-dimensional subspace of $\mathbb{C}^{2m}$. (Now, do we really need that many variables? In other words, might it be true that every 2-homogeneous polynomial in $k < 2m$ variables vanishes on an m-dimensional subspace? In fact, we show in Example 4.2 that $\Theta_2(m) = 2m$.)

The general case follows by induction. For example, if $P : \mathbb{C}^n \to \mathbb{C}$ is 3-homogeneous, then there is $z_1 \in \mathbb{C}^n, z_1 \neq 0$, such that $P(z_1) = 0$. Let $S_1 = \{z \in \mathbb{C}^n : A(z, z_1, z_1) = 0\} = [z_1] \bigoplus \mathbb{C}^{n-2}$. By induction, there is an $\left[\ (n-2)/2\ \right]$-dimensional subspace $T_1 \subset \{z \in \mathbb{C}^{n-2} : A(z, z, z_1) = 0\}$. So if $\left[\ (n-2)/2\ \right] \geq 2$, that is if $n \geq 6$, we can find $z_2 \in T_1, z_2 \neq 0$, such that $P(z_2) = 0$. The argument proceeds along these lines. ∎

QUESTIONS 4.2. (AND AN EXAMPLE) (i) The above argument shows that $\Theta_3(2) \leq 6$. That is, every 3-homogeneous polynomial in 6 variables vanishes on a 2-dimensional subspace. We do not know the exact value of $\Theta_3(2)$. Specifically, might $\Theta_3(2) = 5$? In other words, might it happen that every 3-homogeneous polynomial in 5 variables vanishes on a 2-dimensional subspace? Indeed, we don't know the exact value of $\Theta_m(d)$ for any $d \geq 3$. The only exact estimate we know is given in (ii) below:

(ii) We can show that $\Theta_2(m) = 2m$. Indeed, consider the polynomial $P : \mathbb{C}^{2m} \to \mathbb{C},\ P(z) = \sum_{j=1}^{2m} z_j^2$, which vanishes on the m-dimensional span of $e_1 + ie_2, \ldots, e_{2m-1} + ie_{2m}$. Assume that there were some $k > m$-dimensional subspace X on which P was identically 0. Extend the basis $f_1, \ldots, f_k$ for X to a basis for $\mathbb{C}^{2m}$, and consider the symmetric matrix of A associated to this basis. For any $j, l,\ 1 \le j, l \le k$, we have $2A(f_j, f_l) = P(f_j + f_l) - P(f_j) - P(f_l) = 0$. Thus, the $2m \times 2m$ matrix A has a square block of size greater than m consisting only of $0'$s, making it singular. This is a contradiction, since the matrix of P with respect to the standard basis is just the identity.

Cheerfully ignoring the polynomial $P : \mathbb{R}^m \to \mathbb{R},\ P(x) = \sum_{j=1}^{m} x_j^2$, whose zero set is well-understood, we next consider the question of finding 'large' subspaces in the zero set of *real* polynomials. Surprisingly perhaps, there are quite a few things which can be said, even in finite dimensions.

PROPOSITION 4.3. [7] *Let $P : \mathbb{R}^m \to \mathbb{R}$ be a symmetric d-homogeneous polynomial, where d is odd. Then $P^{-1}(0)$ contains a subspace of dimension $[m/2]$.*

Proof. First, recall that P is said to be *symmetric* if $P(x_1, \ldots, x_m) = P(x_{\sigma 1}, \ldots, x_{\sigma m})$ for every permutation σ of $\{1, \ldots, m\}$. It is not difficult to see that P can be factored as $P(x) = Q(\sum_j x_j, \sum_j x_j^2, \ldots, \sum_j x_j^d)$ for some polynomial $Q(y)$ in d variables. In other words, $P(x) = \sum_\alpha a_\alpha (\sum_j x_j)^{\alpha_1} (\sum_j x_j^2)^{\alpha_2} \cdots (\sum_j x_j^d)^{\alpha_d}$, where the sum is taken over all $\alpha = (\alpha_1, \ldots, \alpha_d)$ such that $1\alpha_1 + 2\alpha_2 + \cdots + d\alpha_d = d$. Now, for each fixed α, it cannot happen that $\alpha_j = 0$ whenever j is odd. Otherwise, the above sum could not be the odd number d. Hence, in every case, at least one of the α_j is odd for some odd j, which means that the corresponding summand vanishes on any vector of the form $e_1 - e_2, e_3 - e_4, \ldots$ Hence, P vanishes on $\mathrm{span}\{e_1 - e_2, \ldots, e_{2[m/2]-1} - e_{2[m/2]}\}$, which completes the proof. ∎

Thus, for at least some very special polynomials, there are large subspaces contained in their zero sets, the dimension of which is independent of the (odd) degree of the polynomial. Further, this argument extends to Banach spaces with symmetric basis, yielding an infinite dimensional subspace in the zero set. But, how about more general polynomials? In the 2-homogeneous case, we have a complete answer whose proof is very simple.

PROPOSITION 4.4. [7] *Let $P : \mathbb{R}^m \to \mathbb{R}$ be a 2-homogeneous polynomial. Then $P^{-1}(0)$ contains a subspace of dimension $r = \min\{p, n\} + z$, where p, n,* *and z are the respective number of positive, negative, and zero eigenvalues of the symmetric $m \times m$ matrix A associated to P.*

Proof. Let A be the $m \times m$ matrix associated to P. Since A is symmetric, there is a basis of $\mathbb{R}^m$ with respect to which A is diagonal. There is no loss in scaling this basis so that A has the following form:

$$\begin{pmatrix} \lambda_1 & 0 & . & . & . & . & . & 0 \\ . & . & . & . & . & . & . & . \\ 0 & . & \lambda_p & 0 & . & . & . & . \\ 0 & . & . & -\lambda_{p+1} & 0 & . & . & . \\ . & . & . & . & . & . & . & . \\ 0 & 0 & . & . & . & -\lambda_{p+n} & . & . \\ . & . & . & . & . & . & 0 & 0 \\ . & . & . & . & . & . & . & . \\ 0 & . & . & . & . & . & 0 & 0 \end{pmatrix},$$

with all the $\lambda_j > 0$. Since

$$\mathrm{span}\{\mathrm{e}_1 + \sqrt{\frac{\lambda_1}{\lambda_{\mathrm{p}+1}}}\mathrm{e}_{\mathrm{p}+1}, \mathrm{e}_2 + \sqrt{\frac{\lambda_2}{\lambda_{\mathrm{p}+2}}}\mathrm{e}_{\mathrm{p}+2}, \dots, \mathrm{e}_{\mathrm{p}+\mathrm{n}+1}, \dots, \mathrm{e}_{\mathrm{k}}\} \subset \mathrm{P}^{-1}(0),$$

the result follows. ∎

Thus, for example, if $P(x) = \sum_{j=1}^m x_j^2$, then $p = m, n = z = 0$, and so $r = 0$ as expected. On the other hand, if $P(x) = \sum_{j=1}^m (-1)^j x_j^2$, then $r = [m/2]$.

Of course, for any real polynomial P of odd homogeneity on $\mathbb{R}^m, m \geq 2$, $P^{-1}(0)$ always contains a line: Choosing x so that $P(x) > 0$, it follows that $P(-x) < 0$, and since $m > 1$ there is a path from x to $-x$ which avoids the origin. Hence, by the intermediate value theorem, we can find $y \neq 0$ such that $P(y) = 0$. Hence $P(ty) = 0$ for every $t \in \mathbb{R}$. One can say more:

PROPOSITION 4.5. [7] *If $P : \mathbb{R}^m \to \mathbb{R}$ is 3-homogeneous and if $m \geq \frac{1}{4}(3^n(6n+5)-1)$, then there is a subspace of $P^{-1}(0)$ of dimension $[n/2]$.*

This result is rather ugly, to say the least, and far from giving a good estimate. For instance, as we observed, any odd polynomial in $m = 2$ variables will yield a 1-dimensional subspace in its zero set. To satisfy the conditions in the above proposition with $n = 2$, the number of variables m must be at least 38. To say that this result can be improved is an understatement! We remark that subspaces of $P^{-1}(0)$, for P homogeneous of odd degree, were studied nearly fifty years ago in a different setting (see, e.g, [9]).

We believe that the following questions are worthy of investigation.

QUESTIONS 4.6. (1) Let $P : \mathbb{R}^m \to \mathbb{R}$ be a 4-homogeneous polynomial. Is there a formula similar to that of Proposition 4.4 which enables us to calculate the dimension of the largest possible subspace of $P^{-1}(0)$?

(2) Let E be an infinite dimensional real Banach space and let $P : E \to \mathbb{R}$ be a k-homogeneous polynomial, where k is odd. Is there an infinite dimensional subspace of E on which P vanishes? Proposition 4.5 notwithstanding, we do not even know the answer to this question when $k = 3$.

We turn next to the problem of finding subspaces of $P^{-1}(0)$, for homogeneous real polynomials P on infinite dimensional Banach spaces. We will find the following concept useful. A function $f : E \to \mathbb{R}$ is said to be *positive definite* if $f(x) > 0$ for all $x \in E$ except $x = 0$, for which $f(0) = 0$. It is easy to see that E admits a positive definite 2-homogeneous polynomial if and only if there is a continuous linear injection of E into some Hilbert space. (For instance, if such a polynomial exists with associated symmetric form A, defining $< x, y >\equiv A(x, y)$ produces an inner product on E.)

THEOREM 4.7. [6] *Suppose that E does not admit a positive definite 2-homogeneous polynomial. Then for every $P \in \mathcal{P}(^2E)$, there is an infinite dimensional subspace $F \subset E$ such that $P|_F \equiv 0$.*

Before proving this, we give some examples where the theorem applies and where it doesn't apply.

EXAMPLE 4.8. Let E be a separable space with countable dense subset $\{x_n\}$ of the unit sphere of E. For each n, let $\phi_n \in E^*$ have norm one and be such that $\phi_n(x_n) = 1$. Define the 2-homogeneous polynomial $P : E \to \mathbb{R}$ by $P(x) = \sum_{n=1}^{\infty} 1/2^n \phi_n(x)^2$. If $P(y) = 0$ for some non-zero vector y, then we may suppose that the same occurs for a unit vector x. This means that $\phi_n(x) = 0$ for every n, even for those n for which $||x - x_n||$ is very small, which is a contradiction. The same type of argument works for ℓ_∞ (and more generally for any $\mathcal{C}(K)$ where K is a separable, compact set) and any Hilbert space H. (For the last assertion, take a complete orthonormal set $\{u_\alpha\}$ and set $P \in \mathcal{P}(^2H)$, $P(x) = \sum_\alpha x_\alpha^2$.) On the other hand, $c_0(\Gamma)$ and $\ell_p(\Gamma), p > 2$, do not admit positive definite 2-homogeneous polynomials, provided Γ is an uncountable index set.

Proof of Theorem 4.7. Let $\mathcal{S} = \{S \subset E$: S is a subspace of E and $P|_S = 0\}$. It is a straightforward exercise to verify that $\mathcal{S}$ is inductive, so that we may apply Zorn's Lemma to find a maximal element $S \in \mathcal{S}$. We will show that $\dim S = \infty$. If this is not the case, then S is spanned by a finite number of linearly independent vectors, say $v_1, \ldots, v_n$.

Let $T = \cap_{x \in S} \ker A_x$, where $A_x : E \to \mathbb{R}$ is the functional: $y \in E \to A(x, y)$. Note that T is also equal to $\cap_{i=1}^n \ker A_{v_i}$, and that T has finite codimension in E.

Claim 1: $S \subset T$. To see this, fix $y \in S$. For any $x \in S$, $0 = P(x + y)$, since S is a vector space, $= P(x) + 2A_x(y) + P(y) = 2A_x(y)$. Therefore $y \in \ker A_x$ for every $x \in S$, which means that $y \in T$.

Thus, we may write $T = S \bigoplus Y$ for some infinite dimensional subspace Y.

Claim 2: If $y \in T$ and $P(y) = 0$, then $y \in S$. To see this, for any $x \in S, P(x + y) = P(x) + 2A_x(y) + P(y) = 0$. Because of the maximality of S, it follows that $y \in S$.

Therefore, either $P|_Y$ or $-P|Y$ is positive definite. (If not, there are $w, z \in Y$ such that $P(w) < 0 < P(z)$. Since $\dim Y = \infty$, there is a path in Y joining w to z which misses the origin. Thus there is a zero of P on this path, which contradicts Claim 2.) Let's agree that $P|_Y$ is positive definite. We may therefore choose a finite number of functionals $\{\phi_1, \ldots, \phi_n\}$ on E^* so that $P + \sum_{i=1}^n \phi_i^2$ is positive definite on T. Finally, if $\Pi : E \to T$ is a linear projection, then $Q(x) \equiv (P(x) + \sum_{i=1}^n \phi_i(x)^2) \circ \Pi(x) + \sum_{i=1}^n A_{v_i}(x)^2$ is a 2-homogeneous positive definite polynomial on E. But this is a contradiction to our initial hypothesis, and the proof is finished. ∎

QUESTION 4.9. Since the Banach space E in Theorem 4.7 must of necessity be non-separable, it is natural to ask whether the space F found in this theorem is also non-separable. In certain situations, for example if $E = \ell_p(\Gamma)$ where Γ is uncountable and $p > 4$, this question has a positive solution, whose proof we omit:

THEOREM 4.10. [6] *Suppose that E does not admit a positive definite 4-homogeneous polynomial. Then, every 2-homogeneous and every 3-homogeneous polynomial on E vanishes on a non-separable subspace of E.*

5. EXTENSION OF POLYNOMIALS

Our interest here will be focussed on the following two questions: Let $E \subset F$ be two Banach spaces, and let $P \in \mathcal{P}(^n E)$.

(1) Is there an extension $\tilde{P} \in \mathcal{P}(^nF)$ of P?

(2) Assuming that we can find such an extension, can we further arrange for it to have the same norm as $||P||$?

As we will see, the answer to (1) is 'no' in general, but 'yes' in the most important situation, when $F = E^{**}$. And, the answer to (2) will be seen to be 'yes' in this same situation. We begin with two general methods for constructing counterexamples to Question 1, and we then present a counterexample to Question 2.

EXAMPLES 5.1. (i) Consider a pair of Banach spaces $E \subset F$ with the property that for some weakly null sequence $(x_n) \subset E$ and some polynomial $P : E \to \mathbb{K}$, $P(x_n) \not\to 0$, but that for every polynomial $Q : F \to \mathbb{K}$, $Q(x_n) \to 0$. Then, it is clear that P is not extendible to F. One instance of this occurs with $E = \ell_2 \subset F = \mathcal{C}[0,1]$, the null sequence being the unit vector basis $(e_n) \subset \ell_2$. The non-extendible polynomial here is $P(x) = \sum_{j=1}^{\infty} x_j^2 \in \mathcal{P}(^2\ell_2)$, using the fact [22] that since $\mathcal{C}[0,1]$ has the (polynomial) Dunford Pettis property, every polynomial $Q : \mathcal{C}[0,1] \to \mathbb{K}$ is such that $Q(e_n) \to 0$.

(ii) I am indebted to Ray Ryan for pointing out the following 'recipe' for a counterexample. Let $E \subset F$ and G be Banach spaces, and let $T \in \mathcal{L}(E,G)$ be a continuous linear operator which does not admit an extension to $\mathcal{L}(F, G^{**})$. Define $P \in \mathcal{P}(^2(E \times G^*))$ by $P(e, g^*) = g^*(T(e))$. Then P cannot be extended to any $\tilde{P} \in \mathcal{P}(^2(F \times G^*))$. Indeed, suppose that $\tilde{P}$ exists, and let $\tilde{A} : (F \times G^*) \times (F \times G^*) \to \mathbb{K}$ be the associated symmetric bilinear form. Let's identify $\tilde{A}$ with the linear operator $L_{\tilde{A}} : (F \times G^*) \to (F \times G^*)^* = (F^* \times G^{**})$. Define $S \in \mathcal{L}(F, G^{**})$ by $S(f)(g^*) = \pi_2 \circ L_{\tilde{A}}(f, g^*)$, where π_2 is the projection onto the second coordinate. It is routine to show that S is an extension of T, which is a contradiction.

One situation in which the above hypotheses are satisfied occurs when $E = G = \ell_2$ and $F = \mathcal{C}[0,1]$. If the identify $\ell_2 \to \ell_2$ had an extension to $\mathcal{C}[0,1]$, this would imply that there is a projection of $C[0,1]$ onto ℓ_2. But this is well-known to be false. Note that the associated non-extendible polynomial in this case is $P : \ell_2 \times \mathcal{C}[0,1]^* \to \mathbb{R}$, $P(x, \phi) = \phi(x)$.

(iii) Let E be the set of all $(z_1, z_2, z_3) \in \mathbb{C}^3$ such that $z_1 + z_2 + z_3 = 0$, considered as a subspace of $F = (\mathbb{C}^3, max)$. It is clear that any polynomial P on E can be extended to a polynomial Q on F, for example by setting $Q(z) = P \circ \Pi(z)$ where $\Pi : F \to E$ is a projection. However, Martin Schottenloher has noted [23] that if $P \in \mathcal{P}(^2E)$ is given by $P(a+b, -a, -b) \equiv a^2 + ab + b^2$, then $||P|| = 1$ but P has no norm preserving extension to F. To see this, let

$v = (1, -1, 0)$ and $w = (1, 0, -1)$ be a basis for E, and define $P \in \mathcal{P}(^2E)$ by $P(av + bw) = a^2 + ab + b^2$. Any extension $Q \in \mathcal{P}(^2F)$ of P can be described as follows: Write an arbitrary $(x, y, z) \in \mathbb{C}^3$ as $av + bw + cu$, where $u = (1, 0, 0)$ is the third basis vector for $\mathbb{C}^3$. Then, Q is of the form $Q(x, y, z) = a^2 + ab + b^2 + c^2\delta_1 + ac\delta_2 + bc\delta_3$, for constants δ_1, δ_2, and δ_3. The proof will be complete once we show the following two claims:

Claim 1. $||P|| = 1$.

Claim 2. $||Q|| > 1$ no matter what extension we take.

For Claim 1, note first that $||P|| \geq P(v) = 1$. In order to prove that $||P|| \leq 1$, let $(x, y, z) \in E$ have norm 1. We first assume that $|x| = |y + z| = 1$. Writing $(x, y, z) = -yv - zw$, we see that $P(x, y, z) = y^2 + yz + z^2$, which we write as $y(y + z) + z^2$. By homogeneity of P, we may assume that $y + z = 1$, and it is easily verified that $|y + z^2| > 1$ is impossible. In a similar way, we consider the cases $|y| = 1$ and $|z| = 1$.

Next, suppose that the second claim is false; that is assume that some extension Q has norm 1. Let's evaluate Q at a number of special points. Evaluating Q at both $(1, 1, -1)$ and $(1, -1, 1)$ we get that $|1+\delta_1| \leq 1$. Similarly, evaluating at $(i, -1, 1)$ and $(i, 1, -1)$, we conclude that $|1 - \delta_1| \leq 1$. Thus, $\delta_1 = 0$. Finally, evaluating Q at $(1, 1, 1)$ and $(-1, 1, 1)$, we see that $|1 - [\delta_2 + \delta_3]| \leq 1/3$ and also that $|3 - [\delta_2 + \delta_3]| \leq 1$. This contradiction concludes the proof.

See also a recent note by Pierre Mazet [18], on the best constant in the extension of 2-homogeneous polynomials from a hyperplane of a Banach space.

For the rest of this paper, we will fix our attention on the case where $F = E^{**}$. We will show three things:

THEOREM 5.2. *Given any n and any $P \in \mathcal{P}(^nE)$, there is an extension $\tilde{P} \in \mathcal{P}(^nE^{**})$.*

THEOREM 5.3. [11] *Given P and $\tilde{P}$ as above, $||P|| = ||\tilde{P}||$.*

Finally, we will show that these results are interesting. *We hope that the reader will agree with this last assertion!*

The idea for 5.2 is originally due to R. Arens [1], who was interested in devising a process to extend bilinear mappings $A : X \times Y \to Z$ to bilinear mappings $X^{**} \times Y^{**} \to Z^{**}$. Arens' technique goes as follows.

Let $A_1 : Z^* \times X \to Y^*$ be given by

$$A_1(z^*, x)(y) = z^*(A(x, y)).$$

Let's follow the same pattern two more times, getting

$$A_2 : Y^{**} \times Z^* \to X^*, \quad A_2(y^{**}, z^*)(x) = y^{**}(A_1(z^*, x)),$$

and finally

$$A_3 : X^{**} \times Y^{**} \to Z^{**}, \quad A_3(x^{**}, y^{**})(z^*) = x^{**}(A_2(y^{**}, z^*)).$$

Let's denote A_3 by $\tilde{A}$. The first thing to verify is that $\tilde{A}|_{X\times Y}$ coincides with the original bilinear form A. Also, $||\tilde{A}|| = ||A||$. Both assertions are easy. Let's do an explicit calculation of $\tilde{A}$ at a point (x^{**}, y^{**}). To do this, we need to recall that from Goldstine's theorem, there are nets $(x_\alpha)_\alpha$ in X, resp. $(y_\beta)_\beta$ in Y, which converge weak$-*$ to x^{**}, resp. y^{**}. For fixed $z^* \in Z^*$, $A_2(y^{**}, z^*) \in X^*$, and so

$$\tilde{A}(x^{**}, y^{**})(z^*) = x^{**}(A_2(y^{**}, z^*)) = \lim_\alpha A_2(y^{**}, z^*)(x_\alpha) = \lim_\alpha y^{**}(A_1(z^*, x_\alpha)).$$

For each α, $A_1(z^*, x_\alpha) \in Y^*$, and once again we use weak$-*$ convergence to get that

$$\lim_\alpha y^{**}(A_1(z^*, x_\alpha)) = \lim_\alpha \lim_\beta A_1(z^*, x_\alpha)(y_\beta) = \lim_\alpha \lim_\beta z^*(A(x_\alpha, y_\beta)).$$

By an application of the Hahn-Banach theorem, we see that

$$\tilde{A}(x^{**}, y^{**}) = \lim_\alpha \lim_\beta A(x_\alpha, y_\beta).$$

In particular, if $Z = \mathbb{C}$ and $X = Y = E$, then we have extended an arbitrary bilinear form on $E \times E$ to one on $E^{**} \times E^{**}$. We use this to extend an arbitrary $P \in \mathcal{P}(^2E)$ to $\tilde{P} \in \mathcal{P}(^2E^{**})$, by $\tilde{P}(x^{**}) = \tilde{A}(x^{**}, x^{**})$.

A number of remarks are in order at this stage.

Remarks 5.4. (1) For any $x^{**}, \tilde{P}(x^{**}) = \lim_{\alpha_1} \lim_{\alpha_2} A(x_{\alpha_1}, x_{\alpha_2})$, and this is not in general equal to $\lim_\alpha A(x_\alpha, x_\alpha) = \lim_\alpha P(x_\alpha)$ (which might not even exist). Although $||A|| = ||\tilde{A}||$, it is by no means clear that $||P|| = ||\tilde{P}||$. Indeed, it appears (at least, at first) that all that can be deduced is that $||P|| \le ||A|| = ||\tilde{A}|| \le 2||\tilde{P}||$. Of course, Theorem 5.3 answers this question.

(2) The same technique can be used to extend n-homogeneous polynomials $P \in \mathcal{P}(^nE)$ and n-linear forms $A : X_1 \times \cdots \times X_n \to Z$, obtaining

$$\tilde{A}(x_1^{**}, \ldots, x_n^{**}) = \lim_{\alpha_1} \ldots \lim_{\alpha_n} A(x_{\alpha_1}, \ldots, x_{\alpha_n}),$$

and thus $\tilde{P}(x^{**}) = \tilde{A}(x^{**}, \dots, x^{**})$. The same unsatisfactory estimate on the norm of $\tilde{P}$ remains, since all that we know, a priori, is $||P|| \leq ||A|| = ||\tilde{A}|| \leq \frac{n^n}{n!}||\tilde{P}||$.

(3) There are many other techniques to extend elements of $\mathcal{P}(^nE)$ in a norm-preserving way. For example, M. Lindström and R. Ryan [17] construct one such extension (or rather very many such extensions!), by using ultrapowers. Galindo, García, Maestre, and Mujica use so-called Nicodemi sequences to extend multilinear mappings in [15]. In their work on the Kobayashi metric, Dineen and Timoney [14] obtain a norm-preserving extension by a somewhat different ultrapower approach.

(4) Our extension technique involved making a choice of which variable to extend first. We could as well have done our extension via iterates of the bilinear mapping $B_1 : Y \times Z^* \to X^*$, given by $B_1(y, z^*)(x) = z^*(A(x, y))$. Although the following example shows that the result would have been different in general, the extended value of P to E^{**} does not vary.

(5) The extension $P \to \tilde{P}$ is linear and continuous.

EXAMPLE 5.5. Let $A : \ell_1 \times \ell_1 \to \mathbb{K}$ be given by

$$A(x, y) = (x_1)y_2 + (x_1 + x_2 + x_3)y_4 + \cdots + (x_1 + \cdots + x_{2n-1})y_{2n} + \cdots$$
$$(y_1)x_2 + (y_1 + y_2 + y_3)x_4 + \cdots + (y_1 + \cdots + y_{2n-1})x_{2n} + \cdots$$

Note that $|A(x, y)| \leq ||x||_1||y||_1$, so that $A \in \mathcal{L}_s(^2\ell_1)$. Let $x^{**} \in \ell_1^{**}$ be a weak$-*$ limit point of the net $(e_{2n})_n$ and let $y^{**} \in \ell_1^{**}$ be a weak$-*$ limit point of $(e_{2k+1})_k$. Then,

$$\tilde{A}(x^{**}, y^{**}) = \lim_n \lim_k A(e_{2k+1}, e_{2n}) = \lim_n 0 = 0,$$

since for each fixed n, $A(e_{2k+1}, e_{2n})$ will be 0 provided k is large. On the other hand, $\tilde{A}(y^{**}, x^{**}) = \lim_k \lim_n A(e_{2k+1}, e_{2n}) = \lim_k 1 = 1$, since for each fixed k, $A(e_{2k+1}, e_{2n}) = 1$ for every large n.

The key here is what is known as regularity: A Banach space E is said to be *regular* if every continuous linear operator $E \to E^*$ is weakly compact. It is known that regularity of E is equivalent to the condition that for every $A \in \mathcal{L}(^2E)$, the two extensions (using $\lim_\alpha \lim_\beta$ and $\lim_\beta \lim_\alpha$ agree).

Let us turn to the proof of the Davie-Gamelin theorem:

Proof of 5.3. We first show for any $x^{**} \in E^{**}$, $||x^{**}|| \leq 1$, any $\epsilon > 0$, and any $N \in \mathbb{N}$, there are N points $x_1, \dots, x_N \in B_E$, such that

$$|A(x_{i_1}, \dots, x_{i_n}) - \tilde{P}(x^{**})| < \epsilon$$

whenever $1 \leq i_1 < \ldots < i_n \leq N$ and are all distinct. Since $\tilde{A}(\cdot, x^{**}, \ldots, x^{**}) \in E^*$, there is $x_1 \in B_E$ such that $|\tilde{A}(x_1, x^{**}, \ldots, x^{**}) - \tilde{P}(x^{**})| < \epsilon/n$. Next, choose $x_2 \in B_E$ which satisfies the two conditions:

$$|\tilde{A}(x_2, x^{**}, \ldots, x^{**}) - \tilde{P}(x^{**})| < \epsilon/n,$$
$$|\tilde{A}(x_1, x_2, x^{**}, \ldots, x^{**}) - \tilde{A}(x_1, x^{**}, \ldots, x^{**})| < \epsilon/n.$$

In general, given $x_1, \ldots, x_{l-1}$, we choose $x_l \in B_E$ satisfying the following:

$$|\tilde{A}(x_{i_1}, \ldots, x_{i_{l-1}}, x_{i_l}, x^{**}, \ldots, x^{**}) - \tilde{A}(x_{i_1}, \ldots, x_{i_{l-1}}, x^{**}, \ldots, x^{**})| < \epsilon/n$$

whenever $1 \leq i_1 < \ldots < i_l \leq N$. Then for any $i_1 < \ldots < i_n$, we have

$$\begin{aligned} |A(x_{i_1}, \ldots, x_{i_n}) - \tilde{P}(x^{**})| \leq |A(x_{i_1}, \ldots, x_{i_n}) \\ - \tilde{A}(x_{i_1}, \ldots, x_{i_{n-1}}, x^{**})| + \cdots \\ + |\tilde{A}(x_{i_1}, x^{**}, \ldots, x^{**}) - \tilde{P}(x^{**})| < \epsilon. \end{aligned}$$

Let's apply this with N very large and with the corresponding point $x = \frac{1}{N}(x_1 + \cdots + x_N) \in B_E$. We claim that $|P(x) - \tilde{P}(x^{**})| < \epsilon$. Once this claim has been proved, it will be obvious that $||\tilde{P}||$ cannot be larger than $||P||$, and the proof wil be finished. The proof of the claim consists of a counting argument:

$$|P(x) - \tilde{P}(x^{**})| = |A(\frac{x_1 + \cdots + x_N}{N}, \cdots, \frac{x_1 + \cdots + x_N}{N}) - \tilde{A}(x^{**}, \ldots, x^{**})|$$
$$\frac{1}{N^n} \sum_{i_j=1,\ j=1,\ldots,n}^{N} |A(x_{i_1}, \ldots, x_{i_n}) - \tilde{A}(x^{**}, \ldots, x^{**})| = \frac{1}{N^n}\{I + II\},$$

where I is the sum of $|A(x_{i_1}, \ldots, x_{i_n}) - \tilde{A}(x^{**}, \ldots, x^{**})|$, over all n-tuples of *distinct* indices and II is the sum of $|A(x_{i_1}, \ldots, x_{i_n}) - \tilde{A}(x^{**}, \ldots, x^{**})|$ over the rest of the indices. By the first part of the argument, $\frac{I}{N^n} < \epsilon$. Next, we count how many terms are contained in the summand II. In I, there are $N(N-1)(N-2)\cdots(N-n+1)$ terms, and so there are $N^n - [N(N-1)(N-2)\cdots(N-n+1)]$ terms in II. Each of these terms is bounded by $2||\tilde{A}|| = 2||A|| \leq 2\frac{n^n}{n!}||P|| = C$, say. Hence,

$$\begin{aligned} \frac{II}{N^n} &\leq \frac{C}{N^n}\left(N^n - [N(N-1)(N-2)\cdots(N-n+1)]\right) \\ &= C[1 - 1(1 - \frac{1}{N})(1 - \frac{2}{N})\cdots(1 - \frac{n-1}{N})], \end{aligned}$$

which tends to 0 as $N \to \infty$. ∎

An examination of the above proof shows that somewhat more has been proved:

COROLLARY 5.6. *Given $x^{**} \in B_{E^{**}}$, there is a net $(x_\alpha) \in B_E$ such that $P(x_\alpha) \to \tilde{P}(x^{**})$ for every $P \in \mathcal{P}(E)$.*

Now, for the long-awaited *interesting* part of this section! First, let's introduce a small bit of notation (cf. Definition 2.8). We will denote by $\mathcal{H}^\infty(B_E)$ the set $\{f \in \mathcal{H}(B_E)\colon ||f||_{B_E} = \sup_{||x|| \leq 1} |f(x)| < \infty\}$. The space $\mathcal{H}^\infty(B_E)$ is the analogue of the classical space $\mathcal{H}^\infty(D)$, where $D \subset \mathbb{C}$ is the open unit disc.

$\mathcal{H}^\infty(B_E)$ is a Banach algebra with identity, since it is clear that $||fg||_{B_E} \leq ||f||_{B_E}||g||_{B_E}$. We denote by $\mathcal{M}(\mathcal{A})$ the *maximal ideal space* of the Banach algebra $\mathcal{A}$. That is, $\mathcal{M}(\mathcal{A}) = \{\phi : \mathcal{A} \to \mathbb{C}\colon \phi$ is a non-trivial homomorphism$\}$; it is a weak$-*$ compact subset of the unit ball of $\mathcal{A}^*$. One can view $\mathcal{A}$ as a subalgebra of $\mathcal{C}(\mathcal{M}(\mathcal{A}))$, via the *Gelfand mapping* $x \in \mathcal{A} \mapsto \hat{x} \in \mathcal{C}(\mathcal{M}(\mathcal{A}))$, where $\hat{x}(\phi) = \phi(x)$ for $\phi \in \mathcal{M}(\mathcal{A})$.

In our situation, when $\mathcal{A} = \mathcal{H}^\infty(B_E)$, we have even more structure, since we can embed $B_E \hookrightarrow \mathcal{M}(\mathcal{A})$ in the obvious way: $\delta : x \rightsquigarrow \delta_x$, where $\delta_x(f) = f(x)$. It is also the case that every homomorphism $\phi \in \mathcal{M}(\mathcal{A})$ lies 'over' a point in the closed ball of E^{**}. This is done via the mapping $\Pi : \mathcal{M}(\mathcal{A}) \to E^{**}$, $\Pi(\phi) = \phi|_{E^*}$. Some observations concerning Π should be made:

(a) $||\Pi(\phi)|| = \sup_{x^* \in B_{E^*}} |\phi(x^*)| \leq 1$ since each $\phi \in \mathcal{M}(\mathcal{A})$ has norm 1.

(b) $\Pi(\delta_x) = x$ for every $x \in B_E$.

(c) Π is weak-$*$ - weak-$*$ continuous.

By Goldstine's theorem, $B_E = \Pi(\delta(B_E))$ is a weak-$*$ dense subset of $B_{E^{**}}$, and consequently $\Pi(\mathcal{M}(\mathcal{A})) = \overline{B_{E^{**}}}$, which is the closed unit ball of E^{**}. This is what we wanted to show.

But, aside from the obvious homomorphisms $\{\delta_x\colon x \in B_E\}$, *what other homomorphisms are there?*

Let's show how the above results imply that every point in B_E^{**} gives rise to a homomorphism. We first show that each $f \in \mathcal{H}^\infty(B_E)$ extends to an $\tilde{f} \in \mathcal{H}^\infty(B_E^{**})$, and that in addition $||f||_{B_E} = ||\tilde{f}||_{B_{E^{**}}}$. First, if $f = \sum_{n=0}^\infty P_n$ is in $\mathcal{H}^\infty(B_E)$, we extend each P_n to $\tilde{P}_n \in \mathcal{P}(^nE^{**})$. Since f is bounded on B_E, an application of the Cauchy-Goursat formula shows that $\limsup ||P_n||^{1/n} = \limsup ||\tilde{P}_n||^{1/n} \leq 1$. Thus, $\tilde{f} = \sum_{n=0}^\infty \tilde{P}_n$ defines an analytic function on $B_{E^{**}}$. The next step is to show that $\tilde{f}$ is bounded on $B_{E^{**}}$. We do this by showing that $||f||_{B_E} = ||\tilde{f}||_{B_{E^{**}}}$. So, fix $x^{**} \in B_{E^{**}}$ with $||x^{**}|| = r < 1, say$, and let $\epsilon > 0$. For any m,

$$|\tilde{f}(x^{**}) - \sum_{n=0}^{m} \tilde{P}_n(x^{**})| \leq \sum_{n=m+1}^{\infty} |\tilde{P}_n(x^{**})| \leq \sum_{n=m+1}^{\infty} ||\tilde{P}_n|| r^n.$$

Therefore, for sufficiently large m, $|\tilde{f}(x^{**}) - \sum_{n=0}^{m} \tilde{P}_n(x^{**})| < \epsilon$. If $(x_\alpha) \in rB_E$ is such that $x_\alpha \to x^{**}$ weak$-*$, then

$$\begin{aligned}|f(x_\alpha) - \tilde{f}(x^{**})| \leq & |f(x_\alpha) - \sum_{n=0}^{m} P_n(x_\alpha)| \\ & + |\sum_{n=0}^{m} P_n(x_\alpha) - \sum_{n=0}^{m} \tilde{P}_n(x^{**})| + |\sum_{n=0}^{m} \tilde{P}_n(x^{**}) - \tilde{f}(x^{**})|.\end{aligned}$$

By Corollary 5.6, $|\sum_{n=0}^{m} P_n(x_\alpha) - \sum_{n=0}^{m} \tilde{P}_n(x^{**})| \to 0$ with α. Consequently, for many α, $|f(x_\alpha) - \tilde{f}(x^{**})| < 3\epsilon$. We have thus proved the following.

COROLLARY 5.7. *There is an extension mapping which takes $f \in \mathcal{H}^\infty(B_E)$ to $\tilde{f} \in \mathcal{H}^\infty(B_{E^{**}})$, such that $||f||_{B_E} = ||\tilde{f}||_{B_{E^{**}}}$.*

Moreover, $f \to \tilde{f}$ is linear, continuous, and multiplicative; that is, $\widetilde{fg} = \tilde{f} \cdot \tilde{g}$ for every $f, g \in \mathcal{H}^\infty(B_E)$. The linearity and continuity have already been proved, and the multiplicativity follows upon sufficient staring at the way the extensions are constructed.

How do such homomorphisms lying over points of $B_{E^{**}}$ arise? The easiest way to construct such a homomorphism is by using Corollary 5.7: For each x^{**} in the open ball of E^{**}, define $\widetilde{\delta_{x^{**}}} \in \mathcal{M}(\mathcal{A})$ by $\widetilde{\delta_{x^{**}}}(f) := \tilde{f}(x^{**})$. In this way, we get a mapping $\tilde{\delta} : B_{E^{**}} \to \mathcal{M}(\mathcal{A})$. This mapping is one-to-one. To see this, note that if $x^{**} \neq y^{**} \in E^{**}$, then there must be $x^* \in E^*$ such that $x^{**}(x^*) \neq y^{**}(x^*)$. Thus, $\widetilde{\delta_{x^{**}}}(x^*) \neq \widetilde{\delta_{y^{**}}}(x^*)$.

At this point, a logical question would be: Why not continue this process? That is, why not apply our reasoning and Corollary 5.7 two, or three, or more times? In this way, we would obtain a function $\tilde{\tilde{\delta}} : B_{E^4} \to \mathcal{M}(\mathcal{A})$, where E^4 denotes the fourth dual of E, and then $\tilde{\tilde{\tilde{\delta}}}$ which would take the ball of the sixth dual into $\mathcal{M}(\mathcal{A})$, etc. The partial answer, whose proof is a bit complicated (and omitted), is given in the following theorem. For it, we need to give a slightly altered definition of regularity: We say that E is *symmetrically regular* if E satisfies the regularity condition with respect to symmetric continuous bilinear forms (rather than arbitrary continuous bilinear forms). With this definition, we can now state our result.

THEOREM 5.8. [3] *A complex Banach space E is symmetrically regular if and only if every homomorphism of the form $\widetilde{\widetilde{\delta_{x^4}}}$, arising from a point of* *the fourth dual of E, is equal to a homomorphism of the form $\widetilde{\delta_{x^{**}}}$ for some $x^{**} \in B_{E^{**}}$.*

In particular, since ℓ_1 is not symmetrically regular by Example 5.5, there are points x^4 in the fourth dual of ℓ_1 such that $\widetilde{\widetilde{\delta_{x^4}}}$ is a homomorphism which does not correspond to any point of the second dual. And, this leads to some interesting questions with which we conclude:

QUESTIONS 5.9. (1) Are there points x^6 of the unit ball of the *sixth* dual of ℓ_1 which do not correspond to any $\widetilde{\widetilde{\delta_{x^4}}}$?

(2) We do not even know the cardinality of the set of homomorphisms $\mathcal{M}(\mathcal{H}^\infty(B_{\ell_1}))$. Is it strictly smaller than the cardinality of ℓ_1^4, which is known to be at least as big as 2^{2^c} (see, for example, [12, p.211])? Can we characterize those points of $B_{\ell_1^4}$ which give rise to homomorphisms which do not already arise from $B_{\ell_1^{**}}$? Is it possible to characterize when two points $x^4, y^4 \in B_{\ell_1^4}$ are such that $\widetilde{\widetilde{\delta_{x^4}}} = \widetilde{\widetilde{\delta_{y^4}}}$?

REFERENCES

[1] ARENS, R., The adjoint of a bilinear operation, *Proc. Amer. Math. Soc.,* **2** (1951), 839–848.

[2] ARON, R.M., Weakly uniformly continuous and weakly sequentially continuous entire functions, in "Advances in Holomorphy", Rio de Janeiro 1977, J.A. Barroso (editor), North Holland, 47-66.

[3] ARON, R.M., GALINDO, P., GARCÍA, D., MAESTRE, M., Regularity and algebras of analytic functions in infinite dimensions, *Trans. Amer. Math. Soc.,* **348**, 2 (1996), 543–559.

[4] ARON, R.M., RUEDA, M.P., A problem concerning zero subspaces of polynomials, *Linear Topological Spaces and Complex Analysis,* **3** (1997), 20–23.

[5] ARON, R.M., BEAUZAMY, B., ENFLO, P., Polynomials in many variables: Real vs complex norms, *J. Approx. Theory,* **74**, 2 (1993), 181–198.

[6] ARON, R.M., BOYD, C., RYAN, R.A., ZALDUENDO, I., Zeros of Polynomials on Banach spaces: The Real Story, (preprint).

[7] ARON, R.M., GONZALO, R., ZAGORODNYUK, A., Zeros of real polynomials, *Linear and multilinear algebra,* **48** (2000), 107–115.

[8] ARON, R.M., PROLLA, J.B., Polynomial approximation of differentiable functions on Banach spaces, *J. Reine Angew. Math.,* **313** (1980), 195–216.

[9] BIRCH, B.J., Homogeneous forms of odd degree in a large number of variables, *Mathematika,* **4** (1957), 102–105.

[10] BENITEZ, C., SARANTOPOULOS, Y., TONGE, A., Lower bounds for norms of products of polynomials, *Math. Proc. Cambridge Philos. Soc.,* **124**, no. 3 (1998), 395–408.

[11] Davie, A.M., Gamelin, T.W., A theorem on polynomial-star approximation, *Proc. Amer. Math. Soc.,* **106**, no. 2 (1989), 351 – 356.

[12] Diestel, J., "Sequences and Series in Banach Spaces", Grad. Text Math., **92**, Springer-Verlag, New York, 1984.

[13] Dineen, S., "Complex Analysis on Infinite Dimensional Spaces", Springer-Verlag, London, 1999.

[14] Dineen, S., Timoney, R., Complex geodesics on convex domains, in "Progress in Functional Analysis", North-Holland Math. Stud., **170**, North-Holland, Amsterdam, 1992, 333 – 365.

[15] Galindo, P., García, D., Maestre, M., Mujica, J., Extension of multilinear mappings on Banach spaces, *Studia Math.,* **108**, no. 1 (1994), 55 – 76.

[16] Gunning, R., Rossi, H., "Analytic Functions of Several Complex Variables", Prentice-Hall, Inc., 1965.

[17] Lindstrom, M., Ryan, R., Applications of ultraproducts to infinite-dimensional holomorphy, *Math. Scand.,* **71**, no. 2 (1992), 229 – 242.

[18] Mazet, P., A Hahn-Banach theorem for quadratic forms, *Math. Proc. Royal Irish Acad.,* **100** (2) (2000), 177 – 182.

[19] Nemirovskii, A.S., Semenov, S.M., The polynomial approximation of functions on Hilbert space, *Mat. Sb. (N.S.),* **92** (134) (1973), 257 – 281.

[20] Pełczyński, A., On weakly compact polynomial operators on B-spaces with Dunford-Pettis property, *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.,* **11** (1963), 371 – 378.

[21] Plichko, A., Zagorodnyuk, A., On automatic continuity and three problems of The Scottish book concerning the boundedness of polynomial functionals, *J. Math. Anal. Appl.,* **220**, no. 2 (1998), 477 – 494.

[22] Ryan, R.A., Dunford-Pettis properties, *Bull. Acad. Polon. Sci. Math.,* **27**, no. 5 (1979), 373 – 379.

[23] Schottenloher, R.M., (private communication, which appeared with permission in R. M. Aron & P. D. Berner, A Hahn-Banach extension theorem for analytic mappings, *Bull. Soc. Math. France,* **106**, no. 1 (1978) 3 – 24).