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Positive Solutions of Systems of Boundary Value Problems

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1. Introduction

Let E be a finite dimensional real vector space ordered by a cone K. A cone K is a nonempty closed convex subset of E with $\lambda K \subseteq K$ $(\lambda \geq 0)$, and $K \cap (-K) = \{0\}$. As usual $x \leq y :\Leftrightarrow y - x \in K$. We assume that K is solid, that is $K^\circ \neq \emptyset$. For $x \leq y$ let $[x, y]$ denote the order interval of all z with $x \leq z \leq y$. Let K^* denote the dual cone of K, that is the set of all $\varphi \in E^*$ with $\varphi(x) \geq 0$ $(x \geq 0)$.

For a continuous function $f : [0,1] \times K \times E \to E$ and $u_0, u_1 \in K$ we consider the Dirichlet boundary value problem

$$u''(t) + f(t, u(t), u'(t)) = 0 \quad (t \in [0,1]), \qquad u(0) = u_0, \quad u(1) = u_1. \tag{1}$$

We will prove the existence of positive solutions of (1), that is $u(t) \geq 0$ $(t \in [0,1])$, under invariance and Perow conditions on f.

We first consider the following invariance condition (I):

$$t \in [0,1],\ x \in \partial K,\ y \in E,\ \varphi \in K^*,\ \varphi(x) = \varphi(y) = 0 \ \Rightarrow\ \varphi(f(t, x, y)) \geq 0.$$

Conditions related to (I) have been used in [9], [10] and [13] to prove the existence of a solution of boundary value problems in convex subsets of E under Lipschitz or Nagumo type growth conditions on f, or in [5] and [16] to prove positivity of solutions of second order differential inequalities. In this paper we combine (I) with a Perow condition of the following type. Let $L(E)$ denote the algebra of all endomorphisms on E.

We consider the condition (P):

There exist $p \in K^\circ$ and $A, B \in C([0,1], L(E))$ such that

$$A(t)p = 0 \quad (t \in [0,1]),$$

and

$$\lim_{x \in K,\ ||x||+||y|| \to \infty} \frac{||f(\cdot, x, y) - A(\cdot)x - B(\cdot)y||_\infty}{||x|| + ||y||} = 0.$$

Remark. We consider E to be normed by $||\cdot||$, the Minkowski functional of $[-p, p]$. Note that $-||x||p \le x \le ||x||p$ $(x \in E)$. By $||\cdot||_\infty$ we denote the corresponding maximum norm on $C([0,1], E)$.

We are now able to state our existence result.

THEOREM 1. *Let f be continuous and satisfy* (I) *and* (P). *Then problem* (1) *has a solution $u : [0,1] \to K$.*

Remarks. 1. If $E = \mathbb{R}$ $(K = [0, \infty))$, then (I) means $f(t, 0, 0) \ge 0$ $(t \in [0,1])$. Moreover in this case $A(\cdot)p{=}0$ means $A(\cdot) = 0$, and the following example shows that we cannot omit this condition: The unique solution of

$$u''(t) + \left(\frac{5\pi}{2}\right)^2 u(t) = 0, \quad u(0) = 0,\ u(1) = 1$$

is $u(t) = \sin((5\pi/2)t)$, and u is not positive on $[0,1]$.

2. The existence of positive solutions of boundary value problems for special cones was studied by various authors and different methods. See for example [1] for the cone of semidefinite matrices in the space of symmetric matrices, or [6], [11], [12] for the coordinate cone in $\mathbb{R}^n$. Of course the method of upper and lower solutions also leads to existence of positive solutions (if the lower solution is in K). For results of this type in ordered vector spaces see for example [1], [3], [8], and [14] p.288 ff.

2. Preliminaries

We first consider the interaction of (I) and (P). Condition (I) is connected with the idea of quasimonotonicity. Let $D \subseteq E$. A function $h : D \to E$ is called quasimonotone increasing (qmi for short) on D, in the sense of Volkmann [17], if

$$x, y \in D,\ x \le y,\ \varphi \in K^*,\ \varphi(x) = \varphi(y) \quad \Rightarrow \quad \varphi(h(x)) \le \varphi(h(y)).$$

In particular if $C \in L(E)$, then $x \mapsto Cx$ is qmi on E if and only if

$$x \geq 0,\ \varphi \in K^*,\ \varphi(x) = 0 \quad \Rightarrow \quad \varphi(Cx) \geq 0.$$

Moreover if $C \in L(E)$, then

$$y \in E,\ \varphi \in K^*,\ \varphi(y) = 0 \quad \Rightarrow \quad \varphi(Cy) = 0$$

is valid if and only if $C = \mu \mathrm{id}_E$ for some $\mu \in \mathbb{R}$. The reason is that K^* is a solid cone (since K is solid and $N := \dim E < \infty$), so we can choose a base $\{\varphi_1, \ldots, \varphi_N\} \subseteq K^*$ of E^* and a predual base $\{y_1, \ldots, y_N\}$ of E with $\varphi_i(y_j) = \delta_{ij}$ $(i, j = 1, \ldots, N)$. With respect to this base it is easy to check that the matrix corresponding to C is μI for some $\mu \in \mathbb{R}$.

These considerations lead to

PROPOSITION 1. *Let f be as in Theorem 1. Then $x \mapsto A(t)x$ is qmi on E $(t \in [0, 1])$, and $B(\cdot) = \mu(\cdot)\mathrm{id}_E$ for a function $\mu \in C([0, 1], \mathbb{R})$.*

Proof. We fix $t \in [0, 1]$. Let $x \in K \setminus \{0\}$, and $\varphi \in K^* \setminus \{0\}$ with $\varphi(x) = 0$. In particular $x \in \partial K$. Condition (P) implies

$$\lim_{\lambda \to \infty} \frac{f(t, \lambda x, 0)}{\lambda ||x||} = A(t) \frac{x}{||x||},$$

and by condition (I) we have $\varphi(f(t, \lambda x, 0)) \geq 0$ $(\lambda \geq 0)$. Hence $\varphi(A(t)x) \geq 0$. Analogously let $y \in E \setminus \{0\}$, and $\varphi \in K^*$ with $\varphi(y) = 0$. Then

$$\lim_{\lambda \to \pm\infty} \frac{f(t, 0, \lambda y)}{|\lambda|\ ||y||} = \pm B(t) \frac{y}{||y||},$$

and $\varphi(f(t, 0, \lambda y)) \geq 0$ $(\lambda \in \mathbb{R})$. Hence $\varphi(B(t)y) = 0$. ∎

Next, let $[a, b] \subseteq \mathbb{R}$. We consider $H \in C([a, b], L(E))$ with $x \mapsto H(t)x$ qmi on E $(t \in [a, b])$, and with $H(\cdot)p = 0$ for some $p \in K^\circ$. Then it is known, see Theorem 1 in [5], that the boundary value problem

$$z''(s) + H(s)z(s) = 0 \quad (s \in [a, b]), \qquad z(a) = 0, \quad z(b) = 0$$

has only the trivial solution. This leads to

PROPOSITION 2. *Let f be as in Theorem 1. Then, the boundary value problem*

$$w''(t) + A(t)w(t) + B(t)w'(t) = 0 \quad (t \in [0, 1]), \qquad w(0) = 0, \quad w(1) = 0$$

has only the trivial solution.

Proof. According to Proposition 1 we have $B(t) = \mu(t)\mathrm{id}_E$. We use the following transformations, see [2, p. 323]. Set

$$q(t) = \exp\Big(\int_0^t \mu(\tau)\, d\tau\Big), \quad r(t) = \int_0^t \frac{1}{q(\sigma)}\, d\sigma \quad (t \in [0,1]).$$

Note that r is strictly increasing and the inverse function $r^{-1} : [0, r(1)] \to [0,1]$ has continuous second derivative. Now if $w : [0,1] \to E$ is a solution of the problem under consideration, then $z : [0, r(1)] \to E$ defined by $z(s) = w(r^{-1}(s))$ solves

$$z''(s) + (q(r^{-1}(s)))^2 A(r^{-1}(s)) z(s) = 0, \qquad z(0) = 0, \quad z(r(1)) = 0.$$

Since q^2 is nonnegative the function $H : [0, r(1)] \to L(E)$ defined by

$$H(s) := (q(r^{-1}(s)))^2 A(r^{-1}(s))$$

has the properties described above. Hence $z = 0$ and therefore $w = 0$. ∎

Proposition 2 is the reason why we can make use of the following multidimensional version [4] of a theorem of Perow [7, p.149]:

Let $g : [0,1] \times E^2 \to E$ be continuous, and assume that there exist $A, B \in C([0,1], L(E))$ such that the boundary value problem $w''(t) + A(t)w(t) + B(t)w'(t) = 0$, $w(0) = w(1) = 0$ has only the trivial solution, and such that

$$\lim_{||x||+||y|| \to \infty} \frac{||g(\cdot, x, y) - A(\cdot)x - B(\cdot)y||_\infty}{||x|| + ||y||} = 0. \tag{2}$$

Then $v''(t) + g(t, v(t), v'(t)) = 0$, $v(0) = v_0$, $v(1) = v_1$ has a solution for each $v_0, v_1 \in E$. Moreover, in this case we have the following a priori estimates:

By the uniqueness of the linear boundary value problem we obtain constants $C_1, C_2 > 0$ such that the solution w of $w''(t) + A(t)w(t) + B(t)w'(t) = r(t)$, $w(0) = v_0$, $w(1) = v_1$ (with v_0, v_1 fixed) satisfies

$$||w||_\infty + ||w'||_\infty \le C_1 ||r||_\infty + C_2$$

for each $r \in C([0,1], E)$.

From (2) we get a constant $M > 0$ such that

$$||g(\cdot, x, y) - A(\cdot)x - B(\cdot)y||_\infty \le M + \frac{1}{2C_1}(||x|| + ||y||) \quad (x, y \in E).$$

Hence v solves

$$v''(t) + A(t)v(t) + B(t)v'(t) = r(t) \quad (t \in [0,1]), \quad v(0) = v_0,\ v(1) = v_1$$

for a function r with

$$||r||_\infty \le M + \frac{1}{2C_1}(||v||_\infty + ||v'||_\infty).$$

Therefore

$$||v||_\infty + ||v'||_\infty \le 2(C_1 M + C_2). \tag{3}$$

Next, we consider the chosen norm which is

$$||x|| = \min\{\lambda \ge 0 : -\lambda p \le x \le \lambda p\}.$$

We set $d(x) := \text{dist}(x, K)$ $(x \in E)$. For each $k \in K$ we get

$$x - k + ||x - k||p \ge 0 \quad \Rightarrow \quad x + ||x - k||p \ge 0,$$

and therefore $x + d(x)p \ge 0$ $(x \in E)$. This, together with

$$||x - (x + d(x))p|| = d(x)$$

proves

$$x + d(x)p \in \partial K \quad (x \in E \setminus K). \tag{4}$$

For the following functional representation of d see for example [5].

$$d(x) = \max\{-\varphi(x) : \varphi \in K^*, ||\varphi|| = 1\} \quad (x \in E \setminus K), \tag{5}$$

where $||\cdot||$ on E^* denotes the corresponding dual norm. Note that

$$||\varphi|| = \varphi(p) \quad (\varphi \in K^*).$$

3. Proof of Theorem 1

For $n \in \mathbb{N}$ we define $g_n : [0,1] \times E^2 \to E$ by

$$g_n(t, x, y) = f(t, x + d(x)p, y) + \frac{1}{n}p.$$

Since f and d are continuous each g_n is continuous. Fix $n \in \mathbb{N}$. In the first step we prove that

$$v''(t) + g_n(t, v(t), v'(t)) = 0 \quad (t \in [0,1]), \qquad v(0) = u_0, \quad v(1) = u_1, \tag{6}$$

has a solution $v : [0,T] \to E$. Since $A(\cdot)p = 0$ we have

$$Q(x,y) := \frac{||g_n(\cdot,x,y) - A(\cdot)x - B(\cdot)y||_\infty}{||x|| + ||y||}$$

$$\leq \frac{||f(\cdot, x + d(x)p, y) - A(\cdot)(x + d(x)p) - B(\cdot)y||_\infty + n^{-1}}{||x|| + ||y||}.$$

Now, consider a sequence $((x_k, y_k))_{k=1}^\infty$ in E^2, and set $\xi_k := x_k + d(x_k)p$.

If $||\xi_k|| \to \infty$ or $||y_k|| \to \infty$, then

$$Q(x_k, y_k) = \frac{||f(\cdot, \xi_k, y_k) - A(\cdot)\xi_k - B(\cdot)y_k||_\infty + n^{-1}}{||\xi_k|| + ||y_k||} \cdot \frac{||\xi_k|| + ||y_k||}{||x_k|| + ||y_k||} \to 0$$

by means of (P), and since $||x + d(x)p|| \leq 2||x||$ $(x \in E)$.

If $(||\xi_k||)$ and $(||y_k||)$ are bounded, and $||x_k|| \to \infty$, then

$$Q(x_k, y_k) = \frac{||f(\cdot, \xi_k, y_k) - A(\cdot)\xi_k - B(\cdot)y_k||_\infty + n^{-1}}{||\xi_k|| + ||y_k|| + 1} \cdot \frac{||\xi_k|| + ||y_k|| + 1}{||x_k|| + ||y_k||} \to 0$$

since the first factor is bounded, and the second factor is tending to 0.

This proves

$$\lim_{||x||+||y|| \to \infty} Q(x,y) = 0,$$

and therefore (6) has a solution $v : [0,1] \to E$.

In the second step we prove that each solution of (6) is positive. Assume that this is not the case. Then $d(v(t))$ has a maximum at $t_0 \in (0,1)$, say, with $0 < d(v(t_0))$. According to (5) there exists $\varphi \in K^*$ such that $||\varphi|| = \varphi(p) = 1$ and $d(v(t_0)) = -\varphi(v(t_0))$. In particular

$$\varphi(v(t_0) + d(v(t_0))p) = \varphi(v(t_0)) + d(v(t_0))\varphi(p) = 0.$$

Moreover, for each $t \in [0,1]$

$$-\varphi(v(t)) \leq -\varphi(v(t) - k) \leq ||v(t) - k|| \quad (k \in K),$$

hence

$$-\varphi(v(t)) \leq d(v(t)) \leq d(v(t_0)) = -\varphi(v(t_0)) \quad (t \in [0,1]),$$

and therefore $\varphi(v'(t_0)) = 0$. According to (4) and condition (I) we obtain

$$\begin{aligned} -\varphi(v''(t_0)) &= \varphi(g_n(t_0, v(t_0), v'(t_0))) \\ &= \varphi(f(t_0, v(t_0) + d(v(t_0))p, v'(t_0)) + p/n) \\ &\geq \varphi(p/n) > 0. \end{aligned} \tag{7}$$

Now, for $h > 0$ sufficiently small,

$$\begin{aligned} &-\frac{d(v(t_0+h)) - 2d(v(t_0)) + d(v(t_0-h))}{h^2} \\ &\le \frac{\varphi(v(t_0+h)) - 2\varphi(v(t_0)) + \varphi(v(t_0-h))}{h^2} \\ &= \varphi\Big(\frac{v(t_0+h) - 2v(t_0) + v(t_0-h)}{h^2}\Big). \end{aligned}$$

By means of (7) and $d(v(t)) \le d(v(t_0))$ $(t \in [0,1])$ we conclude

$$0 \ge \liminf_{h\to 0+} \frac{d(v(t_0+h)) - 2d(v(t_0)) + d(v(t_0-h))}{h^2} \ge -\varphi(v''(t_0)) > 0,$$

a contradiction. Thus $v(t) \ge 0$ $(t \in [0,1])$.

In the last step we choose a solution $v_n : [0,1] \to K$ of (6) for each $n \in \mathbb{N}$. According to the a priori estimate (3) we obtain

$$||v_n||_\infty + ||v_n'||_\infty \le 2C_1(M + n^{-1}) + 2C_2 \le 2C_1(M+1) + 2C_2 \quad (n \in \mathbb{N}),$$

where M is such that

$$||f(\cdot, x + d(x)p, y) - A(\cdot)x - B(\cdot)y||_\infty \le M + \frac{1}{2C_1}(||x|| + ||y||) \quad (x, y \in E).$$

By standard reasoning we can choose a subsequence of (v_n) that converges in $C^2([0,1], E)$ to a solution $u : [0,1] \to K$ of

$$u''(t) + f(t, u(t) + d(u(t))p, u'(t)) = 0, \quad u(0) = u_0,\ u(1) = u_1,$$

and u solves (1) since $d(u(t)) = 0$ $(t \in [0,1])$.

4. Examples

For f we consider a perturbation of a linear function, namely

$$f(t,x,y) = A(t)x + B(t)y + G(t,x,y)x \quad ((t,x,y) \in [0,1] \times K \times E).$$

We assume that

1. $A \in C([0,1], L(E))$, $x \mapsto A(t)x$ is qmi on E $(t \in [0,1])$, and $A(\cdot)p = 0$ for some $p \in K^\circ$,

2. $B(\cdot) = \mu(\cdot)\mathrm{id}_E$, $\mu \in C([0,1], \mathbb{R})$,
   and that the function $G : [0,1] \times K \times E \to L(E)$ is such that
3. G is continuous,
4. $z \mapsto G(t,x,y)z$ is qmi on E $((t,x,y) \in [0,1] \times K \times E)$,
5. $$\frac{||x||}{||x|| + ||y||} G(t,x,y) \to 0 \quad (x \in K,\ ||x|| + ||y|| \to \infty)$$
   in $L(E)$ uniformly on $[0,1]$.

Then obviously (I) and (P) are satisfied. Hence (1) has a solution $u : [0,1] \to K$.

We consider $E = \mathbb{R}^3$ ordered by the ice-cream cone

$$K = \Big\{ x = (x_1, x_2, x_3) :\ x_3 \geq \sqrt{x_1^2 + x_2^2} \Big\}.$$

The linear qmi mappings are characterized in [15], in particular $C \in L(E)$ defines a quasimonotone constant mapping (that is $x \mapsto Cx$ and $x \mapsto -Cx$ are qmi on E) if and only if C has the form

$$C = \begin{pmatrix} \alpha & \beta & \gamma \\ -\beta & \alpha & \delta \\ \gamma & \delta & \alpha \end{pmatrix} \quad \alpha, \beta, \gamma, \delta \in \mathbb{R}.$$

For example let

$$A(t) = \begin{pmatrix} 0 & t^2 & 0 \\ -t^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then 1. is satisfied for $p = (0,0,1) \in K^\circ$, and the corresponding norm is $||x|| = |x_3| + \sqrt{x_1^2 + x_2^2}$. Next, let $\mu = 0$ and let G be defined by

$$G(t,x,y) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{\sqrt[4]{y_1^2 + y_2^2}}{1+t+2x_3} \\ 0 & \frac{\sqrt[4]{y_1^2 + y_2^2}}{1+t+2x_3} & 0 \end{pmatrix}.$$

Then obviously 3. and 4. are valid, and 5. follows from

$$\frac{\sqrt[4]{y_1^2 + y_2^2}}{1 + t + 2x_3} \leq \frac{\sqrt{||y||}}{1 + ||x||} \quad (x \in K,\ y \in E,\ t \in [0,1]).$$

Thus, for example the boundary value problem

$$u_1''(t) + t^2 u_2(t) = 0$$

$$u_2''(t) - t^2 u_1(t) + \frac{\sqrt[4]{(u_1'(t))^2 + (u_2'(t))^2}}{1 + t + 2u_3(t)} u_3(t) = 0$$

$$u_3''(t) + \frac{\sqrt[4]{(u_1'(t))^2 + (u_2'(t))^2}}{1 + t + 2u_3(t)} u_2(t) = 0$$

$$u(0) = (-1, 0, 1), \quad u(1) = (0, 1, 1)$$

has a solution $u : [0,1] \to K$, that is $u_3(t) \geq \sqrt{u_1^2(t) + u_2^2(t)}$ $(t \in [0,1])$.

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